ON A GALOIS CONNECTION BETWEEN ALGEBRAS OF LINEAR TRANSFORMATIONS AND LATTICES OF SUBSPACES OF A VECTOR SPACE

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1. Introduction. Representation theory has contributed much to the study of linear associative algebras. The central problem of representation theory *per se* is the determination for each algebra of all its indecomposable representations. This turns out to be a much deeper problem than the classification of algebras, in the sense that there are algebras for which any "internal question" can be answered but for which the number and nature of representations is almost completely unknown, or if known is much more complicated than the internal theory. This can be illustrated by the example of a commutative algebra of order three for which the representation theory can be shown to be essentially the same as the problem of classifying pairs of rectangular matrices under equivalence. (This algebra has indecomposable representations of every integral degree.)

Detailed study (as yet unpublished) of the representations of certain classes of algebras has led me to consider the possibility of searching for connections between representation theory and lattice theory. The present note is devoted to setting up the machinery for certain phases of such an investigation.

Notation and definitions. Let $\mathfrak{k}$ be a sfield and V a right $\mathfrak{k}$-space of dimension n. If $v_1, \ldots, v_n$ are a basis for V then any vector v in V can be written in the form $v = v_1a_1 + \ldots + v_na_n$ where $a_1, \ldots, a_n$ are uniquely determined scalars (i.e. elements of $\mathfrak{k}$) called the coordinates of v relative to the given basis for V. This can be written in the matrix form as $v = ||v_j||\cdot||a_i||$ where $||v_j||$ denotes the 1 by n (row) matrix made up of the basis vectors and $||a_i||$ denotes the n by 1 (column) matrix made up of the coordinates. (In describing any matrix we shall use the subscript "i" for row index and "j" for column index.) Then for any vector v and scalar a, va is the vector with coordinate matrix $||b_i|| = ||a_ia||$.

We denote by $\mathfrak{T}$ the set of all linear transformations α (i.e., $\mathfrak{k}$-endomorphism) of V into itself. We write the linear transformations as left operators, and then the commutativity of linear transformations with the scalar multiplications take the form $(\alpha v)a = \alpha(va)$. To express the linear transformations in matrix form we use the formula

$$\alpha v = \alpha\,(||v_j||\cdot||a_i||) = (\alpha||v_j||)||a_i|| = ||\alpha v_j||\cdot||a_i||$$
$$= (||v_j||T_\alpha)||a_i|| = ||v_j||(T_\alpha||a_i||).$$

Here T_α is, of course the matrix whose jth column is the coordinate matrix of

Received December 22, 1950.

https://doi.org/10.4153/CJM-1952-021-1 Published online by Cambridge University Press

αv_j. Conversely, the same formula read in reverse shows that every n by n $\mathfrak{k}$-matrix T defines a linear transformation α. We make $\mathfrak{T}$ into a right $\mathfrak{k}$-space of dimension n^2 by the definition $\alpha a = \beta$, where β is the linear transformation for which

$$T_\beta = T_\alpha \left\| \begin{matrix} a & 0 \\ 0 & a \end{matrix} \right\| .$$

The set $\mathfrak{N}$ of all subspaces of V is a complemented modular lattice of (lattice) dimension n. With any subalgebra $\mathfrak{A}$ of $\mathfrak{T}$ we associate the sublattice $\mathfrak{L} = \mathfrak{A}^*$ of $\mathfrak{N}$ consisting of all subspaces of V invariant under $\mathfrak{A}$. With any sublattice $\mathfrak{L}$ of $\mathfrak{N}$ we associate the subalgebra $\mathfrak{A} = \mathfrak{L}^+$ consisting of all linear transformations α in $\mathfrak{T}$ for which each element W of $\mathfrak{L}$ is an invariant subspace. A subalgebra $\mathfrak{A}$ of $\mathfrak{T}$ is said to be *closed* if $(\mathfrak{A}^*)^+ = \mathfrak{A}$. A sublattice $\mathfrak{L}$ of $\mathfrak{N}$ is said to be closed if $(\mathfrak{L}^+)^* = \mathfrak{L}$.

The mappings "*" and "+" constitute a Galois connection [**1**, p.56] between the subalgebras of $\mathfrak{T}$ and the sublattices of $\mathfrak{N}$ (i.e., both * and + invert inclusion and for all $\mathfrak{A}$ we have $(\mathfrak{A}^*)^+ \supseteq \mathfrak{A}$ and for all $\mathfrak{L}$ we have $(\mathfrak{L}^+)^* \supseteq \mathfrak{L}$). The mappings $\mathfrak{A} \to (\mathfrak{A}^*)^+$ and $\mathfrak{L} \to (\mathfrak{L}^+)^*$ are accordingly closure operations [**1**, p. 49] in which the closed elements are just the images under * and +, i.e., $\mathfrak{A}$ is closed if and only if there exists an $\mathfrak{L}$ for which $\mathfrak{A} = \mathfrak{L}^+$ and $\mathfrak{L}$ is closed if and only if there exists an $\mathfrak{A}$ for which $\mathfrak{L} = \mathfrak{A}^*$.

The main purpose of this paper is the beginning of the study of the mappings * and +. Among the problems considered (but not completely solved) are the determination of intrinsic characterizations of closure, and the ways in which properties of the subalgebras and sublattices can be traced in their images under * and + respectively.

The main results of the paper are the two necessary conditions (Theorems 1 and 2) that a lattice be closed, given in §§3 and 4; §§5 and 6 deal with the special case of distributive lattices. Every distributive lattice is closed; the closed algebras whose lattices are distributive are characterized and some sufficient conditions are obtained that an algebra $\mathfrak{A}$ should define a distributive lattice $\mathfrak{A}^*$. Section 7 contains some examples and conjectures.

2. Some elementary properties of mappings + and *. Suppose that $\mathfrak{L} = \mathfrak{A}^*$ is complemented. In the language of representation theory this says that V is a completely reducible representation space for $\mathfrak{A}$. Since $\mathfrak{A}$ is defined as an algebra of linear transformations on V we see that V is space for a faithful representation of $\mathfrak{A}$. Now, any algebra which has a faithful completely reducible representation is semi-simple (for radical elements are mapped into zero by each irreducible representation). Conversely, if $\mathfrak{A}$ is semi-simple then V is completely reducible, that is, $\mathfrak{L} = \mathfrak{A}^*$ is complemented. Hence *$\mathfrak{L} = \mathfrak{A}^*$ is complemented if and only if $\mathfrak{A}$ is semi-simple.*

If $\mathfrak{A} = \mathfrak{L}^+$ then $\mathfrak{A}$ has a unit element. Since the dimension of a lattice is greater than or equal to the dimension of any sublattice we see that the dimension of $\mathfrak{L}$ is less than or equal to the composition length of V considered as an $\mathfrak{A}$-space. The following example shows that the inequality can occur. Let $\mathfrak{k}$ be the rationals

https://doi.org/10.4153/CJM-1952-021-1 Published online by Cambridge University Press

and let $n = 4$. Let $\mathfrak{L}$ have the elements V, S, T, U, Q, R where S is the set of all vectors with coordinates of the form $(a, b, 0, 0)$; T is the set of all vectors with coordinates of the form $(0, 0, a, b)$; U is the set of all vectors with coordinates of the form (a, b, a, b); Q is the set of all vectors with coordinates of the form $(a, b, 2a, 3b)$; and R is the zero space. Then $\mathfrak{A}$ is the set of all matrices of the form

$$\begin{Vmatrix} {}^{c}{}_{d} & 0 \\ 0 & {}^{c}{}_{d} \end{Vmatrix},$$

and $\mathfrak{A}^*$ has dimension 4.

3. Projective closure. Let A and B be subspaces of V, i.e., elements of the lattice $\mathfrak{R}$. We use the symbols $A \cap B$ and $A \cup B$ respectively to indicate the intersection of A and B and the space spanned by A and B. If B is a subspace of A we say that the pair A, B defines a *quotient*, written A/B. (This should not be confused with the residue class space which we denote by $A - B$.) We shall use quotients only in connection with the concepts of transposed quotients and projective quotients [**1**, p. 72].

If A/B and C/D are transposes with $A \cup D = C$ and $A \cap D = B$ then any vector v in A is in C and two vectors in A belong to the same coset modulo B if and only if they belong to the same coset of C modulo D. If v is any vector in C there is a vector v' in the same coset modulo D such that v' is also in A. It is easy to see that the mapping $v + B \to v + D$ defined for all vectors v in A is a non-singular linear transformation of the factor space $A - B$ onto $C - D$. The inverse of this mapping is, of course, also a linear transformation. Hence, with any sequence of transposes leading from an initial quotient A/B to a final quotient C/D we can associate a unique (non-singular) linear transformation of $A - B$ onto $C - D$. We say that such transformations are *lattice induced*. Suppose that in a sublattice $\mathfrak{L}$ of $\mathfrak{R}$ two quotients S/R and T/R are projective with

$$S/R = X_0/Y_0, \quad X_1/Y_1, \quad \ldots, \quad X_k/Y_k = T/R$$

as a sequence of transposes which demonstrate this projectivity [**2**].

Denote by σ: $s + R \to t + R = \sigma(s + R)$ the mapping thus defined from $S - R$ onto $T - R$ by the above given sequence of X_i/Y_i. Then for each pair $a \in \mathfrak{f}$ and $s \in S$ we define $Q_a(s)$ to be the coset $(s + R) + \sigma(s + R)a$ of $S \cup T$ modulo R. If, now, $q_1 \in Q_a(s_1)$ and $q_2 \in Q_a(s_2)$, then $(q_1b_1 + q_2b_2) \in Q_a(s_1b_1 + s_2b_2)$. Hence, the set Q_a consisting of all vectors lying in any one of the cosets $Q_a(s)$ for some $s \in S$ is a subspace of V, that is, an element of $\mathfrak{R}$. If $a \neq 0$, then Q_a has meet R and join $S \cup T$ with both S and T so that $[R; S, T, Q_a; S \cup T]$ is a projective root [**2**, p. 147] in $\mathfrak{R}$. Moreover, the mapping $s + R \to Q_a(s)$ is a linear transformation of $S - R$ onto Q_a. We say that Q_a is *projectively related* to $\mathfrak{L}$.

Definition. We say that $\mathfrak{L}$ is *projectively closed in* $\mathfrak{R}$ if $\mathfrak{L}$ contains every space Q_a projectively related to it.

https://doi.org/10.4153/CJM-1952-021-1 Published online by Cambridge University Press

The above process for defining new spaces Q_a can be generalized in the following manner. Let $W \supset S \supset R$ be a chain in $\mathfrak{N}$, let σ be a linear transformation of $S - R$ into $W - R$, and let $a \in \mathfrak{k}$. Then we define Q_a to be the set of all vectors lying in any one of the cosets $Q_a(s) = (s + R) + \sigma(s + R)a$ for $s \in S$. (Of course, spaces thus obtained need not be projectively related to $\mathfrak{L}$ even if R, S, and W all belong to $\mathfrak{L}$.)

LEMMA 1. *Let $\mathfrak{A}$ be any subalgebra of $\mathfrak{T}$, let $W \supset S \supset R$ be a chain in $\mathfrak{L} = \mathfrak{A}^*$, let σ be an operator homomorphism ($\mathfrak{A}$-homomorphism) of $S - R$ into $W - R$, and let $a \in \mathfrak{k}$. Then $Q_a \in \mathfrak{L}$.*

Proof. Let $\mathfrak{a} \in \mathfrak{A}$. Then we have

$$\mathfrak{a}Q_a(s) = \mathfrak{a}[(s + R) + \sigma(s + R)a] = \mathfrak{a}(s + R) + \mathfrak{a}[\sigma(s + R)a]$$
$$= \mathfrak{a}(s + R) + \sigma(\mathfrak{a}s + R)a = Q_a(\mathfrak{a}s),$$

and hence $\mathfrak{a}Q_a \subseteq Q_a$.

The following theorem which is an immediate consequence of this lemma illustrates the importance of the concept of projective closure.

THEOREM 1. *Every closed lattice $\mathfrak{L} = \mathfrak{A}^*$ is projectively closed.*

Proof. If $\mathfrak{L} = \mathfrak{A}^*$ then every lattice induced linear transformation is an operator ismorphism. Suppose that S/R is projective to T/R in $\mathfrak{L}$ and σ is any lattice induced ismorphism of $S - R$ onto $T - R$. Then apply Lemma 1 with $W = S \cup T$ and we see that $Q_a \in \mathfrak{L}$.

THEOREM 2. *Suppose that $\mathfrak{k}$ is an algebraically closed field, and that $\mathfrak{L}$ is projectively closed in $\mathfrak{N}$. Let $\mathfrak{P} = [R; S, T, U; W]$ be a prime projective root in $\mathfrak{L}$, and let σ be the linear transformation of $S - R$ onto $T - R$ induced by any projectivity (in $\mathfrak{L}$) of S/R and T/R. Then, there exists $a \in \mathfrak{k}$ such that $U = Q_a$. Moreover, if τ is any second linear transformation of $S - R$ onto $T - R$ induced by a projectivity (in L) of S/L and T/R then τ is a scalar multiple of σ.*

Proof. Let $u + R$ be any coset of U modulo R. Since $W - R$ is the direct sum of $S - R$ and $T - R$ there exist unique cosets $s + R$ of S modulo R and $t + R$ of T modulo R such that $u + R = (s + R) + (t + R)$. Since $S \cap T = U \cap S = U \cap T = R$ we see that no one of the vectors u, s, or t can belong to R unless all three do. Moreover, since $S \cup T = U \cup S = U \cup T = W$ we see that every coset of S modulo R and similarly every coset of T modulo R must appear exactly once as $u + R$ runs through all of the cosets of U modulo R. If we denote by $\rho(s + R)$ the (unique) coset $(t + R)$ which is paired with $(s + R)$ in the expression for some $(u + R)$ it is clear that ρ is a non-singular linear transformation of $S - R$ onto $T - R$.

Consider the product $\lambda = \sigma^{-1}\rho$; clearly λ is a non-singular linear transformation of $S - R$ onto itself. Since $\mathfrak{k}$ is algebraically closed, λ has at least one eigenvalue a, and since λ is non-singular $a \neq 0$. Let $(s_0 + R)$ be a non-zero

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eigenvector of λ, that is, s_0 does not belong to R and $\lambda(s_0 + R) = (s_0 a + R)$. Then

$$Q_a(s_0) = (s_0 + R) + \sigma(s_0 a + R) = (s_0 + R) + \rho\lambda^{-1}(s_0 a + R)$$
$$= (s_0 + R) + \rho(s_0 + R) \subset U.$$

The assumption of projective closure requires that $Q_a \in \mathfrak{L}$. Now, $U \supseteq U \cap Q_a \supset R$. But since Q_a and U are both prime over R this requires $U = Q_a$, which establishes the first part of the theorem.

To establish the remaining contention it is clearly sufficient to show that σ is a scalar multiple of ρ. For then the same would be true of τ. To show this we observe that for every $s \in S$ we have $Q_a(s) = (s + R) + \sigma(sa + R)$ as the coset of U modulo R in the form of a sum of a coset of S modulo R and a coset of T modulo R. As we have seen above, such an expression is unique, and hence $\sigma(sa + R) = \rho(s + R)$ for every $s \in S$. From this we conclude that $\sigma a = \rho$, as required.

4. A relative imbedding problem. Consider an l dimensional sublattice $\mathfrak{L}$ of an n-dimensional complemented modular lattice $\mathfrak{N}$. If there exists an l-dimensional complemented sublattice $\mathfrak{M}$ of $\mathfrak{N}$ which contains $\mathfrak{L}$ and which has the same projective structure constants [**2**, §2] as $\mathfrak{L}$ we say that $\mathfrak{L}$ has the *relative imbedding property*. In § 7 below we shall give an example to show that not every $\mathfrak{L}$ has this property.

An algebra $\mathfrak{A}$ is said to be cleft [**3**, p. 499] if its radical $\mathfrak{R}$ has a complement $\mathfrak{B}$ in the lattice of all subalgebras of $\mathfrak{A}$; $\mathfrak{B}$ is then necessarily semi-simple.

THEOREM 3. *Let $\mathfrak{k}$ be a field and let $\mathfrak{A}$ be a cleft subalgebra of $\mathfrak{T}$ with unity element. Then $\mathfrak{L} = \mathfrak{A}^*$ has the relative imbedding property.*

Proof. Let $\mathfrak{R}$ be the radical of $\mathfrak{A}$, let $\mathfrak{B}$ be a semi-simple subalgebra of $\mathfrak{A}$ for which $\mathfrak{A} = \mathfrak{R} + \mathfrak{B}$, and let $V = V_l \supset V_{l-1} \supset \ldots \supset V_0 = 0$, be an $\mathfrak{A}$-composition series for V. We may choose a basis for V adapted to this series for which elements of $\mathfrak{B}$ are represented by matrices with zeros in all non-diagonal blocks, such that equivalent irreducible constituents of $\mathfrak{A}$ are in identical form and such that the elements of $\mathfrak{R}$ are represented by matrices with zeros in all blocks below the main diagonal. If the number of distinct irreducible constituents of $\mathfrak{B}$ is equal to the number r of projective classes of prime quotients in $\mathfrak{L}$ then $\mathfrak{M} = \mathfrak{B}^*$ is complemented and will have the same projective structure constants as $\mathfrak{L}$. Moreover, because of the antitone properties of the mapping * we have $\mathfrak{L} \subseteq \mathfrak{M}$.

We now show that if there are less than r distinct irreducible constituents of $\mathfrak{B}$ then we can replace $\mathfrak{A}$ by a larger cleft algebra $\mathfrak{A}'$ whose semi-simple subalgebra B' has exactly r distinct irreducible constituents and such that $\mathfrak{L} = \mathfrak{A}'^*$. Then $\mathfrak{M}' = \mathfrak{B}'^*$ will serve as the imbedding lattice for $\mathfrak{L}$.

Let $\mathfrak{F}$ be one of the irreducible constituents of $\mathfrak{A}$ and let ϵ be the element of $\mathfrak{B}$ which is represented by the identity matrix in $\mathfrak{F}$ and by zero in all irreducible constituents of $\mathfrak{A}$ which are not equivalent to $\mathfrak{F}$. Suppose that $\mathfrak{J} = \{j_1, \ldots, j_s\}$

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is the set of all indices j for which the factor spaces $V_j - V_{j-1}$ have ϵ as identity operator and suppose that not all of the quotients V_j/V_{j-1} for j in $\mathfrak{J}$ are projective in $\mathfrak{L} = \mathfrak{A}^*$. Partition the set $\mathfrak{J}$ into two non-empty subsets $\mathfrak{J}_1$ and $\mathfrak{J}_2$ in such a way that indices of projective quotients lie in the same subset. Then for $i = 1, 2$ let ϵ_i be the (unique) element of $\epsilon\mathfrak{T}\epsilon$ which induces the identity mapping on the factor spaces $V_j - V_{j-1}$ for all j in $\mathfrak{J}_i$, and which induces the zero mapping on all factor spaces $V_j - V_{j-1}$ for all j not in $\mathfrak{J}_i$.

LEMMA 2. *The mappings ϵ_1 and ϵ_2 belong to $\mathfrak{A}^{*+}$.*

Proof. Clearly ϵ_1 and ϵ_2 are orthogonal indempotents whose sum is ϵ, and hence either both or neither belong to $\mathfrak{A}^{*+}$. Suppose that neither belongs to $\mathfrak{A}^{*+}$. Then there must be an $\mathfrak{A}$-space U of lowest $\mathfrak{k}$-dimension for which $\epsilon_1 U \not\subseteq U$. This space U must be join-irreducible; let U' be its unique maximal $\mathfrak{A}$-subspace.

Let j be the smallest index for which $U \subseteq V_j$. Then since V_j covers V_{j-1}, U covers $V_{j-1} \cap U$, and consequently $V_{j-1} \cap U = U'$. This shows that V_j/V_{j-1} is a transpose of U/U'.

If j does not lie in $\mathfrak{J}$ we have $\epsilon V_j = \epsilon V_{j-1}$ and hence $\epsilon U = \epsilon U' \subseteq U'$. Then since $\epsilon_1 = \epsilon_1\epsilon$ we have $\epsilon_1 U = \epsilon_1\epsilon U \subseteq \epsilon_1 U'$. Now since (dimension U') < (dimension U) we have $\epsilon_1 U' \subseteq U'$ and hence $\epsilon_1 U \subseteq U' \subset U$, contrary to our hypothesis on U. Hence j lies in $\mathfrak{J}$.

Now, since j lies in $\mathfrak{J}$, $\epsilon V_j \not\subseteq \epsilon V_{j-1}$. We may suppose the notation so chosen that $j \in \mathfrak{J}_1$. Then for $v \in V_j$, we have $\epsilon_1 v \in V_{j-1}$ if and only if $v \in V_{j-1}$, and similarly for $u \in U$ we have $\epsilon_1 u \in U'$ if and only if $u \in U'$.

Let u be a vector of U for which $\epsilon_1 u \notin U$. Since ϵ_1 and ϵ induce the identity mapping on $V_j - V_{j-1}$ there exist vectors $u_1 \in V_{j-1}$ such that $\epsilon_1(u - u_1) = u - u_1$. (For example $u_1 = u - \epsilon_1 u$ has this property.) Let k be the smallest index for which there exists a pair of vectors $u \in U$ and $u_1 \in V_k$ such that $\epsilon_1 u \notin U$ and $\epsilon_1(u - u_1) = u - u_1$. Clearly $k < j$. If u, u_1 is such a pair so is ϵu, ϵu_1, hence we may suppose that $u = \epsilon u$ and $u_1 = \epsilon u_1$. Since $u_1 (= \epsilon u_1)$ does not lie in V_{k-1} we see that $k \in \mathfrak{J}$. If $k \in \mathfrak{J}_1$ then

$$u_2 = u_1 - \epsilon_1 u_1 \in V_{k-1}.$$

Moreover,

$$\epsilon_1(u - u_2) = \epsilon_1 u - \epsilon_1 u_1 + \epsilon_1\epsilon_1 u_1 = \epsilon_1(u - u_1) + \epsilon_1 u_1$$

$$= u - u_1 + \epsilon_1 u_1 = u - u_2,$$

contrary to the hypothesis that k is minimal. Hence $k \in \mathfrak{J}_2$.

Set $R = U' \cup V_{k-1}$, $S = U \cup V_{k-1}$, $T = U' \cup V_k$, and $W = U \cup V_k$. We contend that these four $\mathfrak{A}$-spaces are distinct and that $R = S \cap T$, $W = S \cup T$. It is obvious that $S \cup T = W$. Now,

$$S \cap T = (U \cup V_{k-1}) \cap (U' \cup V_k) = [U \cap (U' \cup V_k)] \cup V_{k-1}$$

$$= [(U \cap V_k) \cup U'] \cup V_{k-1} = U' \cup V_{k-1} = R.$$

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(The simplifications used follow from two applications of the modular law and the fact that, since $k < j$, $U \cap V_k \subseteq U'$.)

We cannot have $W = T$ lest the above chosen vector u lie in $U' \cup V_k \subseteq V_{j-1}$ from which it would follow that $u \in U \cap V_{j-1} = U'$ which contradicts the condition $\epsilon_1 u \notin U$. In order to show that all four spaces are distinct it is now sufficient to show that $W \neq S$. Suppose that $W = S$. Then $u_1 \in S$ and so can be written in the form $u_1 = u' + u_2$ where $u' \in U'$, $u_2 \in V_{k-1}$. Now $u' \in U'$, hence $\epsilon_1(u - u') \notin U$. Moreover,

$$\epsilon_1[(u - u') - u_2] = \epsilon_1(u - u_1) = u - u_1 = (u - u') - u_2.$$

Thus the pair of vectors $(u - u')$, u_2 contradict the minimality of k. This contradiction arises from the assumption $W = S$; hence we conclude that $W \neq S$.

Since j and k both lie in $\mathfrak{J}$ we see that $S - R$ is $\mathfrak{A}$-isomorphic to $T - R$ under some mapping σ. Now, by Lemma 1, $\mathfrak{L}$ contains the space $U = Q_1$, and the projective root $[R; S, T, U; W]$ in $\mathfrak{L}$ can be used to show the projectivity of the quotients S/R and T/R and thus of V_j/V_{j-1} and V_k/V_{k-1}. But this contradicts the construction of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. This contradiction arises from the assumption that the $\mathfrak{A}$-space U is not invariant under ϵ_1 and ϵ_2; and thus completes the proof of the lemma.

LEMMA 3. *The algebra $\mathfrak{A}_1$ generated by $\mathfrak{A}$ and ϵ_1, ϵ_2 is cleft with semi-simple subalgebra $\mathfrak{B}_1$ generated by $\mathfrak{B}$ and ϵ_1, ϵ_2.*

Proof. It is clear from the matrix form of $\mathfrak{B}$ that ϵ_1 and ϵ_2 commute with all elements of $\mathfrak{B}$ as well as with each other, and hence that ϵ_1 and ϵ_2 belong to the centre of $\mathfrak{B}_1$. One consequence of this is that $\epsilon_1\mathfrak{B}$ is a two-sided ideal of $\mathfrak{B}_1$. But from the matrix form of $\mathfrak{B}$ it is clear that $\epsilon_1\mathfrak{B}$ is a simple algebra isomorphic to $\epsilon\mathfrak{B}$. Similarly $(1 - \epsilon_1)\mathfrak{B}$ is also a two-sided ideal of $\mathfrak{B}_1$, and reference to the matrix form of $\mathfrak{B}$ shows that $(1 - \epsilon_1)\mathfrak{B}$ is isomorphic to $\mathfrak{B}$ under the mapping $\beta \rightarrow (1 - \epsilon_1)\beta$. Since ϵ_1 is idempotent and lies in the centre of $\mathfrak{B}_1$ the sum $\mathfrak{B}' = (1 - \epsilon_1)\mathfrak{B} + \epsilon_1\mathfrak{B}$ is direct. Clearly $\mathfrak{B}' \supseteq \mathfrak{B}$, and the equalities $(1 - \epsilon_1)\epsilon = \epsilon_2$ and $\epsilon\epsilon_1 = \epsilon_1$ show that $\mathfrak{B}'$ also contains ϵ_1 and ϵ_2. Hence, $\mathfrak{B}_1 = \mathfrak{B}'$ is semi-simple. We have proved incidentally that $\mathfrak{B}_1$ *has exactly one more simple two-sided ideal than* $\mathfrak{B}$.

If we can now find a nilpotent ideal $\mathfrak{R}_1$ in $\mathfrak{A}_1$ for which $\mathfrak{A}_1$ is the direct sum of $\mathfrak{B}_1$ and $\mathfrak{R}_1$ we will have completed the proof of the lemma. The radical $\mathfrak{R}$ of $\mathfrak{A}$ consists of those elements represented by zeros in all of the irreducible constituents, i.e., of those elements whose matrices are zero in all blocks on or below the main diagonal. Since V has composition length l it follows that $\mathfrak{R}^l = 0$. Moreover, the matrix for each element of $\mathfrak{R}\mathfrak{B}_1$ has zeros in all blocks on or below the main diagonal so that $(\mathfrak{R}\mathfrak{B}_1)^l = 0$. Now, consider the subset

$$\mathfrak{R}_1 = \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1) + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^2 + \ldots + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^l.$$

of $\mathfrak{A}_1$. We remark first that since $1 \in \mathfrak{B}_1$, $\mathfrak{R} \subseteq \mathfrak{R}_1$ and so $\mathfrak{B}_1$ and $\mathfrak{R}_1$ generate $\mathfrak{A}_1$.

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Moreover, since $\mathfrak{B}_1^2 = \mathfrak{B}_1$ we have $\mathfrak{B}_1\mathfrak{R}_1 = \mathfrak{R}_1\mathfrak{B}_1 = \mathfrak{R}_1$. An easy induction shows that

$$\mathfrak{R}_1{}^t = \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^t + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^{t+1} + \ldots + \mathfrak{B}_1(\mathfrak{R}\mathfrak{B}_1)^l.$$

This shows both that $\mathfrak{R}_1$ is nilpotent and that it is closed under multiplication. To check closure under addition it suffices to recall that the definition of the product $\mathfrak{C}\mathfrak{D}$ of two subsets $\mathfrak{C}$ and $\mathfrak{D}$ is the set of all sums $c_1d_1 + \ldots + c_sd_s$ for the c_i in $\mathfrak{C}$ and the d_i in $\mathfrak{D}$. Putting together all of the above facts we see that $\mathfrak{R}_1$ is a nilpotent two-sided ideal of $\mathfrak{A}_1$. It remains only to show that the sum $\mathfrak{B}_1 + \mathfrak{R}_1$ is direct and equal to $\mathfrak{A}_1$. The intersection of $\mathfrak{B}_1$ and $\mathfrak{R}_1$ is clearly an ideal in $\mathfrak{B}_1$ and is as part of $\mathfrak{R}_1$ either nilpotent or zero. Since $\mathfrak{B}_1$ is semi-simple this intersection must be zero, so the sum is direct. Now $\mathfrak{A}' = \mathfrak{B}_1 + \mathfrak{R}_1$ is clearly a subalgebra of $\mathfrak{A}_1$. Since $\mathfrak{A}_1$ is generated by $\mathfrak{B}_1$ and $\mathfrak{R}_1$ it follows that $\mathfrak{A}_1 = \mathfrak{A}'$ is cleft with semi-simple component $\mathfrak{B}_1$.

Returning now to the proof of Theorem 3, we observe it is a consequence of Lemma 2 that $(\mathfrak{A}_1)^* = \mathfrak{A}^*$. Moreover, we saw in the proof of Lemma 3 that $\mathfrak{A}_1$ has one more class of irreducible representations than $\mathfrak{A}$. Hence, by repeated applications of our construction we will arrive at a cleft algebra $\mathfrak{A}_s$ having exactly as many classes of irreducible representations as $\mathfrak{L}$ has classes of projective prime quotients and for which $(\mathfrak{A}_s)^* = \mathfrak{A}^*$. This completes the proof.

5. The distributive case. We shall show that every distributive lattice is closed and that a closed algebra corresponds to a distributive lattice if and only if its irreducible constituents are total matrix algebras over $\mathfrak{k}$, no two of which are equivalent. The question as to which non-closed algebras correspond to distributive lattices is not settled although some results are given for the case in which $\mathfrak{k}$ is an algebraically closed field.

We review some of the important properties of a distributive lattice [**1**, Chap. IX]. Let $\mathfrak{L}$ be a distributive sublattice of $\mathfrak{N}$ and let $U_1, U_2, \ldots, U_l$ be the join-irreducible elements of $\mathfrak{L}$ (here $l = \dim \mathfrak{L}$). Let U_j' be the unique element covered by U_j and let $n_j = \dim\ (U_j - U_j')$ $(j = 1, 2, \ldots, l)$. Suppose that the U's have been ordered so that $U_i \subset U_j$ can hold only if $i < j$, and set

$$V_j = U_1 \cup U_2 \cup \ldots \cup U_j \qquad (j = 1, \ldots, l).$$

Then $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_l = V$ is a maximal chain in $\mathfrak{L}$. We choose a basis for V adapted to this chain, and, moreover, such that the jth set of basis elements (i.e., the n_j new ones chosen for V_j in addition to the ones already chosen for V_{j-1}) all lie in U_j. We regard $\mathfrak{A} = \mathfrak{L}^+$ as a matrix algebra in terms of the given basis. Let $A = ||A_{ij}||$ be an n by n $\mathfrak{k}$-matrix partitioned so that A_{ij} is an n_i by n_j $\mathfrak{k}$-matrix.

LEMMA 4. (i) *If A belongs to $\mathfrak{A} = \mathfrak{L}^+$ then so does each matrix B which can be obtained from A by replacing any set of its submatrices A_{ij} by the zero matrix.* (ii) *If $U_i \subseteq U_j$, and A' is any n_i by n_j $\mathfrak{k}$-matrix then the matrix A having $A_{ij} = A'$ and $A_{hk} = 0$ for $h, k \neq i, j$ belongs to $\mathfrak{A}$.* (iii) *If $U_i \not\subseteq U_j$ then $A_{ij} = 0$ for all A in $\mathfrak{A}$.*

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Proof. The lattice dimension of any element W of $\mathfrak{L}$ is the number of join-irreducible elements of $\mathfrak{L}$ which it contains [1, p. 139]. Suppose that

$$U_{h_1}, \ldots, U_{h_t}$$

are the join-irreducible elements contained in W. Then the h_1th, ..., h_tth sets of basis vectors for V taken altogether will be a basis for W. Now let A be as in (ii) and we see that if $U_j \not\subseteq W$ then $AW = 0$ and that if $U_j \subseteq W$ then $AW \subseteq U_i \subseteq U_j \subseteq W$; hence in all cases W is invariant under A, which establishes part (ii) of the lemma. As a consequence of (ii) we see that for each j the idempotent matrix E_j which has i, j component identity and all other components zero is an element of $\mathfrak{A}$. Now (i) follows by consideration of sums of elements E_iAE_j. Finally, to establish (iii) we suppose that A belongs to $\mathfrak{A}$ and observe that for each vector u in U_j the vector $v' = E_iAE_jv$ must again lie in U_j. As an element of the image space of E_i the vector v' must be a linear combination of basis vectors belonging to the ith set. Now, if $A_{ij} \neq 0$ there exist vectors v in U_j for which $v' \neq 0$. Hence, $0 \subset U_i \cap U_j \not\subseteq U_i'$. Since U_i is join-irreducible this requires $U_i \subseteq U_j$. Therefore, we conclude that if $U_i \not\subseteq U_j$ then $A_{ij} = 0$ for all A in $\mathfrak{A}$.

THEOREM 4. *Every distributive lattice is closed. An algebra $\mathfrak{A}$ of linear transformations is closed with $\mathfrak{L} = \mathfrak{A}^*$ distributive if and only if the irreducible constituents of $\mathfrak{A}$ are inequivalent total matrix algebras.*

Proof. Let $\mathfrak{A}$ be an algebra of linear transformations whose irreducible constituents are inequivalent total matrix algebras over a sfield $\mathfrak{k}$. Let

$$0 = V_0 \subset V_1 \subset \ldots \subset V_l = V$$

be a composition series for the space V of $\mathfrak{A}$. Then we can select a basis for V adapted to this composition series so that $\mathfrak{A}$ takes the form $\mathfrak{A} = \|\mathfrak{A}_{ij}\|$ where (i) each $\mathfrak{A}_{ij}$ for $i > j$ is zero; (ii) each $\mathfrak{A}_{ij}$ is a total matrix algebra and no two of the $\mathfrak{A}_{jj}$ are equivalent; (iii) each $\mathfrak{A}_{ij}$ for $i < j$ is either zero or the set of all n_i by n_j $\mathfrak{k}$-matrices where n_j = dimension $V_j - V_{j-1}$ = degree $\mathfrak{A}_{jj}$; and (iv) the non-zero $\mathfrak{A}_{ij}$ are completely independent (i.e., they satisfy condition (i) of Lemma 4).

That $\mathfrak{L} = \mathfrak{A}^*$ is distributive of dimension l follows from the fact that the l irreducible constituents of $\mathfrak{A}$ are inequivalent and hence that $\mathfrak{L}$ can contain no projective root. We now search for the join-irreducible elements of $\mathfrak{L}$.

We subdivide the basis vectors for V in terms of which $\mathfrak{A}$ is written into l sets with n_j in the jth set and in such a manner that the elements of the first j sets form a basis for V_j $(j = 1, \ldots, l)$. Let $\mathfrak{J}_j = \{i_1, \ldots, i_t(= j)\}$ be the set of all indices i for which $A_{ij} \neq 0$. Then for each j let U_j be the subspace of V generated by all of the elements in the i_1th, ..., i_tth sets of basis vectors. We now show that $U_1, \ldots, U_l$ are the join-irreducible elements of L, and moreover, that $\mathfrak{L}$ is isomorphic to the lattice $\mathfrak{P}$ of subsets of $\mathfrak{J} = \{1, \ldots, l\}$ generated by the $\mathfrak{J}_j$.

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Suppose that $A_{hk} \neq 0$, let B' be any n_h by n_k $\mathfrak{k}$-matrix, and let B be the element of $\mathfrak{A}$ with $B_{hk} = B'$ and all other components zero. Since by (iv) any A in $\mathfrak{A}$ is a sum of such B, to establish invariance of any space under $\mathfrak{A}$ it is sufficient to test invariance only under all matrices of type B in $\mathfrak{A}$. Now, unless $k \in \mathfrak{J}_j$ we have $BU_j = 0$, and if $k \in \mathfrak{J}_j$ then $BU_j \subseteq U_h$, where of course $h \in \mathfrak{J}_k$. If $\mathfrak{J}_h \subseteq \mathfrak{J}_j$ we would have $U_h \subseteq U_j$ and hence U_j invariant under B. Hence, we conclude that a sufficient condition for invariance of all the U_j is that for each j, $k \in \mathfrak{J}_j$ only if $\mathfrak{J}_k \subseteq \mathfrak{J}_j$. To see that this is true we suppose $h \in \mathfrak{J}_k$, $k \in \mathfrak{J}_j$ and select B as above. Then select an n_k by n_j $\mathfrak{k}$-matrix C' such that $D' = B'C' \neq 0$ and let C be the element of $\mathfrak{A}$ having $C_{kj} = C'$ and all other components zero. Now $D = BC$ has $D_{hj} = D'$ and so $A_{hj} \neq 0$, that is, $h \in \mathfrak{J}_j$. Thus the U_j all belong to $\mathfrak{L}$.

The proof of the join-irreducibility of the U_j rests again on (iv). Let W be any $\mathfrak{A}$-subspace of U_j which is not contained in V_{j-1}. Let w be a vector of W which does not lie in V_{j-1}, let $i \in \mathfrak{J}_j$, and let z be any element of the ith set of basis vectors. Then we can find an n_i by n_j matrix B' such that $Bw = z$ where B is the matrix of $\mathfrak{A}$ with $B_{ij} = B'$ and all other components zero. Hence $W = U_j$. Now in any expression of U_j as a sum of $\mathfrak{A}$-spaces at least one of the summands must contain vectors of U_j which do not lie in V_{j-1} and hence one of the summands is U_j itself. This shows that U_j is join-irreducible.

Let W be any element of $\mathfrak{L}$, let

$$U_{j_1}, \ldots, U_{j_s}$$

be the join-irreducible elements of $\mathfrak{L}$ contained in W, and let $\mathfrak{J}(W) = \{j_1, \ldots, j_s\}$ Then

$$W = U_1 \cup \ldots \cup U_s, \; \mathfrak{J}(W) = \mathfrak{J}_{j_1} \cup \ldots \cup \mathfrak{J}_{j_s}$$

and W has a basis consisting of the elements of the j_1th, $\ldots$, $j_s th$ sets of basis vectors. Moreover, W is uniquely determined by the element $\mathfrak{J}(W)$ of $\mathfrak{P}$. Furthermore, $W_1 \subseteq W_2$ if and only if $\mathfrak{J}(W_1) \subseteq \mathfrak{J}(W_2)$. In other words, the mapping $\sum: W \rightarrow \mathfrak{J}(W)$ of $\mathfrak{L}$ into $\mathfrak{P}$ is 1-1 and isotone. Now, since $\mathfrak{L}$ is a lattice, $\mathfrak{P}' = \sum\mathfrak{L}$ is also a lattice and is isomorphic [**1**, p. 21] to $\mathfrak{L}$. But, since $\mathfrak{L}'$ contains each $\mathfrak{J}_j = \mathfrak{J}(U_j)$, and since $\mathfrak{L}$ is generated by the $\mathfrak{J}_j$, we conclude that $\sum$ is an isomorphism of $\mathfrak{L}$ onto $\mathfrak{P}$.

Now consider $\mathfrak{B} = \mathfrak{A}^{*+} = \mathfrak{L}^{+}$. According to Lemma 3 we see that $\mathfrak{B}_{ij} = 0$ if and only if $U_i \not\subseteq U_j$, and hence if and only if $\mathfrak{J}_i \not\subseteq \mathfrak{J}_j$ and hence if and only if $\mathfrak{A}_{ij} = 0$. Now, since the non-zero components $\mathfrak{A}_{ij}$ of $\mathfrak{A}$ are completely independent, and since $\mathfrak{A} \subseteq \mathfrak{B}$, we conclude that $\mathfrak{A} = \mathfrak{B}$, that is, $\mathfrak{A}$ is closed.

Finally, to see that every distributive lattice $\mathfrak{L}$ is closed we apply Lemma 4 and note that the lattice $\mathfrak{P}$ defined by $\mathfrak{A} = \mathfrak{L}^{+}$ is isomorphic to $\mathfrak{L}$ as well as to $\mathfrak{L}^{+*}$; or, even more simply, observe that the join-irreducible subspaces of $\mathfrak{L}$ are again join-irreducible in $\mathfrak{L}^{+*}$.

6. The distributive case for non-closed algebras. We next consider the question "For what algebras $\mathfrak{A}$ is $\mathfrak{L} = \mathfrak{A}^*$ distributive?" approached from the

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following point of view. Suppose a distributive sublattice $\mathfrak{L}$ of $\mathfrak{N}$ is given. Then we ask "For which subalgebras $\mathfrak{A}$ of $\mathfrak{B} = \mathfrak{L}^+$ is $\mathfrak{A}^* = \mathfrak{L}$?"

If the irreducible constituents $\mathfrak{A}_{jj}$ of $\mathfrak{A}$ are the same as those of $\mathfrak{B}$ then either $\mathfrak{A} = \mathfrak{B}$ or some component $\mathfrak{A}_{ij}$ of $\mathfrak{A}$ is zero whereas $\mathfrak{B}_{ij} \neq 0$. But then, by Theorem 4, $\mathfrak{A}^* \supset \mathfrak{L}$. Hence, the zero components of $\mathfrak{A}$ must be the same as those of $\mathfrak{B}$. Thus in order to have $\mathfrak{A}^* = \mathfrak{L}$ and $\mathfrak{A} \subset \mathfrak{B}$ we must have equivalence between some of the irreducible constituents of $\mathfrak{A}$. We assume (with no loss of generality) that two irreducible constituents of $\mathfrak{A}$ are equal if they are equivalent.

THEOREM 5. *Let $\mathfrak{L}$ be a distributive sublattice of $\mathfrak{N}$. Let the set $\mathfrak{J} = \{1, \ldots, l\}$ be partitioned into subsets $\mathfrak{J}_1, \ldots, \mathfrak{J}_r$ in such a way that*

(1) *h and k belong to the same set $\mathfrak{J}_j$ only if $n_h = n_k$ and*

(2) *the set of all U_h for $h \in \mathfrak{J}_j$ form a chain in $\mathfrak{L}$.*

Then there exists a subalgebra $\mathfrak{A}$ of $\mathfrak{B} = \mathfrak{L}^+$ for which $\mathfrak{A}^* = \mathfrak{L}$ and having $\mathfrak{A}_{hh} = \mathfrak{A}_{kk}$ whenever both h and k belong to one of the sets $\mathfrak{J}_j$. Conversely, suppose $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}^* = \mathfrak{L}$. Let $\mathfrak{J}_1, \ldots, \mathfrak{J}_r$ be the equivalence classes of $\mathfrak{J}$ defined by the equivalence relation $h \sim k$ if and only if $\mathfrak{A}_{hh} = \mathfrak{A}_{kk}$. Then the sets $\mathfrak{J}_j$ satisfy (1) *and* (2).

Proof. For the first part of the theorem we take for $\mathfrak{A}$ the algebra whose components are the same as those of $\mathfrak{B}$ except for the stipulated equalities $\mathfrak{A}_{hh} = \mathfrak{A}_{kk}$. Now either $\mathfrak{A}^*$ is not distributive or by repeated applications of Lemma 2 we get $\mathfrak{A}^{*+} = \mathfrak{B}$ and hence $\mathfrak{L} = \mathfrak{A}^*$.

Suppose that $\mathfrak{A}^*$ is not distributive, then it contains a prime projective root $\mathfrak{P} = [R; S, T, U; W]$ normal [**2**, §4] with respect to the chain $0 = V_0 \subset V_1 \subset \ldots \subset V_l = V$, say V_i/V_{i-1}, W/T, U/R, V_j/V_{j-1} is a sequence of transposes and $V_i \subseteq S \subseteq V_{j-1}$. (We are here using the fact that dim $\mathfrak{A}^* =$ dim $\mathfrak{L}$ and so a maximal chain in $\mathfrak{L}$ is also a maximal chain in the larger lattice $\mathfrak{A}^*$.) Clearly, i and j must lie together in one of the sets $\mathfrak{J}_h$.

We refer once again to the basis chosen for V in § 5 above. For any vector v in V we speak of the first set of n_1 coefficients, . . . , lth set of n_l coefficients. By our choice of basis every vector in S (or in R) has all coefficients arbitrary (but, of course, with all coefficients zero in sets $h > j$). Moreover, any vector v of W which has zero coefficients in the jth set must lie in $W \cap V_{j-1} = S$, and, similarly, if $v \in T$ or U) and has zero coefficients in the jth place then $v \in R = S \cap T$. Also, since $V_i \cap R = V_{i-1}$ any vector u of R which has zero coefficients in all sets except possibly the ith set must be zero.

Now let v be a vector in T with not all zero coefficients in the jth set. Since $U_i \subset U_j$, $A_{ij} = B_{ij} \neq 0$ and there is an n_i by n_j matrix A' such that the matrix A in $\mathfrak{A}$ having $A_{ij} = A'$ and other components zero sends v into a vector $u = Av$ having zero coefficients in all sets except the ith set and having non-zero coefficients in the ith set. This vector u cannot belong to R, but on the other hand if T is invariant under A we have $u = Av \in T \cap V_{j-1} = R$. Thus we see that T cannot be an $\mathfrak{A}$-space. This contradicition arose from our assumption that $\mathfrak{A}^*$ was not distributive. We therefore conclude that $\mathfrak{A}^*$ is distributive, and the first part of the theorem is established.

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Conversely, let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$ having $\mathfrak{A}^* = \mathfrak{L}$. Then either the theorem is true or there is a pair of indices $i < j$ for which $\mathfrak{A}_{ii} = \mathfrak{A}_{jj}$ and $U_i \not\subseteq U_j$. In this case we have from Lemma 3 that $\mathfrak{B}_{ij} = 0$ and hence that $\mathfrak{A}_{ij} = 0$.

Suppose that

$$U_{k_1}, \ldots, U_{k_s}$$

are the join-irreducible elements of $\mathfrak{L}$ properly contained in U_j. Then

$$U_j' = U_{k_1} \cup \ldots \cup U_{k_s}$$

is the unique maximal subspace of U_j. Let $R = V_{i-1} \cup U_j'$; $S = R \cup U_i$; and $T = R \cup U_j$. Since U_j does not contain U_i no U_{k_h} can contain U_i; hence

$$U_{k_h} \cap U_i \subseteq U_i' \subseteq V_{i-1}$$

(here U_i' is the unique maximal subspace of U_i). Since $\mathfrak{L}$ is distributive $R \cap U_i = U_i'$ and so S/R is a transpose of U_i/U_i'. On the other hand since $V_{i-1} \cap U_j \subseteq U_j'$ we see that $R \cap U_j = U_j'$; hence T/R is a transpose of U_j/U_j'.

By our hypothesis that $\mathfrak{A}_{ii} = \mathfrak{A}_{jj}$ we have $T - R$ operator isomorphic to $S - R$. Hence, by Lemma 1 with $W = S \cup T$ and for $a \neq 0$ in $\mathfrak{k}$, we see that the projective root $[R; S, T, Q_a; W]$ is contained in $\mathfrak{A}^*$, contrary to our hypothesis that $\mathfrak{L} = \mathfrak{A}^*$ is distributive. This contradiction arises from the assumption $U_i \not\subseteq U_j$, and so the theorem follows.

Results similar to Theorem 5 can be obtained for "super diagonal" components of $\mathfrak{A}_{ij}$ of $\mathfrak{A}$, but the theory here is not yet complete.

7. Some unsettled problems. We have seen that projective closure and the relative imbedding property are necessary conditions for closure of a lattice. It is not yet known whether these two conditions are also sufficient. The answer to this question may depend on the nature of $\mathfrak{k}$, in particular whether or not it is algebraically closed.

All of the examples known to the author of sublattices which fail to possess the relative imbedding property also fail to be projectively closed. This suggests that projective closure may imply the relative imbedding property. Again the answer may depend on the nature of $\mathfrak{k}$.

We close the present paper with an example of a lattice which does not have the relative imbedding property. Let $\mathfrak{k}$ be the rational field, let Z be a $\mathfrak{k}$-space of dimension 2, and let V be the fourfold Cartesian direct sum of Z with itself, i.e., the vectors v of V are the form (z_1, z_2, z_3, z_4) with z_i in Z. Let β be a linear transformation on Z with eigenvalues 2 and 3 and corresponding eigenvectors z' and z'' (i.e., $\beta z' = z'2$ and $\beta z'' = z''3$).

Let $\mathfrak{N}$ be the lattice of all $\mathfrak{k}$-subspaces of V and let $\mathfrak{L}$ be the finite sublattice of dimension $l = 4$ whose join-irreducible subspaces are

$$\{(z_1,0,0,0)\};\ \{(z_1,z_1,0,0)\};\ \{(z_1,\beta z_1,0,0)\};\ \{(z_1,z_2,0,0)\};$$

$$\{(0,z_1,z_2,0)\};\ \{(z_1,z_2,z_1,0)\};\ \{(0,z_1,z_2,z_3)\}.$$

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(Here, for instance, $\{(z_1, z_2, z_1, 0)\}$ denotes the set of all vectors of the form $(z_1, z_2, z_1, 0)$ obtained as z_1 and z_2 range independently over Z.)

Suppose that there exists a 4-dimensional complemented modular sublattice $\mathfrak{M}$ of $\mathfrak{N}$ which contains $\mathfrak{L}$. Clearly $\mathfrak{L}$ is simple; therefore $\mathfrak{M}$ is simple. Then [**1**, Chap. VIII, Theorem 6] $\mathfrak{M}$ is a projective geometry of dimension 3 over a sfield $\mathfrak{K}$; since $\mathfrak{M} \subset \mathfrak{N}$ and $\mathfrak{k}$ is the rational field, $\mathfrak{K} \supseteq \mathfrak{k}$. This implies, in particular, that $\mathfrak{M}$ contains all elements of $\mathfrak{N}$ projectively related to $\mathfrak{L}$ with respect to $\mathfrak{k}$ (i.e., all Q_a with b in $\mathfrak{k}$). Then the space $\{(z, z2, 0, 0)\}$ belongs to $\mathfrak{M}$. But now $\{(z_1, z_12, 0, 0)\} \cap \{(z_1, \beta z_1, 0, 0)\}$ contains the non-zero vector $(z', z'2, 0, 0)$ and hence has $\mathfrak{k}$-dimension 1. Hence, dim $\mathfrak{M} > 4$.

This example, although closely related to it, does not apply to the Dilworth-Hall problem [**1**, p. 121, Problem 55] since as an abstract lattice $\mathfrak{L}$ can be imbedded in a complemented modular lattice of dimension 4.

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