Canad. Math. Bull. Vol. **31** (1), 1988

ON AN EXTREMAL PROBLEM INVOLVING HARMONIC FUNCTIONS

BY

E. J. P. GEORG SCHMIDT

ABSTRACT. Given a domain D in R^n and two specified points P_0 and P_1 in D we consider the problem of minimizing $u(P_1)$ over all functions harmonic in D with values between 0 and 1 normalised by the requirement $u(P_0) = 1/2$. We show that when D is suitably regular the problem has a unique solution u_* which necessarily takes on boundary values 0 or 1 almost everywhere on the boundary. In the process we prove that it is possible to separate P_0 and P_1 by a harmonic function whose boundary value is supported in an arbitrary set of positive measure. These results depend on the fact that (under suitable regularity conditions) a harmonic function which vanishes on an open subset of the boundary has a normal derivative which is almost everywhere non-vanishing in that set.

Let D be a bounded domain in R^n and let $\mathscr{H}_\infty(D)$ denote the space of real valued and uniformly bounded harmonic functions in D endowed with the supremum norm. We consider the following extremal problem, mentioned to me by Lee Rubel: given two points P_0 and P_1 in D

$$(P)\quad \begin{cases} \text{minimize } u(P_1) \\ \text{subject to } u \in \mathscr{K} = \{u \in \mathscr{H}_\infty(D): 0 \leqq u \leqq 1 \text{ and } u(P_0) = 1/2\}. \end{cases}$$

The existence of a minimizing function $u_* \in \mathscr{K}$ is an easy consequence of the fact that any bounded sequence of harmonic functions in D has a subsequence which converges uniformly on compact subsets of D. In this paper we prove, when the boundary ∂D of D is suitably regular, that u_* is unique and is in fact the harmonic measure of some set in ∂D (or, equivalently, that the "boundary value" of u_* is "bang-bang" i.e. takes on the value 0 or 1 almost everywhere on ∂D).

We recall that D is said to be a Lipschitz (or alternatively C^∞) domain if along the boundary it is locally the epigraph of a Lipschitz (alternatively C^∞) function. For such domains ∂D possesses a Lebesgue surface measure related to

Received by the editors May 22, 1986, and, in revised form June 12, 1987.

This work was supported by the National Science and Engineering Research Council of Canada under Grant A7271.

Key words: harmonic functions, boundary values, boundary control.

AMS Subject Classification: 31A20, 31B20, 49B22.

normal vectors which exist almost everywhere. Deep results of Hunt and Wheeden [5], and Dahlberg [1] (see also Jerison and Kenig [6]) allow one to deduce the following proposition.

PROPOSITION. *Let D be a bounded Lipschitz domain. Then each u in $\mathcal{H}_\infty(D)$ has a nontangential limit in $L_\infty(\partial D)$ (the space of essentially bounded measurable functions on D). The map $u \to f$ is a bijective isometry between $\mathcal{H}_\infty(D)$ and $L_\infty(\partial D)$. Its inverse is given by an integral operator*

$$u(P) = Hf(P) = \int_{\partial D} K(P, Q)f(Q)dS_Q,$$

where dS_Q denotes an element of surface area on ∂D and $K(P, Q)$ is positive and

$$\int_{\partial D} K(P, Q)dS_Q = 1,$$

for each P in D.

The extremal problem can now be reformulated:

$$(P) \quad \begin{cases} \text{minimize } Hf(P_1) \\ \text{subject to } f \in \mathcal{H}_\partial = \{f \in L_\infty(\partial D): 0 \leqq f \leqq 1 \text{ and } Hf(P_0) = 1/2\}. \end{cases}$$

Let f_* denote the boundary value of u_*, the optimal solution to (P). We prove the following theorem using detailed results on conformal mapping in conjunction with the Riesz uniqueness theorem for the case $D \subset R^2$, and a theorem of Weck [10] (see also Schmidt and Weck [9]) for the general case.

THEOREM 1. *Let D be a Lipschitz domain in R^2 or a C^∞ domain in R^n $(n > 2)$. Then (P) has a unique solution u_* whose boundary value f_* takes on the value 0 or 1 almost everywhere on ∂D.*

Note that

$$u_*(P) = Hf_*(P) = \int_{E_*} K(P, Q)dS_Q$$

where $E_* = \{Q \in \partial D: f_*(Q) = 1\}$ so that the solution u_* is simply the harmonic measure of E_* (for properties of harmonic measures see Helms [4] or Hayman and Kennedy [3]).

We state also a "separation theorem" which is a byproduct of the proof of Theorem 1.

THEOREM 2. *Let D be a Lipschitz domain in R^2 or a C^∞ domain in R^n $(n > 2)$. Let $E \subset \partial D$ be a set of positive surface measure. Then, given P_0 and P_1 in D, one can find f in $L_\infty(\partial D)$ vanishing outside E and such that $Hf(P_0) = 0$ while $Hf(P_1) > 0$.*

PROOF OF THE PROPOSITION. This is at best implicit in the previously cited papers. The main facts we need to quote are

https://doi.org/10.4153/CMB-1988-010-6 Published online by Cambridge University Press

(i) the Proposition is true when D is starlike (in the sense of [5]);

(ii) for $u \subset \mathscr{H}_\infty(D)$ the non-tangential limit $f(Q)$ exists for almost every Q in ∂D;

(iii) the harmonic measure $\omega_D^P(E)$ $(P \in D,\ E \subset \partial D)$ associated with the domain D and the surface measure on ∂D are mutually absolutely continuous; in particular "$d\omega_D^P(Q) = K_D(P, Q)dS_Q$" where $K_D(P, Q)$ is positive and

$$\int_{\partial D} K_D(P, Q)dS_Q = 1.$$

The assertions (ii) and (iii) are explicit in [1], [5] and [6]. That (i) holds is a consequence of Section 2 of [5] taken in conjunction with (iii) (which was first proved in [1]).

Given a general Lipschitz domain D let $K(P, Q) = K_D(P, Q)$. For any u in $\mathscr{H}_\infty(D)$ let f be the associated non-tangential limit and define v in $\mathscr{H}_\infty(D)$ by $v = Hf$. To prove the proposition we show that, very plausibly,

(a) the non-tangential limit of v is indeed f; and

(b) two functions in $\mathscr{H}_\infty(D)$ (in this case u and v) having the same tangential limits are necessarily identical.

To prove (a) let D_1 be any starlike subdomain of D obtained locally at a point of ∂D as the epigraph of a Lipschitz function. Then let "$d\omega_{D_1}^P(Q) = K_1(P, Q)dS_Q$" and $\partial_1 D_1 = \partial D_1 \backslash \partial D$, $\partial_2 D_1 = \partial D_1 \cap \partial D$. The kernels $K(P, Q)$ and $K_1(P, Q)$ are related as follows: when $P \in D_1$

$$K(P, Q) = \begin{cases} \displaystyle\int_{\partial_1 D_1} K_1(P, R)K(R, Q)dS_R + K_1(P, Q), & \text{for } Q \in \partial_2 D_1 \\ \displaystyle\int_{\partial_1 D_1} K_1(P, R)K(R, Q)dS_R, & \text{for } Q \in \partial D \backslash \partial D_1. \end{cases}$$

This is easily seen by using "test functions" ϕ in $C(\partial D)$ and by representing the harmonic functions $H\phi$ (which are continuous on the closure $\bar{D}$ of D), restricted to D_1, in terms of their values on ∂D_1 using the kernel $K_1(P, Q)$. Now it follows that for P in D_1

$$v(P) = \int_{\partial D} K(P, Q)f(Q)dS_Q = \int_{\partial_1 D_1} K_1(P, Q)v(Q)dS_Q + \int_{\partial_2 D_1} K_1(P, Q)f(Q)dS_Q.$$

From assertion (i) it follows that v has non-tangential limit f almost everywhere in $\partial_2 D_1$; since D_1 can be located anywhere along ∂D, (a) follows.

To show (b) let $w = u - v$. Then w has non-tangential limit 0. If w does not vanish identically we can suppose that it has a positive supremum S on D. One can choose a sequence $\{P_n\}_{n=1}^\infty$ convergent in R^n to P^*, and such that $S = \lim w(P_n)$ as $n \to \infty$. Necessarily, by the maximum principle, P^* is in ∂D. One chooses a starlike domain D_1 about P^*; then P^* is in $\partial_2 D_1$.

It follows from (i), and from an elementary argument along the lines of Section 2 of [5], that since the tangential limit of w on $\partial_2 D_1$ is 0 in fact w is continuous on $D \cup \partial_2 D_1$ and 0 on $\partial_2 D_1$. This then leads to the contradiction $S = \lim w(P_n) = w(P^*) = 0$, which proves (b).

PROOF OF THEOREM 1 (AND THEOREM 2) FOR C^∞ DOMAINS IN R^n ($n > 2$). Noting that any convex combination of solutions of (P_∂) is again a solution, uniqueness follows easily once one has proved that every solution necessarily takes on the values 0 or 1 almost everywhere.

Suppose now that some solution f_* of (P_∂) does not take on the values 0 or 1 almost everywhere. Then one can find a measurable subset E of ∂D having positive measure and a δ between 0 and 1 such that $\delta < f_*(Q) < 1 - \delta$ for Q in E. To obtain a contradiction we shall prove Theorem 2 and then define $g = f_* - \delta \bar{f}_\infty^{-1} \bar{f}$; then $g \in \mathscr{K}_\partial$ and $Hg(P_1) < Hf_*(P_1)$, contradicting the optimality of f_*.

The proof of Theorem 2 is also by contradiction! For this purpose we define two linear functionals on $L_\infty(E)$ by $l_0(f) = Hf(P_0)$ and $l_1(f) = Hf(P_1)$. These are non trivial by the mutual absolute continuity of surface and harmonic measure. If Theorem 2 were false one would have that $l_0(f) = 0$ implies $l_1(f) = 0$, and consequently that $l_1(f) = cl_0(f)$ for some constant c. The latter implies that $K(P_1, Q) - cK(P_0, Q) = 0$ for almost every Q in E. This leads to the main idea of the proof.

For smooth domains $K(P, Q) = -\upsilon_Q \cdot \nabla_Q G(P, Q)$ where υ_Q is the unit outward normal to ∂D at Q, and where $G(P, Q)$ is the Green's function of D. $G(P, Q)$ is symmetric in P and Q, harmonic in both variables when $P \neq Q$, has an appropriate singularity at $P = Q$, and satisfies the boundary condition $G(P, Q) = 0$ if P is in D and Q in ∂D. Now the falseness of Theorem 2 would imply that $\upsilon_Q \cdot \nabla_Q v(Q) = 0$ for almost every Q in E, where $v(Q) = G(P_1, Q) - cG(P_0, Q)$ is harmonic in $D \setminus \{P_0, P_1\}$ and also satisfies the boundary condition $v(Q) = 0$ on ∂D. It follows directly from [10] or [9] that v vanishes identically in $D \setminus \{P_0, P_1\}$, which is impossible because of the singularities at P_0 and P_1. This completes the proof.

We remark that the C^∞ requirement was used seriously only in the last step of the proof; the result could be expected to hold under weaker conditions (as it does for $n = 2$), but this would require a different argument. One can weaken the hypothesis for the case $n > 2$ in a somewhat frivolous way (which does at least cover the situation where, for example, D is a polyhedron) by requiring D to be Lipschitz and also C^∞ on the complement of a (closed) subset of ∂D having measure zero.

PROOF OF THEOREM 1 (AND THEOREM 2) FOR LIPSCHITZ DOMAINS IN R^2. When D is conformally equivalent to the open unit disc U the result is trivial since any conformal map of a Lipschitz domain (indeed of a domain with rectifiable boundary) onto the open disc extends to a homeomorphism between $\bar{D}$ and $\bar{U}$ and, moreover sets up a correspondence between sets of measure zero in ∂D and ∂U. This transfers the problem to the C^∞ domain U to which the previous argument applies. In the general case much more care is needed.

The proof is exactly as in the case $n > 2$ until one reaches the conclusion that if Theorem 2 is false $K(P_1, Q) - cK(P_0, Q) = 0$ for almost every Q in E. Now it follows from Dahlberg [1] (Theorem 3 and a subsequent remark) that for almost every Q in ∂D

$$K(P, Q) = -\lim_{t \to 0} v_Q \cdot \nabla G(P, Q - tv_Q)$$

where $G(P, Q)$ is the Green's function of D and $\nabla G(P, Q - tv_Q)$ is to be interpreted as the gradient of $G(P, Q)$ with respect to Q evaluated at $Q - tv_Q$. Then $v(Q) = G(P_1, Q) - cG(P_0, Q)$ is harmonic in $D\{P_0, P_1\}$, continuous in $\bar{D}\backslash\{P_0, P_1\}$ and satisfies the boundary conditions $v(Q) = 0$ on ∂D as well as

$$\lim_{t \to 0} v_Q \cdot \nabla v(Q - tv_Q) = 0$$

for almost every Q in E.

If D were sufficiently regular to ensure that $G(P, Q)$ was continuously differentiable in Q for Q in $\bar{D}\backslash\{P_0, P_1\}$ one could now, using coordinates (ξ, η) (identified with the complex number $\zeta = \xi + i\eta$) for Q, define an analytic function

$$\Psi(\zeta) = \frac{\partial v}{\partial \eta} + i\frac{\partial v}{\partial \xi}$$

continuous in $\bar{D}$ and vanishing almost everywhere in E. Choosing a starlike subdomain D_1 of D with $E \cap \partial D_1$ of positive measure and P_0, P_1 not in D_1, composition of Ψ with a conformal map $f{:}U \to D$ would yield an analytic function $f(\Psi(z))$ ($z = x + iy$ in U) continuous in $\bar{U}$ and vanishing on a subset of ∂U having positive measure. That function would have to vanish identically by the Riesz uniqueness theorem (see Rudin [7], page 373) and hence v would be identically constant, in fact zero, on D_1 and thus on D. When D is merely Lipschitz, the argument is similar but more intricate.

As above we introduce D_1 and a conformal map $f{:}U \to D$. Then $w(z) = v(f(z))$ is in $\mathscr{H}_\infty(U) \cap C(\bar{U})$ with $w(z) = 0$ for z in the arc $\Gamma = f^{-1}(\partial D_1 \cap \partial D)$. Moreover w is in $C^\infty(U \cup \Gamma)$ (see Gilbarg and Trudinger [2], Theorem 6.19 and the subsequent remark. Now consider

$$\Phi(z) = \frac{\partial w}{\partial y} + i\frac{\partial w}{\partial x},$$

which is analytic and continuous in $\bar{U}$. To complete the proof as before it is enough to show that $\Phi(z)$ vanishes on a subset of Γ having positive measure.

Since the tangential derivative of w vanishes on Γ, it is then sufficient to verify that at almost all points of $F = f^{-1}(E \cap \partial D_1)$ w has normal derivative zero. To do this we consider the curve $\zeta(t) = \zeta - t\upsilon_\zeta$ for $\zeta \in E \cap \partial D_1$ (with $t \geqq 0$ and small enough to ensure that $\zeta(t) \in D_1$). Then we define the curve $z(t) = f^{-1}(\zeta(t)) = x(t) + iy(t)$, which is contained in D_1 for $t > 0$ and has $z(0) = z = f^{-1}(\zeta)$ in F. Now differentiating $w(z(t)) = v(\zeta(t))$ with respect to t, one finds

$$\lim_{t\downarrow 0} \nabla w(z(t)) \cdot \frac{d}{dt}(x(t), y(t)) = \lim_{t\downarrow 0} \nabla v(\zeta(t)) \cdot \upsilon_\zeta = 0$$

for almost all ζ in $E \cap \partial D_1$. The proof will then be complete once we show that for almost all ζ in $E \cap \partial D_1$ (i.e. z in F) $(d/dt)(x(t), y(t))$ has a limit (μ_x, μ_y) which is non-zero and not tangential to ∂U at z. For then one obtains in the limit that the non-tangential derivative $(\mu_x, \mu_y) \cdot \nabla w(z) = 0$ for almost all z in F.

We lean heavily on results to be found in Pommerenke's book [7]. Whenever ∂D has a tangent at $\zeta = f(z)$ one has (by Theorem 10.4, page 302) "conformality" at z; hence $z(t)$ approaches z normally (since $\zeta(t)$ approaches ζ normally). More specifically one has

$$\lim_{t\downarrow 0} \arg[z - z(t)] = \arg[z].$$

Furthermore (as a consequence of Theorem 10.5, page 305 and Exercise 2, page 329 which is an easy corollary of the deep Theorem 10.15, page 326) for almost all z such that ∂D has a tangent at $\zeta = f(z)$,

$$a = \lim_{t\downarrow 0} \frac{df}{dz}(z(t))$$

exists and is not zero. Now

$$\frac{dz}{dt}(t) = \frac{d}{d\zeta} f^{-1}(\zeta(t)) \frac{d\zeta}{dt}(t) = -\upsilon_\zeta / \frac{df}{dz}(z(t)),$$

and hence indeed

$$\lim_{t\downarrow 0} \frac{dz}{dt}(t) = -\upsilon_\zeta / a = \mu_x + i\mu_y \neq 0.$$

It follows from the normal approach of $z(t)$ to z that this limit is normal to ∂U at z, and thus the proof is complete.

References

1. B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rat. Mech. Anal. **65** (1977), pp. 275-288.

2. D. Gilbarg, and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, Heidelberg (1983).

https://doi.org/10.4153/CMB-1988-010-6 Published online by Cambridge University Press

3. W. K. Hayman, and P. B. Kennedy, *Subharmonic functions* (*Volume* 1) Academic Press, London (1976).

4. L. L. Helms, *Introduction to potential theory*, Wiley-Interscience, New York (1969).

5. R. A. Hunt, and R. L. Wheeden, *On the boundary values of harmonic functions*, A.M.S. Transactions **132** (1968), pp. 307-322.

6. D. S. Jerison, and C. S. Kenig, *An identity with applications to harmonic measure*, A.M.S. Bulletin (New Series) **2** (1980), pp. 447-451.

7. Chr. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, Gottingen (1975).

8. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York (1974).

9. E. J. P. G. Schmidt, and N. Weck, *On the boundary behaviour of solutions to elliptic and parabolic equations — with applications to boundary control for parabolic equations*, SIAM J. Cont. and Opt. **16** (1978), pp. 593-598.

10. N. Weck, *Uber das prinzip der eindeutigen Fortsetzbarkeit in der Kontrolltheorie*, *Optimisation and optimal control*, Lecture Notes in Mathematics 477, Springer-Verlag, Berlin (1975), pp. 276-284.

Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, P.Q.
Canada H3A 2K6

https://doi.org/10.4153/CMB-1988-010-6 Published online by Cambridge University Press