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THE CONTINUITY OF DERIVATIONS FROM GROUP ALGEBRAS: FACTORIZABLE AND CONNECTED GROUPS

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Abstract

A group is said to be *factorizable* if it has a finite number of abelian subgroups, $H_1, H_2, \ldots, H_n$, such that $G = H_1 H_2 \ldots H_n$. It is shown that, if G is a factorizable or connected locally compact group, then every derivation from $\mathcal{L}^1(G)$ to an arbitrary $\mathcal{L}^1(G)$-bimodule $\mathcal{X}$ is continuous.

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Introduction

This paper is to provide a partial answer to a question in [3]. In order to state the question, let $\mathcal{L}^1(G)$ be the Banach algebra which is the Lebesgue space of the locally compact group G with convolution product. It is called the *group algebra* of G. A *derivation* from $\mathcal{L}^1(G)$ is a linear map $D : \mathcal{L}^1(G) \to \mathcal{X}$, where $\mathcal{X}$ is a Banach bimodule over $\mathcal{L}^1(G)$, such that $D(F_1 * F_2) = F_1 \cdot D(F_2) + D(F_1) \cdot F_2$ for every F_1 and F_2 in $\mathcal{L}^1(G)$. Then question 22 [3] asks for which, if any, groups G is there a discontinuous derivation from $\mathcal{L}^1(G)$.

A complete answer to this question would probably require much more to be known about the structure of group algebras than is known at present and attempts to answer the question can generate interesting problems concerning

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the structure of group algebras. However, some partial answers are known where a restriction is placed on the bimodule $\mathcal{X}$ or on the group G. For example, since $\mathcal{L}^1(G)$ is semisimple for every locally compact group G, an immediate corollary of [11]is that every derivation from $\mathcal{L}^1(G)$ to itself is continuous. Also, every derivation from $\mathcal{L}^1(G)$ to a commutative Banach $\mathcal{L}^1(G)$-bimodule is continuous, see [20, Theorem 4.3]. The continuity of derivations from $\mathcal{L}^1(G)$ to an arbitrary $\mathcal{L}^1(G)$-bimodule may be deduced from [9, Theorem 2] if G is abelian or compact.

The contribution of this paper is to show that derivations from $\mathcal{L}^1(G)$ are continuous if G is factorizable or connected. In the course of doing so, some factorization results for finite codimensional ideals in $\mathcal{L}^1(G)$ when G is factorizable or connected will be proved, (see Section 2). Such factorization results are an example of the sort of information about the structure of group algebras which is required to answer the automatic continuity question. In the cases where G is abelian or compact, finite codimensional ideals in $\mathcal{L}^1(G)$ have bounded approximate units because in these cases G is amenable, see [12], and the required factorizations follow from Cohen's theorem, see [2, Theorem 11.10]. However, many factorizable and connected groups are not amenable and other methods have to be used to prove the required factorization results.

There are some abuses of notation which will occur throughout the paper, as various algebras which are shown in [7]to be isomorphic will be identified. Let $\mathcal{M}(G)$ denote the algebra of bounded measures on G with convolution product. Then $\mathcal{L}^1(G)$ will be identified with the subalgebra, $\mathcal{M}_a(G)$, of $\mathcal{M}(G)$ consisting of measures which are absolutely continuous with respect to Haar measure, see [7, Theorem 19.18], and the discrete group algebra, $\ell^1(G)$, will be identified with $\mathcal{M}_d(G)$, the subalgebra of $\mathcal{M}(G)$ consisting of discrete measures, see [7, Theorem 19.15]. If H is a subgroup of G, then $\ell^1(H)$ will be identified with the subalgebra of $\ell^1(G)$ consisting of functions which are supported on H. Each measure, μ, belonging to $\mathcal{M}(G)$ defines a left multiplier on $\mathcal{L}^1(G)$ by convolution, that is, the map $F \mapsto \mu * F$ is a left multiplier on $\mathcal{L}^1(G)$. Thus, with these identifications, each function in $\mathcal{L}^1(G)$, $\ell^1(G)$ or $\ell^1(H)$ defines a multiplier on $\mathcal{L}^1(G)$.

1. Factorizable groups

1.1 DEFINITION. A group G, is said to be factorizable if there are abelian subgroups, $H_1, H_2, \ldots, H_n$ of G such that $H_1H_2\ldots H_n \equiv \{h_1h_2\ldots h_n | h_i \in H_i, i = 1, 2, \ldots n\}$ is equal to G.

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The abelian subgroups, $H_1, H_2, \ldots, H_n$, may be regarded as 'parametrizing' G. In later sections we shall use this idea to derive information about the structure of $\mathcal{L}^1(G)$ when G is factorizable from well-known theorems about the structure of commutative group algebras. This information will be used in the proof of continuity of derivations from the group algebras.

The class of factorizable groups is quite large. As an almost immediate consequence of a theorem of Iwasawa we have the following

1.2 THEOREM. *If G is a connected Lie group, then G is factorizable.*

PROOF. By [8, Theorem 6], there are abelian subgroups, $H_1, H_2, \ldots, H_r$, and a compact, connected subgroup, K, of G such that $G = H_1H_2\ldots H_rK$. Now K is a Lie group and so has one-parameter subgroups, $V_1, V_2, \ldots, V_k$, such that $V_1V_2\ldots V_k$ covers an open neighbourhood, U, of the identity element e. Since K is connected, $\bigcup_{n=1}^{\infty} U^n$ covers K. Then, since K is compact, there is an n such that $K = U^n$ and it follows that K is factorizable.

Connected Lie groups are uncountable and so, as discrete groups, their group algebras are not separable. However, they have many countable, factorizable subgroups.

1.3 THEOREM. *Let G be a factorizable group and S be a countable subset of G. Then G has a countable, factorizable subgroup $H \supseteq S$.*

PROOF. Let $H_1, H_2, \ldots, H_n$ be abelian subgroups of G such that $G = H_1H_2\ldots H_n$. Construct subsets $R_m, T_m, m = 1, 2, \ldots$ of G recursively as follows. For each $s \in S$ choose $h_i \in H_i$, $i = 1, 2, \ldots, n$ such that $s = h_1h_2\ldots h_n$ and define R_1 to be the set of all h_i's chosen. Define T_1 to be the subgroup generated by R_1. Then T_1 is a countable subgroup of G. Next, supposing that T_m has been constructed and is countable for some m, repeat the construction with T_m in place of S. That is, for each $s \in T_m$ choose $h_i \in H_i$, $i = 1, 2, \ldots, n$ such that $s = h_1h_2\ldots h_n$ and define R_{m+1} to be the set of all h_i's chosen. Define T_{m+1} to be the subgroup of G generated by R_{m+1}, so that T_{m+1} is countable.

Put $H = \bigcup_{m=1}^{\infty} T_m$. Then H will be a countable, factorizable subgroup of G containing S.

Subgroups of Lie groups are not the only infinite factorizable groups. For example, if R is a ring with unit, then the group of upper triangular $n \times n$ matrices over R is factorizable. Compact polythetic groups, as defined in [14], are factorizable also.

Some of the properties for $\mathcal{L}^1(G)$ when G is factorizable which are proved in the next section will hold if it is supposed only that there are abelian subgroups $H_1, H_2, \ldots, H_n$ of G such that $H_1H_2\ldots H_n$ is dense in G. This suggests the following question.

1.4 PROBLEM. Let G be a locally compact group which has abelian subgroups $H_1, H_2, \ldots, H_n$ such that $H_1H_2\ldots H_n$ is dense in G. Must G be factorizable?

It is easily seen that, if G is supposed also to be compact, then the answer to this question is yes. The non-compact polythetic groups might be an interesting class of groups on which to begin to answer this question.

2. Factorization in ideals of group algebras

It may be shown that factorization in the finite codimensional ideals of a group algebra is a necessary condition for the continuity of all derivations from the algebra, see [4]or the section on point derivations in [3]. It is not surprising then that an important part of the proof of continuity of derivations from certain group algebras will be a proof of some factorization results for finite codimensional ideals in the group algebras. These are established in Theorems 2.5 and 2.6 below and the proof of continuity of derivations will be completed in the next section. Theorems 2.5 and 2.6 will be required also in the proof of the automatic continuity of left $\ell^1(G)$-module homomorphisms from $\mathcal{L}^1(G)$ in [21].

The proofs of factorization will be carried out first for discrete group algebras. Suppose that H is an abelian subgroup of G and $\mathcal{J}$ is an ideal with finite codimension in $\ell^1(H)$. Then, by the main theorem in [12], $\mathcal{J}$ has bounded approximate units and so, by Cohen's factorization theorem, $\mathcal{J} * \ell^1(G) \equiv \{h * f | h \in \mathcal{J}, f \in \ell^1(G)\}$ is a closed $\mathcal{J}$-submodule of $\ell^1(G)$. Hence, if $\sum_n f_n$ is a convergent sum of elements belonging to $\mathcal{J} * \ell^1(G)$, then this sum is in the submodule. This fact will be used repeatedly in the proof of the following lemma.

2.1 LEMMA. *Let G be a group and suppose that $G = H_1H_2\ldots H_n$ where $H_1, H_2, \ldots H_n$ are abelian subgroups of G. Let, for each k, $\mathcal{J}_k$ be an ideal with finite codimension in $\ell^1(H_k)$. Then there are subalgebras $\mathcal{K}_1, \mathcal{K}_2, \ldots,$ $\mathcal{K}_M$ of $\ell^1(G)$, each of the form $\delta_x * \mathcal{J}_k * \delta_{x^{-1}}$ for some k and some x in G, such that*

$$\sum_{m=1}^{M} \mathcal{K}_m * \ell^1(G)$$

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is closed and has finite codimension in $\ell^1(G)$.

PROOF. We will show that there are elements $x_1, x_2, \dots, x_p$ in G such that every f in $\ell^1(G)$ is of the form

$$(\alpha) \qquad f = h + \sum_{j=1}^{p} c_j \delta_{x_j},$$

where h is in $\sum_{m=1}^{M} \mathcal{K}_m * \ell^1(G)$ and each c_j depends continuously on f. To begin, choose, for each k between 1 and n, a finite set, S_k, from H_k such that $\ell^1(H_k) = \mathcal{J}_k + \operatorname{span}\{\delta_x | x \in S_k\}$.

Let Z_1 be a set of representatives of the right cosets of H_1 chosen from $H_2 \dots H_n$. Then each f in $\ell^1(G)$ has the form $f = \sum_{z \in Z_1} f^{(z)} * \delta_z$, where $f^{(z)}$ belongs to $\ell^1(H_1)$. For each z in Z_1, $f^{(z)} = g^{(z)} + \sum_{x \in S_1} c_x^{(z)} \delta_x$ for some functions $g^{(z)}$ in $\mathcal{J}_1$ and scalars $c_x^{(z)}$ which depend continuously on f. Now $\sum_{z \in Z_1} g^{(z)} * \delta_z$ belongs to $\mathcal{J}_1 * \ell^1(G)$ and $\sum_{z \in Z_1} (\sum_{x \in S_1} c_x^{(z)} \delta_x) * \delta_z$ belongs to $\sum_{x \in S_1} \delta_x * \ell^1(H_2 \dots H_n)$ whence

$$f = g + \sum_{x \in S_1} \delta_x * h_x,$$

where g is in $\mathcal{J}_1 * \ell^1(G)$ and h_x belongs to $\ell^1(H_2 \dots H_n)$ and depends continuously on f for each x in S_1.

Next, since $H_2 \dots H_n$ is a set of cosets of H_2, we may choose a set, Z_2, of representatives of these cosets which is contained in $H_3 \dots H_n$. For each h_x above we then have $h_x = \sum_{z \in Z_2} h_x^{(z)} * \delta_z$, where $h_x^{(z)}$ belongs to $\ell^1(H_2)$. Now, for each z in Z_2, $h_x^{(z)}$ has the form $h_x^{(z)} = g_x^{(z)} + \sum_{y \in S_2} c_{y,x}^{(z)} \delta_y$ for some functions $g_x^{(z)}$ in $\mathcal{J}_2$ and scalars $c_{y,x}^{(z)}$. Hence, as above, we have

$$h_x = g_x + \sum_{y \in S_2} \delta_y * h_{y,x},$$

where g_x is in $\mathcal{J}_2 * \ell^1(G)$ and $h_{y,x}$ belongs to $\ell^1(H_3 \dots H_n)$ for each y and x. It is easily seen that $\delta_x * g_x$ belongs to $(\delta_x * \mathcal{J}_2 * \delta_{x^{-1}}) * \ell^1(G)$ for each x and so, combining with the above, we have

$$f = h + \sum_{x \in S_1} \left(\sum_{y \in S_2} \delta_x * \delta_y * h_{y,x} \right),$$

where $h = g + \sum_{x \in S_1} \delta_x * g_x$, which belongs to a subspace of the form $\sum_{m=1}^{L} \mathcal{K}_m * \ell^1(G)$.

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This argument may be repeated a further $n-2$ times to show that f is as described in (α). The group elements x_j appearing in (α) will belong to $S_1 S_2 \dots S_n$ and the subalgebras $\mathcal{K}_m$ will be of the form $\delta_x * \mathcal{J}_k * \delta_{x^{-1}}$ where x belongs to $S_1 \dots S_{k-1}$, $k = 1, 2, \dots, n$. It is clear that each c_j will depend continuously on f and so $\sum_{m=1}^{M} \mathcal{K}_m * \ell^1(G)$ is closed.

For a Banach algebra, $\mathcal{A}$, define $\mathcal{A}^2 = \operatorname{span}\{ab | a, b \in \mathcal{A}\}$. Then $\mathcal{A}$ is said to *factor weakly* if $\mathcal{A}^2 = \mathcal{A}$. The next lemma shows that finite codimensional ideals in $\ell^1(G)$ factor weakly if G is discrete and factorizable.

2.2 LEMMA. *Let G be a group as in Lemma 2.1 and let $\mathcal{J}$ be a two-sided ideal with finite codimension in $\ell^1(G)$. Then $\mathcal{J}$ factors weakly.*

In particular, for each k, let $\mathcal{J}_k = \mathcal{J} \cap \ell^1(H_k)$, so that $\mathcal{J}_k$ has finite codimension in $\ell^1(H_k)$. Suppose that $\mathcal{K}_m$, $m = 1, \dots, M$, are the subalgebras of $\ell^1(G)$ constructed in Lemma 2.1. Then there are P elements, a_p, which are products in $\mathcal{J}$, such that

$$\mathcal{J} = \sum_{p=1}^{P} \mathbb{C} a_p + \sum_{m=1}^{M} \mathcal{K}_m * \ell^1(G).$$

Furthermore, there is a $K > 0$ such that every f in $\mathcal{J}$ satisfies $f = \sum_{p=1}^{P} c_p a_p + \sum_{m=1}^{M} f_m$ where $\sum_{p=1}^{P} |c_p| + \sum_{m=1}^{M} \|f_m\|_1 < K\|f\|_1$ and f_m belongs to $\mathcal{K}_m * \ell^1(G)$ for each m.

PROOF. Since $\mathcal{J}$ has finite codimension in $\ell^1(G)$, $\mathcal{J}_k$ has finite codimension in $\ell^1(H_k)$ for each k. Hence, as constructed in Lemma 2.1, there are subalgebras $\mathcal{K}_m$ of $\ell^1(G)$ such that $\sum_{m=1}^{M} \mathcal{K}_m * \ell^1(G)$ is closed and has finite codimension in $\ell^1(G)$. Each subalgebra $\mathcal{K}_m$ is of the form $\delta_x * \mathcal{J}_k * \delta_{x^{-1}}$ for some k and some x in G and so is a finite codimensional ideal in $\ell^1(xH_k x^{-1})$, which is a commutative group algebra. Hence $\mathcal{K}_m$ has bounded approximate units for each m and so, by Cohen's factorization theorem, $\mathcal{K}_m^2 = \mathcal{K}_m$. Since $\mathcal{J}$ is an ideal, $\mathcal{K}_m$ is contained in $\mathcal{J}$ and $\sum_{m=1}^{M} \mathcal{K}_m * \ell^1(G)$ is contained in $\mathcal{J}^2$. It follows that $\mathcal{J}^2$ is closed and has finite codimension in $\mathcal{J}$.

Equip the quotient space $\ell^1(G)/\mathcal{J}^2$ with the quotient norm. It is easily seen that $\mathcal{J}^2$ is a two-sided ideal in $\ell^1(G)$ and so $\ell^1(G)/\mathcal{J}^2$ may also be equipped with the quotient product. Hence the representation of G on $\ell^1(G)$ by translation, the regular representation, induces a representation of G as a group of isometries on $\ell^1(G)/\mathcal{J}^2$. Now any bounded, finite dimensional group representation is equivalent to a unitary representation and so

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the representation of $\ell^1(G)$ on $\ell^1(G)/\mathcal{J}^2$ is equivalent to a self-adjoint representation. Since $\ell^1(G)$ has a unit and $\mathcal{J}^2$ is a two-sided ideal, the kernel of this representation of $\ell^1(G)$ is $\mathcal{J}^2$ and so $\ell^1(G)/\mathcal{J}^2$ is isomorphic to a finite dimensional C^*-algebra, whence $\ell^1(G)/\mathcal{J}^2$ is semisimple. Clearly, $\mathcal{J}/\mathcal{J}^2$ is contained in the radical of $\ell^1(G)/\mathcal{J}^2$. Therefore $\mathcal{J} = \mathcal{J}^2$.

We have seen that $\sum_{m=1}^M \mathcal{K}_m * \ell^1(G)$ has finite codimension in $\mathcal{J}$ and so there are P elements a_p in $\mathcal{J}$ such that $\mathcal{J} = \text{span}\{a_p | p = 1, 2, \dots, P\} \oplus \sum_{m=1}^M \mathcal{K}_m * \ell^1(G)$. Since $\mathcal{J} = \mathcal{J}^2$, these P elements may be chosen to be products. The complementary subspaces $\text{span}\{a_p | p = 1, 2, \dots, P\}$ and $\sum_{m=1}^M \mathcal{K}_m * \ell^1(G)$ are closed and so the projection onto $\sum_{m=1}^M \mathcal{K}_m * \ell^1(G)$ along $\text{span}\{a_p | p = 1, 2, \dots, P\}$ is bounded. Furthermore, since $\sum_{m=1}^M \mathcal{K}_m * \ell^1(G)$ is closed, the Open Mapping Theorem tells us that the map

$$\bigoplus_{m=1}^M \mathcal{K}_m * \ell^1(G) \mapsto \sum_{m=1}^M \mathcal{K}_m * \ell^1(G)$$

is open. Therefore there is a constant, K, as asserted.

The hypothesis that the groups, H_k, should be abelian is used in only one way in the above arguments. That is to ensure that finite codimensional ideals in $\ell^1(H_k)$ have bounded approximate units so that Cohen's factorization theorem can be applied. In order to ensure this, it would suffice, in the statement of Lemma 2.1, to suppose only that the subgroups H_k are amenable, see [12]. However the abelian case will suffice for the automatic continuity proofs and it is not obvious that the lemmas would apply in significantly greater generality if 'abelian' were replaced by 'amenable'.

The next lemma will allow this factorization result for discrete groups to be transferred to non-discrete, factorizable groups. Recall that we are identifying $\mathcal{M}(G)$ with the multiplier algebra of $\mathcal{L}^1(G)$ and the group algebra, $\ell^1(G)$, with the algebra of discrete measures in $\mathcal{M}(G)$. With these identifications, the convolutions in what follows are well-defined.

2.3 Lemma. *Let G be a locally compact group and $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{L}^1(G)$ with codimension n. Then*

$$\mathcal{J} \equiv \{f \in \ell^1(G) | f * \mathcal{L}^1(G) \subseteq \mathcal{I}\}$$

is a closed, two-sided ideal with codimension n in $\ell^1(G)$. Furthermore, for every F in $\mathcal{I}$ and every $\epsilon > 0$ there are f in $\mathcal{J}$ and U in $\mathcal{L}^1(G)$ such that $\|F - f * U\|_1 < \epsilon$ and $\|f\|_1 \|U\|_1 \le \|F\|_1$.

Proof. It is easily checked that $\mathcal{J}$ is a closed, two-sided ideal. The group algebra $\mathcal{L}^1(G)$ has bounded approximate units of the form $\{U_\lambda\}_{\lambda \in \Lambda}$, where

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U_λ is a non-negative function on G with norm equal to one and so the quotient algebra, $\mathcal{L}^1(G)/\mathcal{I}$, is a finite-dimensional Banach algebra with approximate units bounded by one. It follows that $\mathcal{L}^1(G)/\mathcal{I}$ has a unit with norm one. If $U+\mathcal{I}$ is the unit in $\mathcal{L}^1(G)/\mathcal{I}$, then $\mathcal{J} = \{f \in \ell^1(G) | f * U \in \mathcal{I}\}$. Hence the codimension of $\mathcal{J}$ is at most n.

Choose functions $F_1, F_2, \dots, F_n$ from $\mathcal{L}^1(G)$ such that $\{F_i + \mathcal{I} | i = 1, 2, \dots, n\}$ is a basis for $\mathcal{L}^1(G)/\mathcal{I}$. By [20, Lemma 2.1], we may choose $\lambda \in \Lambda$ and functions $f_1, f_2, \dots, f_k$ in $\ell^1(G)$ such that $f_i * U_\lambda$ is arbitrarily close to F_i for $i = 1, 2, \dots, n$. It follows that they may be chosen such that $\{f_i * U_\lambda + \mathcal{I} | i = 1, 2, \dots, n\}$ is a basis for $\mathcal{L}^1(G)/\mathcal{I}$. Then the map $f \mapsto f * U_\lambda + \mathcal{I}$ has rank n and contains $\mathcal{J}$ in its kernel. Therefore the codimension of $\mathcal{J}$ is at least n.

The above argument shows that, if $U + \mathcal{I}$ is the unit in $\mathcal{L}^1(G)/\mathcal{I}$, then the map

$$f + \mathcal{J} \mapsto f * U + \mathcal{I}$$

is an isomorphism between $\ell^1(G)/\mathcal{J}$ and $\mathcal{L}^1(G)/\mathcal{I}$ and more careful application of [20, Lemma 2.1] shows that it is in fact an isometry. Hence, if $V + \mathcal{I}$ is an invertible element of $\mathcal{L}^1(G)/\mathcal{I}$, then

$$f + \mathcal{J} \mapsto f * V + \mathcal{I} = (f * U + \mathcal{I})(V + \mathcal{I})$$

is an isomorphism and the norm of its inverse is at most $\|(V + \mathcal{I})^{-1}\|$.

Now let F be in $\mathcal{I}$. Then applying [20, Lemma 2.1] again, we see that there are f' in $\ell^1(G)$ and $\lambda \in \Lambda$ such that $f' * U_\lambda$ is arbitrarily close to F, $\|f'\|_1 \leq \|F\|_1$ and $\|U_\lambda\|_1 \leq 1$. It may also be supposed that $U_\lambda + \mathcal{I}$ is sufficiently close to the unit in $\mathcal{L}^1(G)/\mathcal{I}$ that it is invertible and $\|(U_\lambda + \mathcal{I})^{-1}\|_1 \leq 2$. Then $f' * U_\lambda + \mathcal{I}$ will have arbitrarily small norm because F is in $\mathcal{I}$. It follows that a function, g, may be chosen from $\ell^1(G)$ with arbitrarily small norm such that $g * U_\lambda + \mathcal{I} = f' * U_\lambda + \mathcal{I}$. Set $f = f' - g$. Then $f * U_\lambda$ is arbitrarily close to F and f belongs to $\mathcal{J}$.

2.4 THEOREM. *Let G be a locally compact group which has abelian subgroups $H_1, H_2, \dots, H_n$ such that $G = H_1 H_2 \dots H_n$ and let $\mathcal{I}$ be a closed two-sided ideal with finite codimension in $\mathcal{L}^1(G)$. Let $\mathcal{J} = \{f \in \ell^1(G) | f * \mathcal{L}^1(G) \subseteq \mathcal{I}\}$ and $\mathcal{J}_k = \mathcal{J} \cap \ell^1(H_k)$, $k = 1, 2, \dots, n$. Then there are elements $a_1, a_2, \dots, a_P$ which are products in $\mathcal{J}$ and subalgebras $\mathcal{K}_m$, of the form $\delta_x * \mathcal{J}_k * \delta_{x^{-1}}$ for some x in G and k between 1 and n, such that the map*

$$T : \left(\bigoplus_{p=1}^{P} \mathcal{L}^1(G)\right) \oplus \left(\bigoplus_{m=1}^{M} \mathcal{K}_m * \mathcal{L}^1(G)\right) \to \mathcal{I}$$

defined by

$$T(F_1, \ldots, F_P, D_1, \ldots, D_M) = \sum_{p=1}^{P} a_p * F_p + \sum_{m=1}^{M} D_m$$

is a surjection.

PROOF. Choose the elements a_p and subalgebras $\mathcal{K}_m$ to be as constructed in Lemmas 2.1 and 2.2. Since $\mathcal{I}$ is an ideal, it follows from the definitions of the subalgebras $\mathcal{J}_k$ that the range of T is contained in $\mathcal{I}$. To show that T is surjective, it will suffice to show there is a $K > 0$ such that for every $\epsilon > 0$ and F in $\mathcal{I}$, there is $(F_1, \ldots, F_P, D_1, \ldots, D_M)$ in $(\bigoplus_{p=1}^{P} \mathcal{L}^1(G)) \oplus (\bigoplus_{m=1}^{M} \mathcal{K}_m * \mathcal{L}^1(G))$ with norm less than $K\|F\|_1$ such that

$$\|F - T(F_1, \ldots, F_P, D_1, \ldots, D_M)\|_1 < \epsilon.$$

We shall show that this holds with K equal to the constant found in Lemma 2.2.

Let F be in $\mathcal{I}$ and $\epsilon > 0$ be given. Then by Lemma 2.3 there are f in $\mathcal{J}$ and U in $\mathcal{L}^1(G)$ satisfying $\|f\|_1 \le \|F\|_1$, $\|U\|_1 \le 1$ and such that $\|F - f * U\|_1 < \epsilon$. By Lemma 2.2, $f = \sum_{p=1}^{P} c_p a_p + \sum_{m=1}^{M} f_m$ where $\sum_{p=1}^{P} |c_p| + \sum_{m=1}^{M} \|f_m\|_1 < K\|f\|$. Set

$$F_p = c_p U, \ p = 1, 2, \ldots, P \ \text{ and } \ D_m = f_m * U, \ m = 1, 2, \ldots, M.$$

Then $T(F_1, \ldots, F_P, D_1, \ldots, D_M) = f * U$, so that

$$\|F - T(F_1, \ldots, F_P, D_1, \ldots, D_M)\|_1 < \epsilon,$$

and

$$\sum_{p=1}^{P} \|F_p\|_1 + \sum_{m=1}^{M} \|D_m\|_1 \le \sum_{p=1}^{P} |c_p| \, \|U\|_1 + \sum_{m=1}^{M} \|f_m\|_1 \|U\|_1 \le K\|f\|_1 \|U\|_1 \le K\|F\|_1.$$

The factorization results we require may be easily deduced from this theorem.

2.5 THEOREM. *Let G be as in the theorem and let $\mathcal{I}$ be a closed two-sided ideal with finite codimension in $\mathcal{L}^1(G)$. Then there is an integer R such that for every sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathcal{I}$ which converges to zero in norm there are:*

(a) *elements $f^{(1)}, f^{(2)}, \ldots, f^{(R)}$ in $\mathcal{J}$ and sequences $\{F_n^{(1)}\}_{n=1}^{\infty}, \ldots, \{F_n^{(R)}\}_{n=1}^{\infty}$ in $\mathcal{I}$ such that $F_n = \sum_{r=1}^{R} f^{(r)} * F_n^{(r)}$, $n = 1, 2, \ldots$ and $\|F_n^{(r)}\|_1 \to 0$ as $n \to \infty$ for $r = 1, 2, \ldots, R$;*

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(b) *elements* $F^{(1)}, F^{(2)}, \ldots, F^{(R)}$ *in* $\mathcal{I}$ *and sequences* $\{F_n^{(1)}\}_{n=1}^{\infty}, \ldots, \{F_n^{(R)}\}_{n=1}^{\infty}$ *in* $\mathcal{I}$ *such that* $F_n = \sum_{r=1}^{R} F^{(r)} * F_n^{(r)}$, $n = 1, 2, \ldots$ *and* $\|F_n^{(r)}\|_1 \to 0$ *as* $n \to \infty$ *for* $r = 1, 2, \ldots, R$.

PROOF. (a). This follows immediately from the theorem because T is an open map, each of the functions a_p is a product in $\mathcal{J}$ and each of the subalgebras $\mathcal{K}_m$ has a bounded approximate identity for $\mathcal{K}_m * \mathcal{L}^1(G)$.

(b). For this, note first that if f is in $\mathcal{J}$ and F is in $\mathcal{L}^1(G)$, then $F * f$ will be in $\mathcal{I}$. Taking a bounded approximate identity $\{U_\lambda\}_{\lambda \in \Lambda}$ for $\mathcal{L}^1(G)$, we have $F * f = \lim_{\lambda \to \infty} F * f * U_\lambda$ where $f * U_\lambda$ is in $\mathcal{I}$ by definition of $\mathcal{J}$. Now $\mathcal{I}$ is a closed two-sided ideal and so the result follows.

Since $\mathcal{L}^1(G)$ has a left bounded approximate identity, Cohen's factorization theorem implies that the sequence $\{F_n\}_{n=1}^{\infty}$ may be factored as $F_n = U * F_n'$, for $n = 1, 2, \ldots$ where U belongs to $\mathcal{L}^1(G)$, F_n' belongs to $\mathcal{I}$ for each n and $\|F_n'\|_1 \to 0$ as $n \to \infty$. Now by part (a) we have $F_n' = \sum_{r=1}^{R} f^{(r)} * F_n^{(r)}$, $n = 1, 2, \ldots$ where $f^{(r)}$ belongs to $\mathcal{J}$ for each r. Let $F^{(r)} = U * f^{(r)}$. Then, by the above observation, $F^{(r)}$ is in $\mathcal{I}$ and $F_n = \sum_{r=1}^{R} F^{(r)} * F_n^{(r)}$, $n = 1, 2, \ldots$.

These factorization results could be obtained simply by a direct application of Cohen's factorization theorem if it were known that finite codimensional ideals in the group algebras had bounded approximate units. However many factorizable groups, for example $SL(2, \mathbb{R})$ and $SU(2, \mathbb{C})$ as a discrete group, are not amenable and so finite codimensional ideals in their group algebras do not have bounded approximate units, see [19, Theorem 5.2].

The first part of the next theorem will be needed in the proof of continuity of derivations from $\mathcal{L}^1(G)$ when G is connected. The second part will be needed for the proof of continuity of left $\ell^1(G)$-module homomorphisms from $\mathcal{L}^1(G)$ in [21].

2.6 THEOREM. (a). *Let* G *be a connected locally compact group and* $\mathcal{I}$ *be a closed, two-sided ideal with finite codimension in* $\mathcal{L}^1(G)$. *Then there is an integer* R *such that for every sequence* $\{F_n\}_{n=1}^{\infty}$ *in* $\mathcal{I}$ *which converges to zero in norm there are in* $\mathcal{I}$ *elements* F_r *and sequences* $\{F_n^{(r)}\}_{n=1}^{\infty}$ *for* $r = 1, 2, \ldots, R+1$ *such that* $\{F_n^{(r)}\}_{n=1}^{\infty}$ *converges to zero in norm for each* r *and* $F_n = \sum_{r=1}^{R+1} F_r * F_n^{(r)}$ *for each* n.

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(b). *If it is further supposed that* G *is separable, then for each* n, $F_n = \sum_{r=1}^{R+1} f_r * F_n^{(r)}$, *where* f_r *belongs to* $\mathcal{J}$ *and* $\{F_n^{(r)}\}_{n=1}^{\infty}$ *are sequences converging to zero in* $\mathcal{I}$ *for each* r.

PROOF. (a). The proof will be by an 'approximation by Lie groups' argument. The regular representation of G on $\mathcal{L}^1(G)$ induces a representation, $\rho_{\mathcal{I}}$, of G on $\mathcal{L}^1(G)/\mathcal{I}$. Since the regular representation is strongly continuous and $\mathcal{L}^1(G)/\mathcal{I}$ is finite dimensional, $\rho_{\mathcal{I}}$ is norm continuous, that is, the map $x \mapsto \rho_{\mathcal{I}}(x)$ is continuous with respect to the given topology on G and the norm topology for operators on $\mathcal{L}^1(G)/\mathcal{I}$. Choose a neighbourhood, U, of the identity element in G such that $\|I - \rho_{\mathcal{I}}(x)\| < 1$ for every x in U. Then, by the theorem in [13, Section 4.6], there is a compact, normal subgroup N of G contained in U and such that G/N is a connected Lie group. Denote by m_N the normalized Haar measure on N, and regard it as lying in $\mathcal{M}(G)$. Then convolution by m_N determines a projection on $\mathcal{L}^1(G)$ which induces an idempotent operator, $\rho_{\mathcal{I}}(m_N)$ on $\mathcal{L}^1(G)/\mathcal{I}$. Since the support of m_N is contained in U and m_N is a probability measure, $\|I - \rho_{\mathcal{I}}(m_N)\| < 1$. We have then that $I - \rho_{\mathcal{I}}(m_N)$ is an idempotent operator and has norm less than one. It follows that $\rho_{\mathcal{I}}(m_N) = I$.

As shown in [15, 3.5.3 and 3.6.4] the quotient map $G \to G/N$ induces an algebra homomorphism $T_N : \mathcal{L}^1(G) \to \mathcal{L}^1(G/N)$ whose kernel is the closed two-sided ideal $\mathcal{I}_N \equiv \{F - m_N * F | F \in \mathcal{L}^1(G)\}$. That $\mathcal{I}_N$ is a closed two-sided ideal follows from the fact that m_N is a central idempotent in $\mathcal{M}(G)$. It is clear from this characterization that the kernel of T_N is contained in $\mathcal{I}$. Also, since N is compact, it is amenable and so, as shown in [15], $\mathcal{I}_N$ has bounded approximate units.

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{I}$ which converges to zero in norm. Then $T_N(F_n)$ is a sequence in $T_N(\mathcal{I})$ which converges to zero. Since the kernel of T_N is contained in $\mathcal{I}$, $T_N(\mathcal{I})$ is a closed ideal with finite codimension in $\mathcal{L}^1(G/N)$. Now G/N is a connected Lie group and so is factorizable, by Theorem 1.2. Therefore we may apply Theorem 2.5(a) to show that there is an integer R such that this sequence may be factored as

$$T_N(F_n) = \sum_{r=1}^{R} T_N(F_r) * T_N(F_n^{(r)}),$$

where $T_N(F_r)$ and $T_N(F_n^{(r)})$ belong to $T_N(\mathcal{I})$ for each r and n and where $\{T_N(F_n^{(r)})\}_{n=1}^{\infty}$ converges to zero in norm.

The restriction of T_N to $\mathcal{I}$ is an open map onto $T_N(\mathcal{I})$ and so the sequences $\{F_n^{(r)}\}_{n=1}^{\infty}$ may be chosen from $\mathcal{I}$ such that $\|F_n^{(r)}\|_1 \to 0$ as $n \to \infty$.

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Since T_N is an algebra homomorphism $\{F_n - \sum_{r=1}^{R} F_r * F_n^{(r)}\}_{n=1}^{\infty}$ is contained in $\mathcal{I}_N$ and it converges to zero because $\{F_n\}_{n=1}^{\infty}$ and $\{F_n^{(r)}\}_{n=1}^{\infty}$ converge to zero in norm. As remarked above, $\mathcal{I}_N$ has bounded approximate units and so, by Cohen's factorization theorem, there are in $\mathcal{I}_N$ an element F_{R+1} and a sequence $\{F_n^{(R+1)}\}_{n=1}^{\infty}$ which converges to zero such that

$$F_{R+1} * F_n^{(R+1)} = F_n - \sum_{r=1}^{R} F_r * F_n^{(r)}$$

for each n. Rearranging this equation gives the required result.

(b). For this, we shall need some more information about $\mathcal{I}_N$. Since G is separable and N is amenable, the argument used in [22, Proposition 1.3] shows that there is a discrete probability measure, μ, on N such that $\mathcal{I}_N = [(\delta_e - \mu) * \mathcal{L}^1(G)]^-$. Let $\mathcal{A}$ be the closed subalgebra of $\ell^1(N)$ generated by δ_e and μ and define $\mathcal{A}_0 = \{f \in \mathcal{A} | \sum_{x \in N} f(x) = 0\}$. Then $\mathcal{A}_0$ is an ideal in $\mathcal{A}$ which contains $\delta_e - \mu$ and has bounded approximate units $u_J = \delta_e - (1/J)\sum_{j=1}^{J} \mu^j$, $J = 1, 2, 3, \ldots$. By the choice of μ, the u_J's are also bounded approximate units for $\mathcal{I}_N$ and so Cohen's factorization theorem implies that, if $\{D_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{I}_N$ which converges to zero, then there are a sequence $\{D'_n\}_{n=1}^{\infty}$ in $\mathcal{I}_N$ and an element d in $\mathcal{A}_0$ such that $D_n = d * D'_n$ for each n. It is easily checked that $\mathcal{A}_0 \subseteq \mathcal{J}$.

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{I}$ which converges to zero in norm. Then, by Theorem 2.5(a), there is an integer R such that this sequence may be factored as

$$T_N(F_n) = \sum_{r=1}^{R} f_r * T_N(F_n^{(r)}),$$

where: N is the compact subgroup of G found in part (a); $T_N(F_n^{(r)})$ belongs to $T_N(\mathcal{I})$ for each r and n and $\{T_N(F_n^{(r)})\}_{n=1}^{\infty}$ converges to zero in norm; f_r belongs to the closed ideal $\{f \in \ell^1(G/N) | f * \mathcal{L}^1(G/N) \subseteq T_N(\mathcal{I})\}$ for each r.

As in part (a), the functions f_r and $T_N(F_n^{(r)})$ may be pulled back to functions g_r and $F_n^{(r)}$ in $\mathcal{J}$ and $\mathcal{I}$ respectively so that $\{F_n - \sum_{r=1}^{R} g_r * F_n^{(r)}\}_{n=1}^{\infty}$ is contained in $\mathcal{I}_N$ and converges to zero. Then there are a function f_{R+1} in $\mathcal{A}_0$ and a sequence $\{F_n^{(R+1)}\}_{n=1}^{\infty}$ in $\mathcal{I}_N$ which converges to zero in norm such that $F_n - \sum_{r=1}^{R} g_r * F_n^{(r)} = f_{R+1} * F_n^{(R+1)}$ for each n. Since $\mathcal{A}_0$ is contained in $\mathcal{J}$, this equation may be rearranged to give the required factorization.

The final lemma in this section will be used in the automatic continuity arguments in the next section and in [21]to show that certain ideals are finite codimensional and thus allow the application of Theorems 2.5 and 2.6.

2.7 Lemma. *Let G be a locally compact group and let $\mathcal{J}$ be a closed two-sided ideal with codimension n in $\ell^1(G)$. Then $\mathcal{I} \equiv [\text{span}\{f * F | f \in \mathcal{J}, F \in \mathcal{L}^1(G)\}]^-$ is a closed two-sided ideal with codimension at most n in $\mathcal{L}^1(G)$.*

Proof. Suppose that F_i, $i = 1, 2, \dots, n, n+1$ are elements of $\mathcal{L}^1(G)$ which are linearly independent modulo $\mathcal{I}$. Since these functions may be simultaneously approximated arbitrarily closely by functions of the form $f_i * U$ where U is in $\mathcal{L}^1(G)$ and f_i is in $\ell^1(G)$ for $i = 1, 2, \dots, n, n+1$, it follows that there are $n+1$ functions of this form which are linearly independent modulo $\mathcal{I}$. However, if $f_i * U$, $i = 1, 2, \dots, n, n+1$ are linearly independent modulo $\mathcal{I}$, then, by the definition of $\mathcal{I}$, f_i, $i = 1, 2, \dots, n, n+1$ are linearly independent modulo $\mathcal{J}$ which is a contradiction to the hypothesis that the codimension of $\mathcal{J}$ is n. Therefore the codimension of $\mathcal{I}$ is at most n.

Lemma 2.7 looks as though it might be a special case of Lemma 2.3 and indeed, the proof of Lemma 2.7 is essentially just the second paragraph of the proof of Lemma 2.3. However, there are many more finite codimensional ideals in $l^1(G)$ than those occurring in Lemma 2.3 and so Lemma 2.7 applies more generally than Lemma 2.3. In the generality in which Lemma 2.7 applies, the codimension of $\mathcal{I}$ may be strictly less than that of $\mathcal{J}$. For example, if $\mathcal{J}$ is an ideal with codimension one in $\ell^1(\mathbb{R})$ corresponding to a discontinuous character on $\mathbb{R}$, then $\mathcal{I} = \mathcal{L}^1(\mathbb{R})$, i.e. has codimension zero.

3. The continuity of derivations

There is a standard automatic continuity technique which we shall be using known as the

Stability Lemma. *Let $S : \mathcal{X} \to \mathcal{Y}$ be a linear map with separating space $\mathfrak{S}(S)$. Suppose that there are sequences of operators $(R_n)_{n=1}^{\infty}$ and $(T_n)_{n=1}^{\infty}$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively such that $T_nS - SR_n : \mathcal{X} \to \mathcal{Y}$ is a continuous map for each n. Then there is an integer, N, such that $[T_1T_2 \dots T_N\mathfrak{S}(S)]^- = [T_1T_2 \dots T_n\mathfrak{S}(S)]^-$ for every $n \geq N$.*

A proof of this lemma may be found in [18]. The separating space of a linear map $S : \mathcal{X} \to \mathcal{Y}$ is defined by

$$\mathfrak{S}(S) \equiv \{y \in \mathcal{Y} | \exists (x_n)_{n=1}^{\infty} \subseteq \mathcal{X} \text{ such that } \lim_{n\to\infty} \|x_n\| = 0 \text{ and } \lim_{n\to\infty} Sx_n = y\}.$$

The Closed Graph Theorem implies that S is continuous if and only if $\mathfrak{S}(S) = (0)$.

In order to apply the Stability Lemma we require the following fact about group algebras. If G is an abelian or compact group and $\mathcal{I}$ is a closed, two-sided ideal with infinite codimension in $\mathcal{L}^1(G)$, then there are sequences, $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{L}^1(G)$, such that $B_n A_1 A_2 \dots A_n \notin \mathcal{I}$ but $B_n A_1 A_2 \dots A_n A_{n+1} \in \mathcal{I}$, for $n = 1, 2, \dots$.

That the ideals in these group algebras satisfy this condition follows from well-known properties of $\mathcal{L}^1(G)$. In the abelian case, if $\mathcal{I}$ has infinite codimension in $\mathcal{L}^1(G)$, then Wiener's Tauberian theorem, [17, Theorem 7.2.4] implies that

$$h(\mathcal{I}) \equiv \{\gamma \in \hat{G} | \hat{F}(\gamma) = 0, \forall F \in \mathcal{I}\}$$

is an infinite subset of $\hat{G}$, the carrier space of $\mathcal{L}^1(G)$. Since $h(\mathcal{I})$ contains an infinite number of points, there is a sequence $\{W_n\}_{n=1}^{\infty}$ of disjoint, open subsets of $\hat{G}$ each of which contains a point of $h(\mathcal{I})$. Then [17, Theorem 2.6.2] implies that there is a sequence $\{F_n\}_{n=1}^{\infty}$ of functions in $\mathcal{L}^1(G)$, none of which belongs to $\mathcal{I}$, such that $F_n * F_m = 0$ if $m \neq n$. If G is compact, such a sequence of functions may be chosen from the characters on G, [6, Definition 3.4]. To see this note that, if $\mathcal{I}$ has infinite codimension in $\mathcal{L}^1(G)$, then there is an infinite sequence of distinct characters not in $\mathcal{I}$ and, by [6, Theorem 3.6(2)], the convolution product of distinct characters is zero. Once such a sequence of functions, $\{F_n\}_{n=1}^{\infty}$, has been found put $A_n = \sum_{k=n}^{\infty} F_k / (\|F_k\|_1 2^k)$ and $B_n = F_n$. Then $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ will have the required property.

This property is used in conjunction with the Stability Lemma in the proof of continuity of derivations from $\mathcal{L}^1(G)$ if G is abelian or compact, see [9, Theorem 2]. Some earlier results where similar ideas were used are [10, Lemma 1.2], [16, Theorem 2] and [1, corollary 2.6]. The other property of group algebras used in this proof is that, if G is abelian or compact, then finite codimensional ideals in $\mathcal{L}^1(G)$ have bounded approximate units. It may happen that the group algebra of a factorizable or connected group has neither of these properties. However, Lemma 2.7 and Theorems 2.5 and 2.6 will allow essentially the same argument as used in the papers mentioned above to apply to these cases too.

3.1 Theorem. *Let G be a factorizable group. Then every derivation, $D : \mathcal{L}^1(G) \to \mathcal{X}$, where $\mathcal{X}$ is an $\mathcal{L}^1(G)$-bimodule, is continuous.*

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PROOF. To show that derivations into arbitrary bimodules are continuous it will suffice to show that derivations into a particular bimodule are continuous. This bimodule is defined as follows. The Banach space $\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G)$, the projective tensor product of $\mathcal{L}^1(G)$ with itself, may be defined to be an $\mathcal{L}^1(G)$-bimodule by putting $F \cdot (A \otimes B) = (F * A) \otimes B$ and $(A \otimes B) \cdot F = A \otimes (B * F)$, $(F, A, B \in \mathcal{L}^1(G))$, and then extending this action of F to the rest of $\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G)$ by linearity and continuity. The dual space of this bimodule, $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$, becomes a bimodule over $\mathcal{L}^1(G)$ under the dual actions. By [20, Lemma 3.1], we need only show that derivations from $\mathcal{L}^1(G)$ into $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$ are continuous.

Let $D : \mathcal{L}^1(G) \to (\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$ be a derivation with separating space $\mathfrak{S}(D)$. Then, by [20, Lemma 3.1], $\mathfrak{S}(D)$ is a closed submodule of $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$. Put

$$\mathcal{I} \equiv \{F \in \mathcal{L}^1(G) | F \cdot \mathfrak{S}(D) = (0)\}.$$

Then $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{L}^1(G)$ which is called the continuity ideal. The continuity ideal has the property that for every F in $\mathcal{I}$ the map $V \mapsto F \cdot D(V) : \mathcal{L}^1(G) \to (\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$ is continuous. To see this, note that the map is the composite of D and the left action of F on $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$. Since, by definition of $\mathcal{I}$, the action of F annihilates $\mathfrak{S}(D)$, the composite map has zero separating space and so is continuous by [18, Lemma 1.3].

Since the measures in $\mathcal{M}(G)$ act as multipliers on $\mathcal{L}^1(G)$, the action of $\mathcal{L}^1(G)$ on $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$ extends to an action of $\mathcal{M}(G)$ which makes $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$ an $\mathcal{M}(G)$-bimodule. By [20, Lemma 3.4], D extends in a unique way to a derivation

$$\bar{D} : \mathcal{M}(G) \to (\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$$

and it may be shown in the same way as in [20, Lemma 3.1] that $\mathfrak{S}(D)$ is a closed $\mathcal{M}(G)$-submodule of $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$.

Let $H_1, H_2, \ldots, H_n$ be abelian subgroups of G such that $G = H_1 H_2 \ldots H_n$. Then, regarding $\ell^1(G)$ and, for each k, $\ell^1(H_k)$ as a subalgebra of $\mathcal{M}(G)$, we may define

$$\mathcal{J} \equiv \{f \in \ell^1(G) | f \cdot \mathfrak{S}(D) = (0)\}$$

and, for each k,

$$\mathcal{J}_k \equiv \{f \in \ell^1(H_k) | f \cdot \mathfrak{S}(D) = (0)\}.$$

Since $\mathfrak{S}(D)$ is an $\mathcal{M}(G)$-bimodule, $\mathcal{J}_k$ is a closed two-sided ideal in $\ell^1(H_k)$ for each k and $\mathcal{J}$ is a closed two-sided ideal in $\ell^1(G)$.

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Suppose that, for some k, $\mathcal{J}_k$ had infinite codimension in $\ell^1(H_k)$. Then, since H_k is abelian, there would be sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $b_n a_1 a_2 \dots a_n \notin \mathcal{J}_k$ but $b_n a_1 a_2 \dots a_n a_{n+1} \in \mathcal{J}_k$, for $n = 1, 2, \dots$. We could then define, for each integer n, operators R_n, T_n on $\mathcal{L}^1(G)$ and $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$ respectively by

$$R_n F = a_n * F, \ (F \in \mathcal{L}^1(G)) \ \text{and} \ T_n \phi = a_n \cdot \phi, \ (\phi \in (\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*).$$

The properties of a_n and b_n then imply that

$$b_n \cdot T_1 T_2 \dots T_n \mathfrak{S}(D) \neq (0) \ \text{but} \ b_n \cdot T_1 T_2 \dots T_n T_{n+1} \mathfrak{S}(D) = (0).$$

Consequently, we would have that

$$[T_1 T_2 \dots T_n T_{n+1} \mathfrak{S}(D)]^- \subsetneqq [T_1 T_2 \dots T_n \mathfrak{S}(D)]^-$$

for every n. However, $T_n D - D R_n$ is continuous for each n because

$$(T_n D - D R_n)(F) = \bar{D}(a_n) \cdot F$$

and so this would contradict the Stability Lemma. Therefore, $\mathcal{J}_k$ has finite codimension in $\ell^1(H_k)$ for each k.

By definition, $\mathcal{J}_k = \mathcal{J} \cap \ell^1(G)$ for each k. Hence, by Lemma 2.1, $\mathcal{J}$ has finite codimension in $\ell^1(G)$. It is also immediate from the definitions that

$$[\text{span}\{f * F \mid f \in \mathcal{J}, F \in \mathcal{L}^1(G)\}]^- \subseteq \mathcal{I}$$

whence, by Lemma 2.7, $\mathcal{I}$ has finite codimension in $\mathcal{L}^1(G)$.

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{I}$ which converges to zero in norm. Then, by Theorem 2.5, there are: an integer R; elements F_r, $r = 1, 2, \dots, R$ in $\mathcal{I}$; and sequences $\{F_n^{(r)}\}_{n=1}^{\infty}$, $r = 1, 2, \dots, r$ such that $F_n = \sum_{r=1}^{R} F_r * F_n^{(r)}$ for each n and $\|F_n^{(r)}\|_1 \to 0$ as $n \to \infty$ for each r. By the derivation property for D, it follows that, for each n,

$$D(F_n) = \sum_{r=1}^{R} F_r \cdot D(F_n^{(r)}) + D(F_r) \cdot F_n^{(r)}.$$

Now $\|D(F_r) \cdot F_n^{(r)}\|_1 \to 0$ as $n \to \infty$ and, since F_r is in the continuity ideal for each r, $\|F_r \cdot D(F_n^{(r)})\|_1 \to 0$ as $n \to \infty$. It follows that the restriction of D to $\mathcal{I}$ is continuous. Since $\mathcal{I}$ has finite codimension in $\mathcal{L}^1(G)$, D is continuous.

3.2 THEOREM. *Let G be a connected locally compact group. Then every derivation, $D : \mathcal{L}^1(G) \to \mathcal{X}$, where $\mathcal{X}$ is an arbitrary $\mathcal{L}^1(G)$-bimodule, is continuous.*

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PROOF. As in the previous theorem, it will suffice to show that every derivation,

$$D:\mathcal{L}^1(G)\to(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$$

is continuous. Let D be such a derivation with separating space $\mathfrak{S}(D)$ and continuity ideal $\mathcal{I}$. Denote by $\bar{D}$ the unique derivation from $\mathcal{M}(G)$ which extends D. We will begin by showing that there is a compact, normal subgroup N of G such that G/N is a Lie group and $(\delta_e - m_N)*\mathcal{L}^1(G)\subseteq\mathcal{I}$, where m_N denotes the normalized Haar measure on N.

By [13, Theorem 4.6], there is a compact, normal subgroup, K, of G such that G/K is a Lie group. Let m_K be Haar measure on K, regarded as a measure on G and identify $\mathcal{L}^1(K)$ with the subalgebra of $\mathcal{M}(G)$ consisting of measures absolutely continuous with respect to m_K. Since $\mathcal{M}(G)$ acts as an algebra of multilpliers on $\mathcal{L}^1(G)$, this identification induces a left module action of $\mathcal{L}^1(K)$ on $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$. Put $\mathcal{I}_K\equiv\{F\in\mathcal{L}^1(K)|F\cdot\mathfrak{S}(D)=(0)\}$. Then $\mathcal{I}_K$ is a closed two-sided ideal in $\mathcal{L}^1(K)$.

Suppose that $\mathcal{I}_K$ had infinite codimension in $\mathcal{L}^1(K)$. Then there would be in $\mathcal{L}^1(K)$ sequences, $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$, such that

$$B_nA_1A_2\cdots A_n\notin\mathcal{I}_K \text{ but } B_nA_1A_2\cdots A_nA_{n+1}\in\mathcal{I}_K, \text{ for } n=1,2,\dots.$$

We could then define, for each integer n, operators R_n, T_n on $\mathcal{L}^1(G)$ and $(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*$ respectively by

$$R_nF=A_n*F,\ (F\in\mathcal{L}^1(G)) \text{ and } T_n\phi=A_n\cdot\phi,\ (\phi\in(\mathcal{L}^1(G)\hat{\otimes}\mathcal{L}^1(G))^*).$$

The properties of A_n and B_n then imply that, for each n,

$$B_n\cdot T_1T_2\dots T_n\mathfrak{S}(D)\neq(0) \text{ but } B_n\cdot T_1T_2\dots T_nT_{n+1}\mathfrak{S}(D)=(0).$$

Consequently, we would have that

$$[T_1T_2\cdots T_nT_{n+1}\mathfrak{S}(D)]^-\subsetneqq[T_1T_2\dots T_n\mathfrak{S}(D)]^- \text{ for every } n.$$

However, T_nD-DR_n is continuous for each n because $(T_nD-DR_n)(F)=\bar{D}(A_n)\cdot F$ and so this would contradict the stability lemma. Therefore, $\mathcal{I}_K$ has finite codimension in $\mathcal{L}^1(K)$.

The regular representation of K on $\mathcal{L}^1(K)$ induces a representation, ρ_K, of K on the finite dimensional space $\mathcal{L}^1(K)/\mathcal{I}_K$. Since $\mathcal{L}^1(K)/\mathcal{I}_K$ is finite dimensional, ρ_K is continuous with respect to the given topology on K and the norm topology for operators on $\mathcal{L}^1(K)/\mathcal{I}_K$. Choose an open neighbourhood, U, of the identity element in G such that $\|I-\rho_K(x)\|<1$ for every x in $U\cap K$. Then, by the [13, Theorem 4.6], there is a compact, normal subgroup, N, of G which is contained in U such that G/N is a Lie group.

[13, Lemma 4.7.1] shows that, replacing N by $N \cap K$ if necessary, we may suppose that $N \subseteq K$.

Convolution by m_N defines an idempotent operator on $\mathcal{L}^1(K)$ which in turn induces an idempotent operator $\rho_K(m_N)$ on $\mathcal{L}^1(K)/\mathcal{I}_K$. Since m_N is supported in N and is a probability measure, $\|I-\rho_K(m_N)\| < 1$ and so, since it is an idempotent operator, $I-\rho_K(m_N) = 0$. Therefore $(\delta_e - m_N)*\mathcal{L}^1(K) \subseteq \mathcal{I}_K$.

Now let F be in $\mathcal{L}^1(G)$ and $\{U_\lambda\}_{\lambda\in\Lambda}$ be a bounded approximate identity for $\mathcal{L}^1(K)$. Then it is easily checked that $\lim_\lambda \|U_\lambda * F - F\|_1 = 0$ and so, if ϕ belongs to $\mathfrak{S}(D)$, then

$$((\delta_e - m_N) * F) \cdot \phi = \lim_\lambda((\delta_e - m_N) * U_\lambda * F) \cdot \phi = \lim_\lambda((\delta_e - m_N) * U_\lambda) \cdot (F \cdot \phi) = 0,$$

because $(\delta_e - m_N) * U_\lambda \in \mathcal{I}_K$. Therefore $(\delta_e - m_N) \cdot \mathcal{L}^1(G) \subseteq \mathcal{I}$.

Since N is a normal subgroup in G, m_N is central in $\mathcal{M}(G)$. It follows that $(\delta_e - m_N) \cdot \mathcal{L}^1(G)$ is a closed two-sided ideal in $\mathcal{L}^1(G)$. Denote this ideal by $\mathcal{I}_N$. It is shown in [15]that $\mathcal{I}_N$ is the kernel of the algebra homomorphism $T_N : \mathcal{L}^1(G) \to \mathcal{L}^1(G/N)$ induced by the quotient map $G \to G/N$ and so we have shown that this kernel is contained in $\mathcal{I}$. Hence the module action of $\mathcal{L}^1(G)$ on $\mathfrak{S}(D)$ induces an action of $\mathcal{L}^1(G/N)$.

Define $\mathcal{I}_{G/N} = \{F \in \mathcal{L}^1(G/N) | F \cdot \mathfrak{S}(D) = (0)\}$. Then, since G/N is a Lie group and hence factorizable, the same argument as used in the previous theorem shows that $\mathcal{I}_{G/N}$ has finite codimension in $\mathcal{L}^1(G/N)$. Therefore $\mathcal{I} = T_N^{-1}(\mathcal{I}_{G/N})$ has finite codimension in $\mathcal{L}^1(G)$. The continuity of D may now be shown in the same way as in the previous theorem, the only change being that we must now use Corollary 2.6 in the place of Corollary 2.5.

Remark. In these proofs we have shown that the continuity ideal cannot have infinite codimension by working in $\mathcal{M}(G)$ in its guise as the multiplier algebra of $\mathcal{L}^1(G)$. This, and the structure of factorizable and connected groups, has enabled us to use well-known results about the structure of $\mathcal{L}^1(G)$ when G is compact or abelian. We have not needed to know anything about the structure of $\mathcal{L}^1(G)$ when G is connected or factorizable. If it could be shown that $\mathcal{L}^1(G)$ has some of the same structure when G is connected or factorizable as it does when G is compact or abelian, then a more direct proof of the continuity of derivations could be given which would almost be a direct application of [9, Theorem 2]. The structure required is that, for

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each ideal, $\mathcal{I}$, with infinite codimension in $\mathcal{L}^1(G)$ there should be a pair of sequences, $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{L}^1(G)$, such that $B_nA_1A_2\dots A_n \notin \mathcal{I}$ but $B_nA_1A_2\dots A_nA_{n+1} \in \mathcal{I}$, for $n = 1, 2, \dots$. This may be shown if G is discrete and factorizable. In this case, if $\mathcal{I}$ is an ideal with infinite codimension in $\mathcal{L}^1(G)$, then Theorem 2.4 implies that there is an abelian subgroup, H, such that $\ell^1(H) \cap \mathcal{I}$ has infinite codimension in $\ell^1(H)$ and so the pair of sequences may be found in $\ell^1(H) \cap \mathcal{I}$, which is a subalgebra of $\mathcal{L}^1(G)$. If G is connected or factorizable but nondiscrete, then the arguments given above will show that such sequences exist in $\mathcal{M}(G)$ but they will not be in $\mathcal{L}^1(G)$.

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