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ASYMPTOTIC BEHAVIOR OF SKEW CONDITIONAL HEAT KERNELS ON GRAPH NETWORKS

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ABSTRACT In this note, we will consider the heat propagation on locally finite graph networks which satisfy a skew condition on vertices (See Definition of Section 2) For several periodic models, we will construct the heat kernels P_t with the skew condition explicitly, and derive the decay order of P_t as time goes to infinity

Introduction. In previous paper [2], we calculated the heat kernels explicitly on several graph networks. Especially, we have been interested in the case that the coefficients of the heat equation are irregular. In this note we adopt the skew condition, which is a nodal condition concerned with the conservation of heat flow, instead of the irregularity of the coefficients. The precise definition of this condition will be given in Section 2.

One of the motivation of this note is in skew Brownian motion. In one-dimension, Ito and Mckean [5] constructed skew Brownian motion by the following procedure. Let R_t be a reflecting Brownian motion on $[0,\infty)$. Let $0 \leq \alpha \leq 1$, change the sign of each excursion of R_t from 0 independently with probability $1-\alpha$ so that a given excursion is positive with probability α, negative with probability $1-\alpha$. Ito and Mckean derived its scale function and speed measure. J. B. Walsh [9] gave interesting characterizations of skew Brownian motion and derived the transition density. Portenko [7], Harrison and Shepp [3] and Rosenkrantz [8] observed this process in different contexts.

We have an interest in skew Brownian motion on graph networks. Let us recall that the heat kernel is nothing but the transition probability density of the diffusion corresponding to the generator of the heat equation. For an intuitive description of the model Γ_0 in Section 2, it is known that the diffusion corresponding (in above sense) to the skew conditional heat kernel is a natural analogue of one-dimensional skew Brownian motion.

Another motivation is the following, which is a natural continuation of our previous paper [2]. If the graph network Γ is composed of several different materials (*i.e.* the specific heats and the thermal conductivities of edges are different each other), then the heat propagation on Γ is described as the heat equation with a skew condition. Namely, we can replace the irregularity of coefficients with the skew condition in such case.

In this note, we will compute the skew conditional heat kernel and investigate the asymptotic behavior of it as time goes to infinity for several periodic models.

We have a great interest that the decay order of the heat kernel varies with the change of the rates of the skew condition.

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1. **Preliminaries.**

1.1 *Heat kernels (fundamental solutions).* Let $\Gamma = (V, E)$ be a graph network with the set V of vertices and the set E of edges. We suppose Γ is isolated from its surrounding medium and the temperature of each point of the graph Γ is 0° at initial time (*i.e.* $t = 0$). We give a unit quantity of heat at a point $y \in \Gamma$ at $t = 0$, then the temperature $U_t(x) = P_t(x, y)$ at a point $x \in \Gamma$ at a time t, as is well known, satisfies the heat equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{2c(x)^2}\frac{\partial}{\partial x}a(x)^2\frac{\partial}{\partial x}\right)U_t(x) = 0, \tag{1}$$

on each edge, where $2c^2$ is the specific heat and a^2 is the thermal conductivity. Then $U_t = P_t$ is called the *heat kernel* or the *fundamental solution* of the heat equation (1). In probabilistic interpretation, $P_t(x, y)$ is the probability density of finding out the Brownian particle at $y \in \Gamma$ at a time t which started from $x \in \Gamma$ at $t = 0$.

In this note, we assume that $c(x)$ and $a(x)$ are constants in each edge $I \in E$ and assume that there are neither injection nor absorption of heat at each vertex $P \in V$.

Let $P \in V$ and $I_1, I_2, \ldots, I_r$ be edges joined at P, then, from the above assumption, we deduce at each $P \in V$,

$$\sum_{j=1}^{r} a_j^2 \lim_{\substack{x_j \to x_0 \\ x_j \in I_j}} \frac{dU}{dx_j}(x_j) = 0, \tag{2}$$

$$\lim_{\substack{x_j \to x_0 \\ x_j \in I_j}} U(x_j) \quad j = 1, 2, \ldots, r \text{ are mutually equal,} \tag{3}$$

where a_j^2 is the thermal conductivity of I_j and x_0 is the coordinate of P. The condition (2) means the conservation of heat flow, and (3) means the continuity of the temperature.

1.2 *Green function.* We set $L = (1/2c^2)(d/dx)a^2(d/dx)$ and define a natural domain of L as $\mathcal{D}(L) = \{f \,; f \in C(\Gamma) \cap L^2(\Gamma, 2c^2 dx), Lf \in L^2(\Gamma, 2c^2 dx)$, and (2) is valid at each $P \in V\}$. Then L is a closed and self-adjoint operator on $\mathcal{D}(L)$, and it has non-positive spectrum (see [2]). For $\lambda \in \mathfrak{C}\text{-}\mathfrak{R}$, we consider the Dirichlet problem

$$\begin{cases} (\lambda - L)G(x, y, \lambda) = \delta_y(x) \\ G(x, y, \lambda) \longrightarrow 0 \text{ as } |x - y| \longrightarrow \infty. \end{cases} \tag{4}$$

We know the existence of the solution of (4) since $(\lambda - L)^{-1}$ is bounded on $L^2(\Gamma)$. We call the solution $G(x, y, \lambda)$ the *Green function* of the Dirichlet problem (4) as usual. Then it is well known that the heat kernel $P_t(x, y)$ is given by

$$P_t(x, y) = \frac{1}{2\pi i}\int_\Omega e^{\lambda t} G(x, y, \lambda)\, d\lambda, \tag{5}$$

where Ω is a contour around $\mathfrak{R}_-$ in the complex λ-plane as in the Figure 1.1.

1.3 *Skew Brownian motion.* Let Γ_0 be the radial graph composed of r half lines $I_i = [0_i, +\infty)$, $i = 1, 2, \ldots, r$, which are connected at a point 0 (every 0_i is identified with 0). We choose coordinates $x_i \in I_i = [0, +\infty)$ as in the Figure 1.2.

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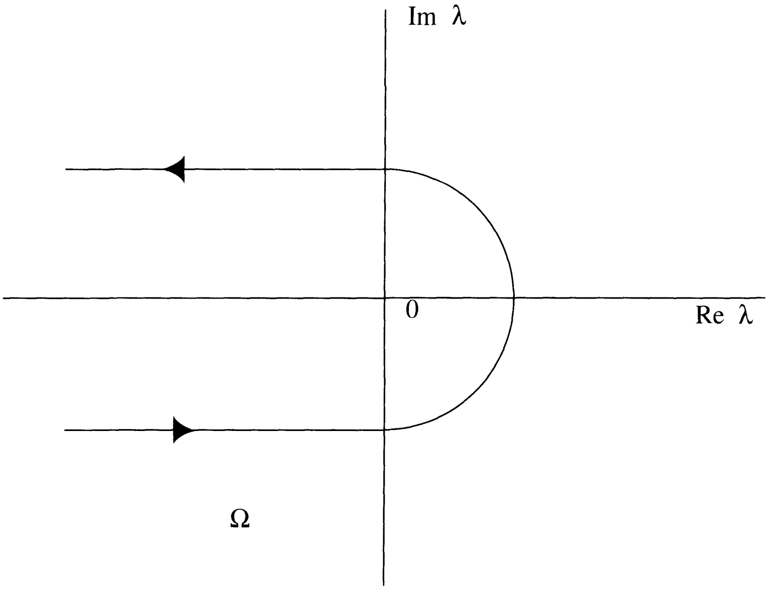

FIGURE 1 1

Let us construct a process X_t on Γ_0 which is an analogy of one-dimensional skew Brownian motion.

Let R_t be a reflecting Brownian motion on $I_1 = [0_1, +\infty)$ and turn away each excursion of R_t from the origin to I_j independently with probability α_j ($j = 2, 3, \ldots, r$) where each α_j is non-negative and $0 \le \sum_{i=2}^{r} \alpha_i \le 1$. We put $\alpha_1 = 1 - \sum_{i=2}^{r} \alpha_i$. Then we call the resulting process X_t *skew Brownian motion* with rates $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ to the directions $(I_1, I_2, \ldots, I_r)$ at the origin.

2. **Basic skew model.** In Γ_0, let the specific heat and the thermal conductivity of each I_j be $2c_j^2$ and a_j^2 respectively, $j = 1, 2, \ldots, r$. Then the heat equation (1) is as follows

$$\left(\frac{\partial}{\partial t} - \frac{1}{2c_j^2}\frac{\partial}{\partial x_j} a_j^2 \frac{\partial}{\partial x_j}\right) U_t(x_j) = 0, \tag{6}$$

for $x_j \in I_j, x_j \neq 0, j = 1, 2, \ldots, r$.

Then, the fundamental solution $P_t(x, y)$ of the heat equation (6) with nodal conditions (2) and (3) is given as follows

LEMMA 1.

$$P_t(x,y) = \begin{cases} (2c_j^2 / \sum_k a_k c_k) g(t, \frac{c_i}{a_i}x + \frac{c_j}{a_j}y) & \textit{for } x \in I_i,\ y \in I_j\ (i \neq j) \\ \frac{c_i}{a_i} g\big(t, \frac{c_i}{a_i}(x-y)\big) + \\ \frac{c_i}{a_i}\{(a_i c_i - \sum_{k \neq j} a_k c_k) / \sum_k a_k c_k\} g\big(t, \frac{c_i}{a_i}(x+y)\big) & \textit{for } x, y \in I_i, \end{cases} \tag{7}$$

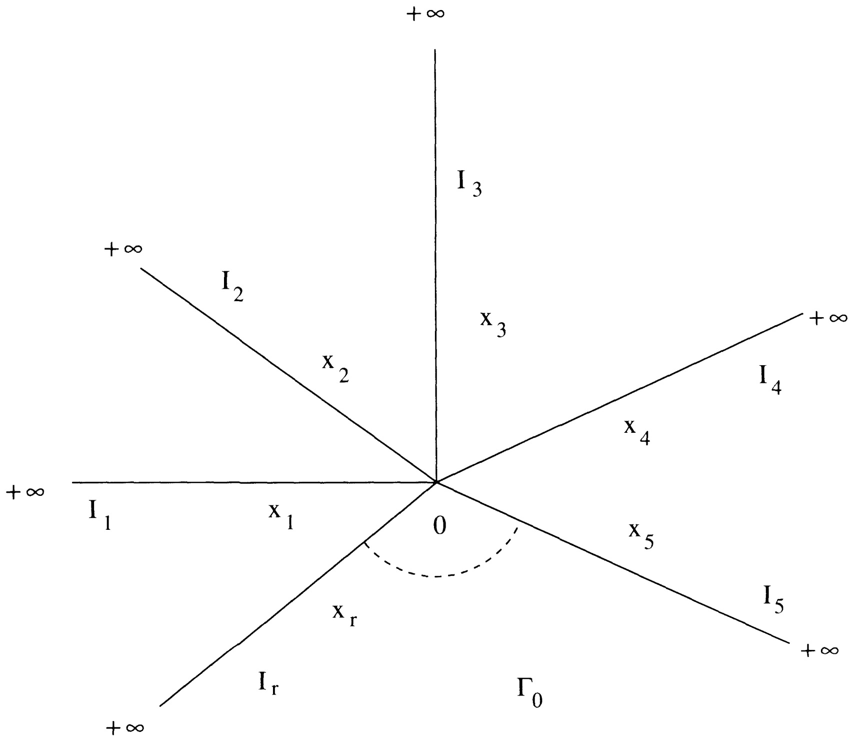

FIGURE 1 2

where $g(t,z)$ is the Gaussian kernel i.e. $g(t,z) = \exp(-z^2/2t)/(2\pi t)^{1/2}$.

SKETCH OF THE PROOF. For $y \in I_j$, we can put from (4)

$$G(x,y,\lambda) = \begin{cases} p_i e^{-(2\lambda)^{1/2} c_i x/a_i} & \text{for } x \in I_i,\ i \neq j, \\ q e^{(2\lambda)^{1/2} c_j x/a_j} + r e^{-(2\lambda)^{1/2} c_j x/a_j} & \text{for } x \in I_j,\ 0 < x < y, \\ s e^{-(2\lambda)^{1/2} c_j x/a_j} & \text{for } x \in I_j,\ y < x. \end{cases}$$

with unknown constants p_i, q, r, s. These constants are found by (2), (3) and (4). And we get

$$p_i = 2^{1/2} c_j^2 \big(\lambda^{1/2} \textstyle\sum a_k c_k \gamma\big)^{-1} (\equiv p),$$
$$q = \textstyle\sum a_k c_k p (2a_j c_j)^{-1},$$
$$r = \big(a_j c_j - \sum_{k \neq i} a_k c_k\big) p (2a_j c_j)^{-1},$$
$$s = q\gamma^2 + r,$$

where $\gamma = \exp\big((2\lambda)^{1/2} c_j y/a_j\big)$. For instance, we choose $x \in I_i$ $(i \neq j)$. Then, $G(x,y,\lambda) = \{2^{1/2} c_j^2/\lambda^{1/2} \sum a_k c_k\} \exp\{-(2\lambda)^{1/2}(\frac{c_i}{a_i} x + \frac{c_j}{a_j} y)$. Therefore, from the equality

$$\frac{1}{2\pi i} \int_\Omega e^{\lambda t + a\lambda^{1/2}} \lambda^{-1/2}\, d\lambda = (\pi t)^{-1/2} e^{-a^2/4t},$$

we derive (7). In the case $x \in I_j$, (7) is immediate in the same way.

Next, we calculate the transition density $P_t(x, y)$ of the skew Brownian motion X_t with rates $(\alpha_1, \alpha_2, \ldots, \alpha_r)$. By a similar calculation (see J. B. Walsh [9]), we have

LEMMA 2.

$$P_t(x,y) = \begin{cases} 2\alpha_j g(t, x+y) & \text{if } x \in I_i,\ y \in I_j \\ g(t, x-y) + (2\alpha_i - 1) g(t, x+y) & \text{if } x, y \in I_i. \end{cases} \tag{8}$$

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ with $\beta_i > 0$, $i = 1, 2, \ldots, r$ and let $\Phi_\beta(x) = \beta_i x$ if $x \in I_i$. Then the transition density $P_t^\beta(x, y)$ of the stretched process $\Phi_\beta(X_t)$ is given as follows

$$P_t^\beta(x,y) = \begin{cases} \frac{2\alpha_j}{\beta_j} g(t, \frac{1}{\beta_i}x + \frac{1}{\beta_j}y) & \text{if } x \in I_i,\ y \in I_j \\ \frac{1}{\beta_i}\Big\{g\big(t, \frac{1}{\beta_i}(x-y)\big) + (2\alpha_i - 1) g\big(t, \frac{1}{\beta_i}(x+y)\big)\Big\} & \text{if } x, y \in I_i. \end{cases} \tag{9}$$

To compare (9) with (7), we obtain the next result.

PROPOSITION 1. *The fundamental solution of the equation (6) with conditions (2) and (3) could be constructed as the transition density of the stretched skew Brownian motion $\Phi_\beta(X_t)$ with $\beta_i = a_i/c_i$ and skew rates $\alpha_i = a_i c_i / \sum_k a_k c_k$, $i = 1, 2, \ldots, r$.*

Namely, skew rates α_i are proportioned to the square root of the product of the specific heat and the thermal conductivity.

Here we consider the case $a_i^2 = 2c_i^2$ on each I_i. Then the heat equation (6) is as follows

$$\Big(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\Big) U_t(x) = 0. \tag{10}$$

And the nodal conditions are (3) and (11) given below,

$$\sum_{j=1}^{r} \alpha_j \lim_{\substack{x_j \to 0 \\ x_j \in I_j}} \frac{\partial U}{\partial x_j}(x_j) = 0. \tag{11}$$

Therefore, the heat kernel of this equation ((10), (3), and (11)) coincides with the transition density of the skew Brownian motion X_t defined above. So far, we have studied the basic graph Γ_0, and this may be generalized to all locally finite graphs.

From now on, we will consider the heat equation (10) with the nodal conditions (12) and (13) on vertices as below. Let $P \in V$ and its coordinate be x_0 and let $I_1, I_2, \ldots, I_r$ be the adjacent edges (Figure 2.1 in the case $r = 3$); then the nodal conditions are

$$\lim_{\substack{x_i \to x_0 \\ x_i \in I_i}} U(x_i), \quad i = 1, 2, \ldots, r \text{ are mutually equal} \tag{12}$$

$$\sum_{i=1}^{r} \alpha_i \lim_{\substack{x_i \to x_0 \\ x_i \in I_i}} \frac{\partial U}{\partial x_i}(x_i) = 0 \quad \Big(\sum_{i=1}^{r} \alpha_i = 1,\ 0 \le \alpha_i \le 1\Big). \tag{13}$$

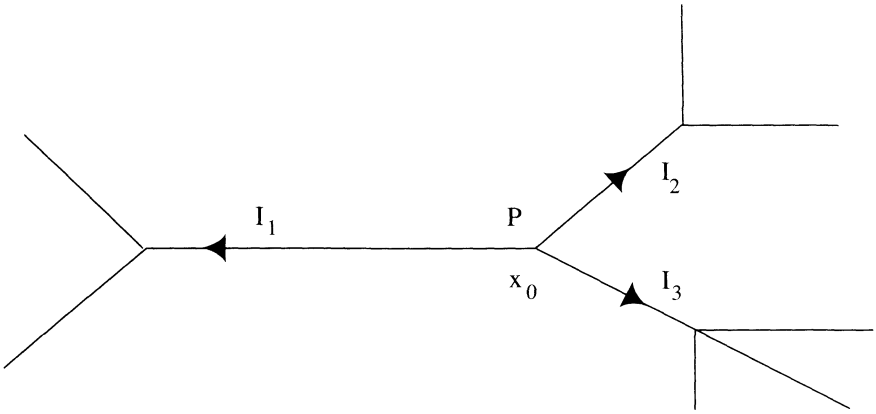

FIGURE 2.1

DEFINITION. We call (13) the *skew condition* with rate α_i to the direction of I_i, $i = 1, 2, \ldots, r$, at the vertex P.

In this situation, the heat kernel should be equal to the transition density of the process which behave at each edge similarly to skew Brownian motion on Γ_0 at the origin.

In the following sections, we calculate the heat kernels $P_t(x, y)$ with the skew condition on some periodic graph networks. And we investigate the local asymptotic decay order of P_t as time goes to infinity.

3. Periodic line and its interpretation. Let Γ_1 be a line to join the unit segments with skew rate α ($0 \leq \alpha \leq 1$) at all of these joints to the opposite direction from the origin as in the Figure 3.1.

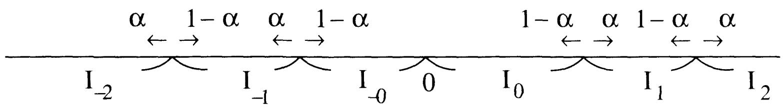

FIGURE 3.1

We could interpret this model as below. The line is composed of different materials I_i, $i = 0, \pm 1, \pm 2, \ldots$, and the specific heat and the thermal conductivity of I_i are $\alpha^{|i|}$ together. Let us compute the Green function $G(x, y, \lambda)$. By periodicity of Γ_1, we can set from (4),

$$G(x, 0, \lambda) = \begin{cases} c_1 r_1^n(\theta_1 e^{\lambda^{1/2}(x-n)} + e^{-\lambda^{1/2}(x-n)}) & \text{for } x \in I_n \\ c_1 r_1^n(\theta_1 e^{-\lambda^{1/2}(x+n)} + e^{\lambda^{1/2}(x+n)}) & \text{for } x \in I_{-n}, \end{cases} \tag{14}$$

$n = 0, 1, 2, \ldots$, with unknown constants c_1, r_1, θ_1, where I_0 stands for the positive part of I_0 anew and I_{-0} is the negative part of I_0. We will use the expression I_0 and I_{-0} in

Section 4 and 5 too. The constants c_1, r_1, θ_1 will be determined as below. From the condition (12) we have

$$\theta_1 \exp(\lambda^{1/2}) + 1 = r_1\big(\theta_1 + \exp(\lambda^{1/2})\big). \tag{15}$$

Next, from the condition (13), we have

$$(1-\alpha)\big(\theta_1 \exp(\lambda^{1/2}) - 1\big) = \alpha r_1\big(\theta_1 - \exp(\lambda^{1/2})\big). \tag{16}$$

At first, we assume $\alpha \neq 1/2$. If $\alpha \neq 0$, then, paying attention to the fact that G should tend to 0 at infinity, we get

$$\begin{aligned} \theta_1 &= \frac{\gamma - \gamma^{-1} - \{(\gamma-\gamma^{-1})^2 + 4(1-2\alpha)^2\}^{1/2}}{2(2\alpha-1)} \\ r_1 &= \frac{\gamma + \gamma^{-1} - \{(\gamma+\gamma^{-1})^2 + 16\alpha(\alpha-1)\}^{1/2}}{4\alpha}, \end{aligned} \tag{17}$$

where $\gamma = \exp(\lambda^{1/2})$. If $\alpha = 0$, then we have $\theta_1 = \gamma^{-1}$ and $r_1 = 2/(\gamma+\gamma^{-1})$. And to find c_1, it suffices to recall the condition (4) at $y = 0$. We can rewrite the equation (4) at $y = 0$ as follows

$$\frac{\partial}{\partial x} G(y-0, y, \lambda) - \frac{\partial}{\partial x} G(y+0, y, \lambda) = 1.$$

To substitute (14) into the above expression, we derive

$$c_1 \lambda^{1/2}(\theta_1 - 1) - c_1\lambda^{1/2}(-\theta_1 + 1) = 1.$$

Therefore we get $c_1 = 1/\big(2\lambda^{1/2}(1-\theta_1)\big)$. Consequently, we have

LEMMA 3. *If* $\alpha \neq 1/2$,

$$G(x,0,\lambda) = \begin{cases} r_1^n\{2\lambda^{1/2}(1-\theta_1)\}^{-1}(\theta_1 e^{\lambda^{1/2}(x-n)} + e^{-\lambda^{1/2}(x-n)}) & \textit{for } x \in I_n \\ r_1^n\{2\lambda^{1/2}(1-\theta_1)\}^{-1}(\theta_1 e^{-\lambda^{1/2}(x+n)} + e^{\lambda^{1/2}(x+n)}) & \textit{for } x \in I_{-n}. \end{cases} \tag{18}$$

In particular, $G(0,0,\lambda) = (1+\theta_1)/2\lambda^{1/2}(1-\theta_1)$, and hence

$$P_t(0,0) = \frac{1}{2\pi i}\int_\Omega \frac{1+\theta_1}{2\lambda^{1/2}(1-\theta_1)} e^{\lambda t}\, d\lambda. \tag{19}$$

Now, let us estimate the decay order of $P_t(0,0)$ as $t \longrightarrow \infty$. To that end, in (19), we examine the integrand for λ as $|\lambda|$ is small enough. From (17), we derive

$$\begin{aligned} \frac{1+\theta_1}{1-\theta_1} &= \frac{\{(\gamma-\gamma^{-1})^2 + 4(2\alpha-1)^2\}^{1/2} - 2(2\alpha-1)}{\gamma-\gamma^{-1}} \\ &= \frac{1}{\gamma-\gamma^{-1}}\Big[2|2\alpha-1|\Big\{1 + \Big(\frac{\gamma-\gamma^{-1}}{2(2\alpha-1)}\Big)^2\Big\}^{1/2} - 2(2\alpha-1)\Big] \\ &= \frac{2|2\alpha-1| - 2(2\alpha-1)}{\gamma-\gamma^{-1}} + \frac{\gamma-\gamma^{-1}}{2|2\alpha-1|} + k_1(\gamma-\gamma^{-1})^3 + k_2(\gamma-\gamma^{-1})^5 + \cdots \end{aligned}$$

with constants $k_1, k_2, \dots$. Hence we consider the next two cases,

(I) If $\alpha < 1/2$, then we have $(1+\theta_1)/(1-\theta_1) \sim 2(1-2\alpha)\lambda^{-1/2}$ as $|\lambda|$ is small, where $f(\lambda) \sim g(\lambda)$ means $\{g(\lambda)\}^{-1}f(\lambda) - 1 \longrightarrow 0$ as $|\lambda| \longrightarrow 0$. To the change of the variable $\lambda^{1/2} = i\xi$, we derive

$$\begin{aligned} P_t(0,0) &= \frac{1}{2\pi i}\int_\Omega e^{\lambda t} G(0,0,\lambda)\, d\lambda \\ &\sim \frac{1}{2\pi i}\int_{-\delta-0\imath}^{+\delta-0\imath} \frac{e^{-\xi^2 t}}{2i\xi}\Big(\frac{2(1-2\alpha)}{i\xi}\Big)(-2\xi)\, d\xi \\ &= \frac{1-2\alpha}{\pi}\int_{-\delta-0\imath}^{+\delta-0\imath} \frac{e^{-\xi^2 t}}{i\xi}\, d\xi \\ &= 1-2\alpha. \end{aligned}$$

(II) If $\alpha > 1/2$, as $|\lambda|$ is small, we have

$$\frac{1+\theta_1}{1-\theta_1} \sim \frac{1}{2\alpha-1}\lambda^{1/2} + k_1'\lambda^{3/2} + k_2'\lambda^{5/2} + \cdots,$$

with constants $k_1', k_2', \dots$. Then we derive

$$\frac{1}{2\pi i}\int_{-\delta-0\imath}^{+\delta-0\imath} \frac{e^{-\xi^2 t}}{2i\xi}\Big(\frac{1}{2\alpha-1} i\xi + k_1'(i\xi)^3 + k_2'(i\xi)^5 + \cdots\Big)(-2\xi)\, d\xi = 0.$$

Since $\gamma - \gamma^{-1} = 2i\sin\xi$, we have $(1+\theta_1)/(1-\theta_1) = [\{4(2\alpha-1)^2 - 4\sin^2\xi\}^{1/2} - 2(2\alpha-1)]/2i\sin\xi$. Therefore by the similar computation to [2], we have $P_t(0,0) = O\big(t^{-3/2}\exp(-|\sin^{-1}(2\alpha-1)|^2 t)\big)$, as $t \longrightarrow \infty$. Here $f(t) = O\big(g(t)\big)$ as $t \longrightarrow \infty$ means $\{g(t)\}^{-1}f(t) \longrightarrow c \neq 0$, a constant as $t \longrightarrow \infty$.

If $\alpha = 1/2$, we have $\theta_1 = 0$ and $r_1 = \gamma^{-1}$ from (15), (16). Therefore we derive for $x \in I_n$

$$\begin{aligned} P_t(x,0) &= \frac{1}{2\pi i}\int_\Omega e^{\lambda t}\frac{\gamma^{-n}e^{-\lambda^{1/2}(x-n)}}{2\lambda^{1/2}}\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Omega \frac{e^{\lambda t}e^{-\lambda^{1/2}x}}{2\lambda^{1/2}}\, d\lambda \\ &= \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}}, \qquad (\textit{i.e.}\ \text{the heat kernel is canonical}). \end{aligned}$$

Consequently, we have

THEOREM 1. *In the model Γ_1, as $t \longrightarrow \infty$, we have*

$$P_t(0,0)\begin{cases} = O\big(t^{-3/2}\exp\big(-|\sin^{-1}(2\alpha-1)|^2 t\big)\big), & \text{if } \alpha > \frac{1}{2} \\ \longrightarrow 1-2\alpha, & \text{if } \alpha < \frac{1}{2} \\ = (4\pi t)^{-1/2}, & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

REMARK. As $\int_0^\infty P_t(0,0)\, dt = \infty$ $(\alpha \leq 1/2)$, $< \infty$ $(\alpha > 1/2)$, the diffusion corresponding to this model is recurrent if and only if $\alpha \leq 1/2$.

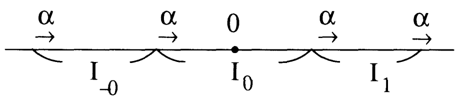

FIGURE 3 2

REMARK. In the model Γ_1, let the skew rate to positive direction at each joint be α (in the Figure 3.2.).

Then, by the similar calculation, we have

$$P_t(0,0) = O\big(t^{-3/2}\exp\big(-|\sin^{-1}(2\alpha-1)|^2 t\big) \text{ as } t \longrightarrow \infty.$$

4. **Cayley tree with skew.** Let Γ_2 be a homogeneous tree of order 3 with a skew condition on each vertex with rates α, β to the opposite directions of the origin and with rate $1-\alpha-\beta$ toward the origin, where α, β, $1-\alpha-\beta$ are non negative constants. The length of each edge is 1 as in the Figure 4.1.

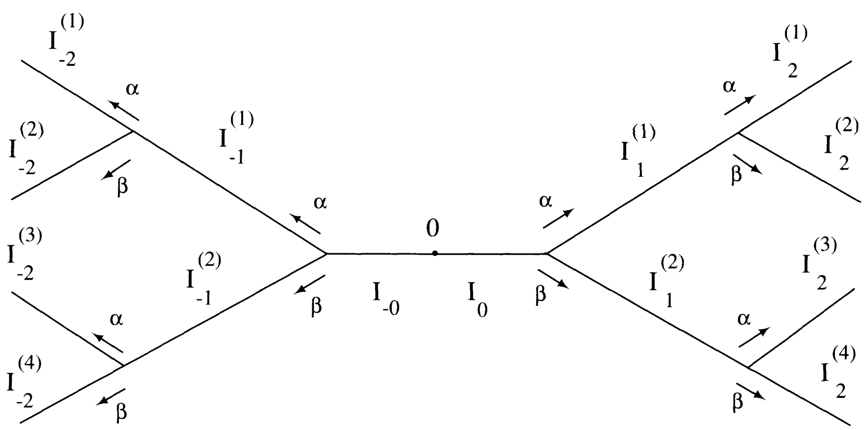

FIGURE 4 1

Let us compute the Green function $G(x,y,\lambda)$. For simplicity, we shall not repeat similar arguments as in the model Γ_1. Let $I_n^{(k)}$ ($I_{-n}^{(k)}$) be the $n+1$-st edges from I_0 to the positive (negative, respectively) direction, $k = 1, 2, \ldots, 2^n$. Then we have as before

$$(20) \qquad G(x,0,\lambda) = \begin{cases} c_2 r_2^n(\theta_2 e^{\lambda^{1/2}(x-n)} + e^{-\lambda^{1/2}(x-n)}) & \text{for } x \in I_n^{(k)} \\ c_2 r_2^n(\theta_2 e^{-\lambda^{1/2}(x+n)} + e^{\lambda^{1/2}(x+n)}) & \text{for } x \in I_{-n}^{(k)}, \end{cases}$$

$n = 0, 1, 2, \ldots$, with unknown constants c_2, r_2, θ_2 which are independent of k and will be determined similarly to the model Γ_1.

If $\alpha+\beta \neq 0,\ 1/2$, then we have

$$\theta_2 = \frac{\gamma-\gamma^{-1}-\{(\gamma-\gamma^{-1})^2+4(1-2\alpha-2\beta)^2\}^{1/2}}{2(2\alpha+2\beta-1)} \tag{21}$$

$$r_2 = \frac{\gamma+\gamma^{-1}-\{(\gamma+\gamma^{-1})^2+16(\alpha+\beta)(\alpha+\beta-1)\}^{1/2}}{4(\alpha+\beta)}$$

$$c_2 = \frac{1}{2\lambda^{1/2}(1-\theta_2)}.$$

Then, we get the Green function $G(x,0,\lambda)$ with the constants θ_1 and r_1 in (14) replaced by θ_2 and r_2 respectively.

If $\alpha+\beta = 1/2$, we have $\theta_2 = 0$ and $r_2 = \gamma^{-1}$.

If $\alpha+\beta = 0$, we have $\theta_2 = \gamma^{-1}$ and $r_2 = 2/(\gamma+\gamma^{-1})$.

In particular, we get

$$G(0,0,\lambda) = (1+\theta_2)/2\lambda^{1/2}(1-\theta_2). \tag{22}$$

Therefore, by the similar computation to the model Γ_1, we get

THEOREM 2. *In the model Γ_2, we have, as $t \longrightarrow \infty$,*

$$P_t(0,0)\begin{cases} = O\big(t^{-3/2}\exp\big(-|\sin^{-1}(2\alpha+2\beta-1)|^2 t\big) & \textit{if } \alpha+\beta > \frac{1}{2} \\ \longrightarrow 1-2\alpha-2\beta & \textit{if } \alpha+\beta < \frac{1}{2} \\ = (4\pi t)^{-1/2} & \textit{if } \alpha+\beta = \frac{1}{2}. \end{cases}$$

In this model, if x and x' are located at equal distances from the origin, then $P_t(x,0) = P_t(x',0)$. Therefore, in order to calculate $P_t(x,0)$, we can regard this model as Γ_1 of Section 3 with rate α replaced by $\alpha+\beta$, be means of bundling all $I_n^{(k)}$, $k = 1,2,\ldots,2^n$, as in the Figure 4.2.

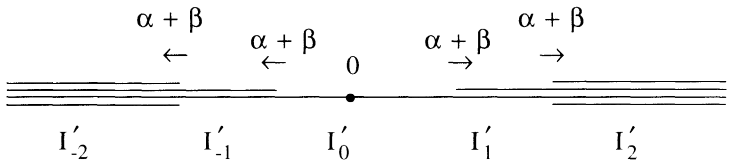

FIGURE 4.2.

But we can not calculate $P_t(0,y)$ from such a viewpoint. Let us compute $P_t(0,y)$ by the next way. Let $I_n^{k,n-k}$ be an edge such that

I) $I_n^{k,n-k}$ is the $(n+1)$-st edge from I_0.

II) On the trail from I_0 to $I_n^{k,n-k}$, there are k $(n-k)$ vertices with skew rates α (β, respectively) to positive direction from the origin. The sequence of the skew rates is arbitrary.

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Here we will compute the decay order of $P_t(0,y)$ for $y \in I_n^{k,n-k}$, as $t \longrightarrow \infty$. By periodicity, we can set the Green function,

$$G(x,y,\lambda) = \begin{cases} r_3(e^{\lambda^{1/2}x} + \theta e^{-\lambda^{1/2}x}) & \text{for } x \in I_0 \\ r_3(a_k e^{\lambda^{1/2}(x-k)} + b_k e^{-\lambda^{1/2}(x-k)}) & \text{for } x \in I'_k \\ c_k(\theta e^{\lambda^{1/2}(x-k)} + e^{-\lambda^{1/2}(x-k)}) & \text{for } x \in I^*_k \\ r_3(a_n e^{\lambda^{1/2}(x-y)} + b_n e^{-\lambda^{1/2}(x-y)}) & \text{for } x \in I'_n, \text{ and } x < y \\ c_n(\theta e^{\lambda^{1/2}(x-y)} + e^{-\lambda^{1/2}(x-y)}) & \text{for } x \in I'_n, \text{ and } y < x \end{cases} \tag{23}$$

with unknown constants r_3, a_k, b_k, c_k $(k = 1,2,\ldots,n)$, and known constant θ which is equal to θ_2, where $I'_0, I'_1, I'_2, \ldots, I'_n$ are the edges on the trail from I_0 to $I_n^{k,n-k}$ (*i.e.* $I'_0 = I_0$, $I'_n = I_n^{k,n-k}$), and I^*_k is the edge joining I'_{k-1} and I'_k. We can set the Green function on other edges to satisfy (4), (12) and (13). From the condition at point y, we get

$$\begin{gathered} r_3(a_n + b_n) = c_n(\theta + 1) \\ \lambda^{1/2} r_3(a_n - b_n) - \lambda^{1/2} c_n(\theta - 1) = 1. \end{gathered} \tag{24}$$

Therefore, by an easy calculation, we have

$$r_3 = -\frac{\theta + 1}{2\lambda^{1/2}(\theta b_n - a_n)}, \tag{25}$$

here, a_n, b_n are unknown constants, and in the case of $|\lambda| \sim 0$, we can get $\theta b_n - a_n$ by the following procedure.

For instance, we consider the case that there are skew rates $(\alpha, \beta, 1-\alpha-\beta)$ to the directions to $(I'_{k+1}, I^*_{k+1}, I'_k)$ respectively, as in the Figure 4.3.

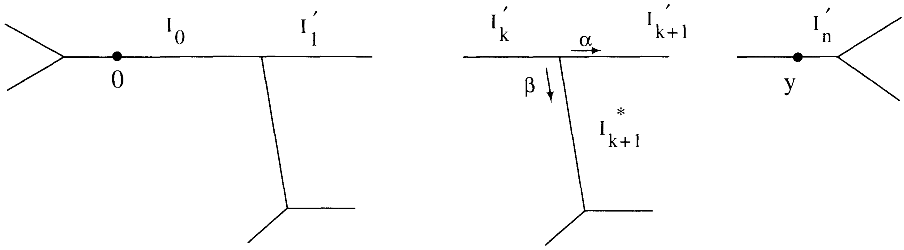

FIGURE 4 3

We get by the conditions (12), (13) at the vertex P where I'_k, I'_{k+1} and I^*_{k+1} are adjacent,

$$\begin{gathered} r_3(a_k\gamma + b_k) = r_3(a_{k+1} + b_{k+1}\gamma) = c_k(\theta + \gamma) \\ (1-\alpha-\beta) r_3(a_k\gamma - b_k) = \alpha r_3(a_{k+1} - b_{k+1}\gamma) + \beta c_k(\theta - \gamma). \end{gathered} \tag{26}$$

We assume $\alpha\beta \neq 0$ (*i.e.* non-trivial case). Then by an easy calculation, we obtain

$$\begin{aligned} \theta b_{k+1} - a_{k+1} &= \frac{\gamma - (2\alpha + 2\beta - 1)\theta}{2\alpha}\Big(\frac{(2\alpha + 2\beta - 1)\gamma - \theta}{\{(2\alpha + 2\beta - 1)\theta - \gamma\}\gamma\theta}\theta b_k - a_k\Big) \\ &= \frac{\gamma - (2\alpha + 2\beta - 1)\theta}{2\alpha}(\theta b_k - a_k), \end{aligned} \tag{27}$$

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where the second equality is derived from (21).

If $\alpha+\beta<1/2$, we get $\theta\sim 1+\lambda^{1/2}/(2\alpha+2\beta-1)$ and $\gamma\sim 1+\lambda^{1/2}$ as $|\lambda|\longrightarrow 0$. Therefore, we derive from (27), $\theta b_{k+1}-a_{k+1}\sim\frac{1-\alpha-\beta}{\alpha}(\theta b_k-a_k)$.

If $\alpha+\beta>1/2$, we get $\theta\sim -1+\lambda^{1/2}/(2\alpha+2\beta-1)$ and $\gamma\sim 1+\lambda^{1/2}$ as $|\lambda|\longrightarrow 0$. Then we have $\theta b_{k+1}-a_{k+1}\sim\frac{\alpha+\beta}{\alpha}(\theta b_k-a_k)$.

If $\alpha+\beta=1/2$, we have $\theta=0$. Then we derive from (25), $r_3=1/2\lambda^{1/2}a_n$ and $a_{k+1}=(1/2\alpha)a_k$ (*i.e.* coincide with the case that $\alpha+\beta<1/2$). Consequently, we get by inductive method,

$$\theta b_{k+1}-a_{k+1}\sim\begin{cases}\frac{(1-\alpha-\beta)^n}{\alpha^k\beta^{n-k}}(\theta^2-1) & \text{for } \alpha+\beta\leq 1/2\\ \frac{(\alpha+\beta)^n}{\alpha^k\beta^{n-k}}(\theta^2-1) & \text{for } \alpha+\beta>1/2.\end{cases}$$

Therefore, owing to (23), (25), we derive

$$\begin{aligned}G(0,y,\lambda)&=-\frac{(1+\theta)^2}{2\lambda^{1/2}(\theta b_n-a_n)}\\ &=\begin{cases}\frac{\alpha^k\beta^{n-k}}{(1-\alpha-\beta)^n}G(0,0,\lambda) & \text{for } \alpha+\beta\leq 1/2\\ \frac{\alpha^k\beta^{n-k}}{(\alpha+\beta)^n}G(0,0,\lambda) & \text{for } \alpha+\beta>1/2.\end{cases}\end{aligned}$$

Then we have the next result.

THEOREM 3. *For $y\in I_n^{k,n-k}$, as $t\longrightarrow\infty$,*

$$P_t(0,y)\sim\begin{cases}\frac{\alpha^k\beta^{n-k}}{(1-\alpha-\beta)^n}P_t(0,0) & \textit{for } \alpha+\beta\leq 1/2\\ \frac{\alpha^k\beta^{n-k}}{(\alpha+\beta)^n}P_t(0,0) & \textit{for } \alpha+\beta>1/2.\end{cases}$$

In particular, if $\alpha+\beta<\frac{1}{2}$, we have, as $t\longrightarrow\infty$,

$$P_t(0,y)\longrightarrow\frac{\alpha^k\beta^{n-k}}{(1-\alpha-\beta)^n}(1-2\alpha-2\beta)\quad \textit{for } y\in I_n^{k,n-k}.$$

5. **Comb type.** Let Γ_3 be a comb type model with a skew condition on each vertex with rates α, β, and $1-\alpha-\beta$ to the transverse opposite direction from the origin, vertical direction and toward the origin respectively, as illustrated in the Figure 5.1.

Let $I_n=(n-\frac{1}{2},n+\frac{1}{2})$ $n=0,\pm1,\pm2,\ldots$, and J_n be an upper branch at $x=n\mp\frac{1}{2}$. In this case we can set the Green function

$$G(x,0,\lambda)=\begin{cases}cr_4^n(\theta_4e^{\lambda^{1/2}(x-n)}+e^{-\lambda^{1/2}(x-n)}) & \text{for } x\in I_n\ (n\geq 0)\\ br_4^ne^{-\lambda^{1/2}(x-n)} & \text{for } x\in J_n\ (n\geq 0)\end{cases}\tag{28}$$

with unknown constants b, c, r_4 and θ_4. If $x\in I_{-n}\cup J_{-n}$, then we replace x by $-x$ in (28). From the conditions (12) and (13) we have

$$\begin{gathered}\theta_4\gamma+1=r_4(\theta_4+\gamma),\quad c(\theta_4+\gamma)=b\gamma,\\ (1-\alpha-\beta)(\theta_4\gamma-1)=\alpha r_4(\theta_4-\gamma)-\beta r_4(\theta_4+\gamma).\end{gathered}\tag{29}$$

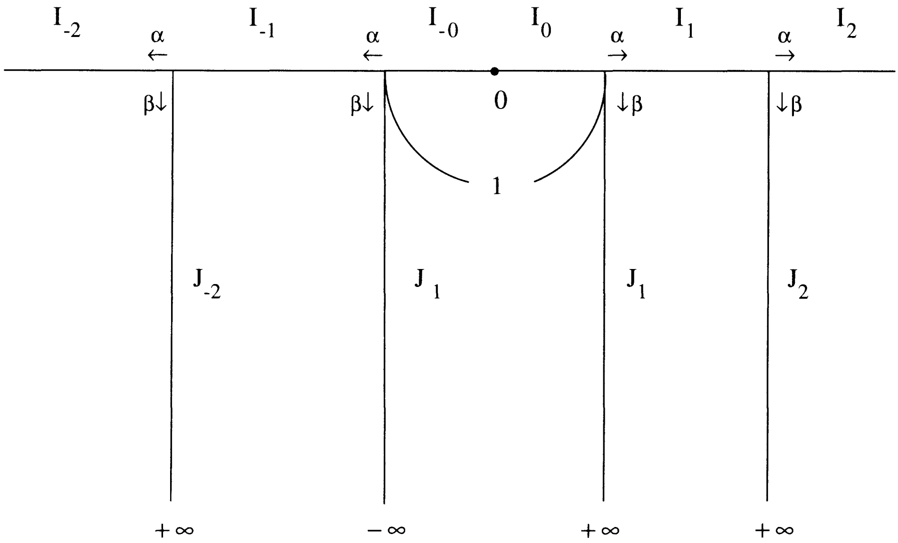

FIGURE 5 1

From (29) and the condition (4) at the origin, these constants will be determined as below At first we assume $\alpha \neq 0, 1/2$, then we get by the similar calculation,

$$(30) \qquad r_4 = \frac{\gamma + (1-2\beta)\gamma^{-1} - [\{\gamma + (1-2\beta)\gamma^{-1}\}^2 - 16\alpha(1-\alpha-\beta)]^{1/2}}{4\alpha}$$

$$\theta_4 = \frac{\gamma - (1-2\beta)\gamma^{-1} - [\{\gamma - (1-2\beta)\gamma^{-1}\}^2 - 4(2\alpha-1)(1-2\alpha-2\beta)]^{1/2}}{2(2\alpha-1)}$$

$$b = \frac{(4\alpha-1)\gamma - (1-2\beta)\gamma^{-1} - [\{\gamma - (1-2\beta)\gamma^{-1}\}^2 - 4(2\alpha-1)(1-2\alpha-2\beta)]^{1/2}}{2(2\alpha-1)\gamma}$$

$$c = \frac{1}{2\lambda^{1/2}(1-\theta_4)}$$

Therefore, we get the Green function $G(x,0,\lambda)$, in the same way as in the model Γ_1 We recall $G(0,0,\lambda) = (1+\theta_4)/(2\lambda^{1/2}(1-\theta_4)\}$ By an easy calculation from (30), we derive

$$\frac{1+\theta_4}{1-\theta_4} = \frac{[\{\gamma - (1-2\beta)\gamma^{-1}\}^2 - 4(2\alpha-1)(1-2\alpha-2\beta)]^{1/2} - 2(2\alpha-2\beta-1)}{\gamma - (1-2\beta)\gamma^{-1} + 2\beta}$$

$$\sim \begin{cases} \dfrac{2|2\alpha+\beta-1|\left\{1+\frac{2\beta(1-\beta)\lambda^{1/2}}{(2\alpha+\beta-1)^2}+\frac{2\beta^2-2\beta+1}{(2\alpha+\beta-1)^2}\lambda\right\}^{1/2}-2(2\alpha+\beta-1)}{4\beta+2(1-\beta)\lambda^{1/2}+\beta\lambda} & \text{for } 2\alpha+\beta-1 \neq 0, \\ \dfrac{\{8\beta(1-\beta)\lambda^{1/2}+4(2\beta^2-2\beta+1)\lambda\}^{1/2}}{4\beta+2(1-\beta)\lambda^{1/2}+\beta\lambda} & \text{for } 2\alpha+\beta-1 = 0 \end{cases}$$

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Hence we get next estimations.

(I) If $\beta \neq 0$, we derive, as $|\lambda| \longrightarrow 0$,

$$G(0,0,\lambda)$$

$$\sim \begin{cases} \frac{\{2\beta(1-\beta)\}^{1/2}}{4\beta}\lambda^{-1/4} & \text{for } 1-\alpha-\beta=\alpha,\ \alpha\neq 0 \\ \frac{1-2\alpha-\beta}{2\beta}\lambda^{-1/2} & \text{for } 1-\alpha-\beta>\alpha,\ \alpha\neq 0 \\ \frac{1-\beta}{4(2\alpha+\beta-1)}\left\{1+\frac{2\alpha\beta(\alpha+\beta-1)}{(1-\beta)(2\alpha+\beta-1)^2}\lambda^{1/2}\right\} & \text{for } 1-\alpha-\beta<\alpha,\ a+\beta\neq 1,\ \alpha\neq\frac{1}{2}. \end{cases} \tag{31}$$

(II) If $\beta = 0$, we have from (29)

$$\theta_4 = \frac{\gamma-\gamma^{-1}-\{(\gamma-\gamma^{-1})^2+4(1-2\alpha)^2\}^{1/2}}{2(2\alpha-1)},$$

as $|\lambda| \longrightarrow 0$, which coincides with (17) of the model Γ_1.

If $\alpha = 1/2$, we have $\theta_4 = 2\beta/\{(1-2\beta)\gamma^{-1}-\gamma\}$ from (29), then the Green function $G(0,0,\lambda) \sim \frac{1-\beta}{4\beta}\{1-\frac{1-2\beta}{2\beta(1-\beta)}\lambda^{1/2}\}$ (for $\beta\neq 0$), $2\lambda^{-1/2}$ (for $\beta = 0$), as $|\lambda| \longrightarrow 0$.

If $\alpha = 0$, we have $\theta_4 = (1-2\beta)\gamma^{-1}$ from (29); then

$$G(0,0,\lambda) \sim \begin{cases} \frac{1-\beta}{2\beta}\lambda^{-1/2} & (\beta\neq 0), \\ 2\lambda^{-1/2} & (\beta = 0). \end{cases}$$

In order to examine the decay order of P_t, we recall the next remark.

REMARK. If $G = O(\lambda^a)$ as $|\lambda| \longrightarrow 0$, then $P_t = O(t^{-a-1})$ as $t \longrightarrow \infty$, for $a \neq 0, 1, 2, \ldots$.

Indeed it is immediate by an elementary Tauberian Theorem. And we have the next result.

THEOREM 4. *In the model Γ_3, we have, as $t \longrightarrow \infty$,*

$$P_t(0,0)$$

$$\begin{cases} = O(t^{-1/2}) & \textit{for } 1-\alpha-\beta>\alpha,\ \beta\neq 0 \textit{ or } (\alpha,\beta)=(1/2,0) \\ = O(t^{-3/4}) & \textit{for } 1-\alpha-\beta=\alpha,\ \alpha\neq 0,1/2 \\ = O(t^{-3/2}) & \textit{for } 1-\alpha-\beta<\alpha,\ \alpha+\beta\neq 1,\ \beta\neq 0 \\ \longrightarrow 1-2\alpha & \textit{for } \beta=0,\ 0\leq\alpha<1/2 \\ = O(t^{-3/2}e^{-|\sin^{-1}(2\alpha-1)|^2 t}) & \textit{for } \beta=0,\ \frac{1}{2}<\alpha<1 \\ = O(t^{-3/2}e^{-\pi^2 t/4}) & \textit{for } \alpha+\beta=1, \end{cases}$$

as in the Figure 5.2.

REMARK. If $\beta \neq 0$ and $\alpha+\beta\neq 1$, we derive $\int_0^\infty P_t(0,0)\,dt = \infty$. Then the diffusion corresponding to this model is recurrent.

REMARK. In the model Γ_3, if the skew rate α to the transverse direction turn to the same direction on all vertices, then we have $P_t(0,0) = O(t^{-3/2})$, by the similar calculation above.

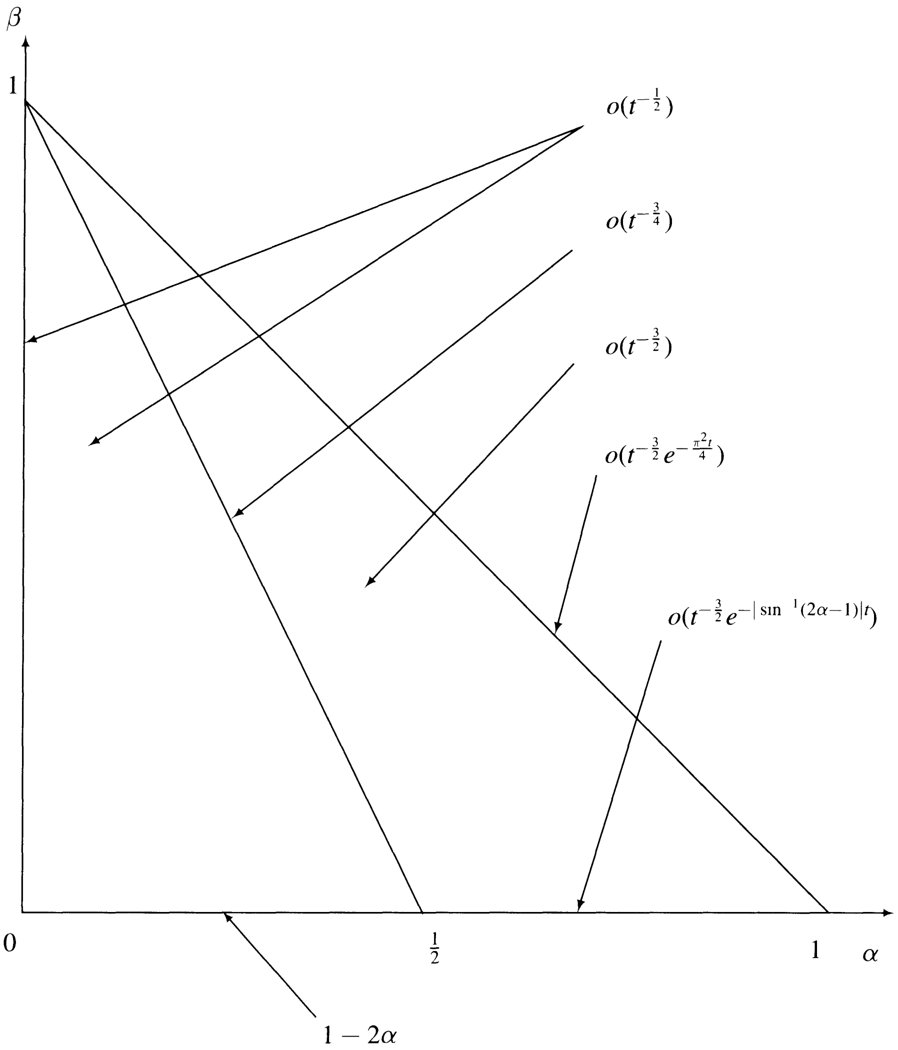

FIGURE 5.2.

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