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SIX PAIRWISE ORTHOGONAL LATIN SQUARES OF ORDER 69

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Abstract

In this note it is proved that a $BIB(v, k, 1)$ implies $N(v-4) \geqslant \min(N(k-2), N(k-1)-1, N(k)-1)$ and that $N(69) \geqslant 6$.

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A *latin square* of order v is a $v \times v$ matrix whose every row and every column is a permutation of a v-set Σ. Two latin squares of order v are said to be *orthogonal* if when we superpose the squares every symbol in first square meets every symbol in second square exactly once. Denote by $N(v)$ the maximum number of pairwise orthogonal latin squares of order v. A *pairwise balanced design of index unity*, denoted by $PB(v; K; 1)$, is a design comprising a set of v elements arranged in some subsets (called blocks) with block size in K such that any pair of the v distinct elements occur together in exactly one block. A *balanced incomplete block design of index unity*, denoted by $BIB(v, k, 1)$, is a $PB(v; K; 1)$ with $K = \{k\}$.

Using an idea of brushes due to W. D. Wallis [3], we prove in this note that a $BIB(v, k, 1)$ implies $N(v-4) \geqslant \min(N(k-2), N(k-1)-1, N(k)-1)$ and that $N(69) \geqslant 6$.

Let a brush with centre x be a set of some blocks which all contain the common element x but are otherwise disjoint. Two brushes are disjoint if and only if their sets of elements are disjoint. Denote by $P_n(k)$ the set of all integers v such that there are n pairwise orthogonal latin squares of order v with orthogonal subsquares of order k. In this way we can write $v \in P_n(1)$ to indicate $N(v) \geqslant n$. Similarly to Theorem 9 in [3] we have

THEOREM 1. *Suppose that there is a* $PB(v; K; 1)$ *in which* B^* *is a (possibly empty) distinguished set of blocks comprising a union of disjoint brushes. Suppose further that the size of any block in* B^* *is in* $P_n(1)$ *and that the size of any block not in* B^* *is in* $P_{n+1}(1)$. *Then* $v \in P_n(k)$ *where* k *is the size of any block which is distinguished or none of whose elements is a non-central element of a brush.*

Using Theorem 1, we get the following theorem which is a generalization of Theorem 13.3.4 in [2].

THEOREM 2. *If there is a* $BIB(v, k, 1)$, *then* $N(v - 4) \geqslant \min(N(k - 2), N(k - 1) - 1, N(k) - 1)$.

PROOF. Delete from $BIB(v, k, 1)$ four elements y_1, y_2, y_3 and y_4, such that any three of them are not in the same block. Let l_{ij} be the block containing y_i and y_j and $l^*_{ij} = l_{ij} \backslash \{y_i, y_j\}$. We now get a $PB(v - 4; k - 2, k - 1, k; 1)$ with six distinguished $(k - 2)$-blocks l^*_{ij}, $1 \leqslant i \neq j \leqslant 4$. It is easy to see that the three groups of the six blocks $\{l^*_{12}, l^*_{34}\}$, $\{l^*_{13}, l^*_{24}\}$ and $\{l^*_{14}, l^*_{23}\}$ are pairwise disjoint. We consider a group as a brush if its two blocks have a common element, otherwise we consider each block in a group as a brush if its two blocks have no common element. Then we get a disjoint set of brushes. Let $n = \min(N(k - 2), N(k - 1) - 1, N(k) - 1)$. From Theorem 1 we have $v - 4 \in P_n(k - 2)$ and then the proof is complete.

Up until now the list of Brouwer [1] indicates that five is the last known lower bound for $N(69)$. The following theorem improves this lower bound to six.

THEOREM 3. *There are six pairwise orthogonal latin squares of order* 69 *with orthogonal subsquares of order* 7.

PROOF. From [2, page 293] there is a $BIB(73, 9, 1)$. Since $\min(N(7), N(8) - 1, N(9) - 1) = 6$ (see [2, Theorem 13.2.2]), we have from Theorem 2 that $69 \in P_6(7)$.

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