ANALYTIC FUNCTIONS WITH AN IRREGULAR LINEARLY MEASURABLE SET OF SINGULAR POINTS

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Introduction. V. V. Golubev, in his study [6], has constructed, by using definite integrals, various examples of analytic functions having a perfect nowhere-dense set of singular points. These functions were shown to be single-valued with a bounded imaginary part. In attempting to extend his work to the problem of constructing analytic functions having perfect, nowhere-dense singular sets under quite general conditions, he posed the following question: Given an arbitrary, perfect, nowhere-dense point-set E of positive measure in the complex plane, is it possible to construct, by passing a Jordan curve through E and by using definite integrals, an example of a single-valued analytic function, which has E as its singular set, with its imaginary part bounded.

In the present investigation, we shall require the set E, which is bounded and closed, to belong to the class of irregular sets of finite linear measure.[1] Hence, we wish to determine the possibility of obtaining, by using definite integrals, a function $\phi(z)$ having the following properties:

1. $\phi(z)$ analytic in the extended z-plane (except the points of an irregular, bounded and closed point-set E of essential singularities of positive linear measure);
2. $\phi(z)$ single-valued;
3. $|\mathfrak{I}\phi(z)|$ bounded.

The current problem is one that has evolved as a result of researches made by various authors. D. Pompeiu [10, pp. 914-915] was the first to exhibit an interest in constructing, with the help of definite integrals, an analytic function having a perfect, nowhere-dense, bounded set E of essential singular points. He proved that if E is of two-dimensional positive Lebesgue measure, there exist functions continuous and analytic in the extended z-plane with singularities in E.

Employing definite integrals, A. Denjoy [5, pp. 258-260] constructed an example of an analytic function $f(z)$ having a perfect, nowhere-dense set E of essential singularities of positive Lebesgue measure in the linear interval $0 \leqslant x \leqslant 1$. $f(z)$ was single-valued with a bounded imaginary part.

Golubev [6, p. 122] extended Denjoy's result to the case in which E was a perfect, nowhere-dense set of positive measure on any rectifiable curve and formed

Received August 10, 1951.

[1]Point-sets of finite linear measure are divided into two classes: the first consisting of regular sets, and the second of irregular sets. Regular sets are completely analogous to rectifiable curves; irregular sets are dissimilar to rectifiable curves in fundamental geometrical properties. Cf. [1, pp. 424-426; 3, pp. 142-143].

https://doi.org/10.4153/CJM-1952-038-9 Published online by Cambridge University Press

the bounded function[2] $F(z) = e^{-if(z)}$. Moreover, he assumed the existence of any perfect, nowhere-dense set E of positive measure in the z-plane. He constructed, by passing a Jordan curve $g = g(t), h = h(t)$ through E and by using definite integrals, an example of a single-valued analytic function, which has E as its singular set, with its imaginary part bounded. (The set E then corresponded to some perfect, nowhere-dense set E_t of values t.) Golubev, however, obtained questionable results.[3]

In §1 of this paper, we establish, for measurable functions defined and bounded on E, the general integral representation using Carathéodory linear measure. By means of the integral representation, we define, in §2, an analytic function as a function of its singular set. In §3, we generalize Golubev's technique of constructing a curvilinear integral of a function defined and continuous for a regular set E on any rectifiable curve to the case where E is an irregular set on any Jordan curve [**6**, p. 122]. We give, in §4, a characterization of the curvilinear integral and $\mathfrak{T}\phi(z)$.

1. Integral representation. We consider, in the complex plane, a point-set E satisfying the foregoing requirements. Let P denote any point of E, and $f(P)$ a single-valued function of a point defined and bounded on E. We enclose f in a finite or denumerable number of convex point-sets,

$$u_0, u_1, u_2, \ldots, u_n, \ldots$$

satisfying the following conditions:

(a) $f(P)$, for each P, is interior to at least one of the sets $u_0, u_1, u_2, \ldots$.
(b) The diameter du_i of each u_i is smaller than a positive number ρ chosen in advance.[4]

Now $f(P)$ is measurable with regard to the covering u_i, that is,[5] $E(|f| > \mu) = E(f > \mu)$, for $\mu > 0$, forms a measurable set[6] E_μ of points P. We insert between

[2]The analytic function $F(z)$ is single-valued and bounded in a domain whose boundary in a regular set E of positive measure. According to Fatou's Theorem, as generalized by Golubev, $F(z)$ has on nearly all points of E a definite value, a fact from which we conclude E to be removable. [**6**, p. 44; **8**, pp. 154-157; **11**, p. 40].

[3]Golubev [**6**, pp. 127-129] noted that his construction of this function was weakened because E_t had no direct connection with E, and could be selected arbitrarily.

[4]The convex point-set u_i is orthogonally projected on a plane, the result of which is another convex point-set whose area depends upon the position of the particular plane in space. We call the upper bound of the areas for all possible planes the two-dimensional Carathéodory diameter du_i of u_i [**4**, p. 426; **7**, p. 162].

[5]$E(|f| > \mu)$ is the sum of two measurable sets $E(f > \mu)$ and $E(f < -\mu)$. However, the latter set does not exist for the measure under consideration. ($E(|f| > \mu)$ denotes the subset of E for which $|f(P)| > \mu$.)

[6]A plane set E is called measurable with respect to Carathéodory linear measure if the relation

$$L_2(A + B) = L_2(A) + L_2(B)$$

for every pair of sets A and B contained in E and its complement, C(E) [**4**, pp. 404-426].

the upper bound M and the lower bound m of f the following numbers;

$$\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \ldots \leqslant \mu_{n-1},$$

and establish, for the integral of $f(P)$ over E, the representation,

$$\int_E f(P)du = \liminf_{\rho\to 0} \sum_{i=0}^{n} \mu_i \, du_i \qquad (\mu_0 = m, \mu_n = M).$$

E is a bounded point-set; hence, as ρ approaches zero, there exists a finite limit. We denote the linear measure of f by $L_2(f)$, where[7]

$$L_2(f) = \int_E du_i = \liminf_{\rho\to 0} \sum_{j=0}^{n} du_i.$$

2. Analytic function as function of its singular set. The integral of $f(P)$, by the nature of its construction, is a number that depends upon the point-set E. This function of a set,

$$F(E) = \int_E f(P)du$$

is defined as the definite integral of $f(P)$ over E, and

$$|F(E)| = \left|\int_E f(P)du\right| \leqslant \int_E |f(P)|du \leqslant ML_2(f),$$

where M denotes the maximum value of $f(P)$ over E.

We consider now the function

$$\phi(z) = \int_E f(P, z)du,$$

a function defined by a definite integral which contains in the integrand a parameter. Let $f(P, z)$ be a single-valued, bounded function defined when P lies in E and z in the complementary set, $C(E)$. We first prove an extension of a well-known integral theorem in the Theory of Functions to integrals of the class under consideration. The theorem will serve as a nucleus for current developments.

THEOREM 1. *If $f(P, z)$ is a continuous function of P and z together, the function*

$$\phi(z) = \int_E f(P, z)du$$

is continuous in $C(E)$.

Moreover, if $f(P, z)$ has for each z a partial derivative $f_z(P, z)$ continuous in P and z together, the function $\phi(z)$ is analytic in $C(E)$, that is,

$$\phi'(z) = \int_E f_z(P, z)du.$$

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[7]The measure $L_2(f)$ is non-negative and single-valued. For the conditions that $L_2(f)$ must satisfy, see [**7**, pp. 158-159].

https://doi.org/10.4153/CJM-1952-038-9 Published online by Cambridge University Press

The first part of the theorem is valid because of the continuity of $f(P, z)$ in the two variables (P, z).

From the existence and continuity of the partial derivative $f_z(P, z)$, we derive the continuity of the partial derivatives $f_x(P, z)$ and $f_y(P, z)$. Further, these derivatives satisfy the Cauchy-Riemann differential equations. We have

$$f_x(P, z) = - if_y(P, z) = f_z(P, z)$$

and therefore $\phi(z)$ possesses a partial derivative with respect to x and y continuous in $C(E)$. Hence,

$$\phi_x(z) = \int_E f_x(P, z)du = - i\int_E f_y(P, z)du = - i\phi_y(z).$$

These derivatives likewise fulfil the requirements of the Cauchy-Riemann differential equations. Consequently, $\phi(z)$ is analytic in $C(E)$, and

$$\phi'(z) = \int_E f_z(P, z)du.$$

This proves the second part.

In compliance with the hypothesis of Theorem 1 we select

$$f(P, z) = \frac{1}{P - z}$$

and construct the function

$$\phi(z) = \int_E \frac{du}{P - z}.$$

The integral in the right member is a function of an irregular set of singular points in a sense analogous to that in which a function of a set of singular lines has been constructed with the help of definite integrals.[8] Thus, $\phi(z)$ represents at most a denumerable set of functions analytic except for certain singular points.

In $C(E)$, $\phi(z)$ is an analytic function. Its first derivative is given by the formula

$$\phi'(z) = \int_E \frac{du}{(P - z)^2}.$$

Moreover, $\phi(z)$ possesses derivatives of every order analytic in $C(E)$ and they are given by the formulae

$$\phi''(z) = 2!\int_E \frac{du}{(P - z)^3}$$

and, in general,

$$\phi^{(n)}(z) = n!\int_E \frac{du}{(P - z)^{n+1}} \qquad (n = 1, 2, 3, \ldots).$$

From the well-known fact that a function can be represented by a power

[8]For examples of functions of a set of singular lines, cf. [6, pp. 92-97].

series in the neighbourhood of any point of a domain in which it is analytic, we have the following result:

COROLLARY 1. *If $z = z_0$ be any fixed point in* C(E),

$$\phi(z) = \int_E \frac{du}{P - z}$$

can be represented, in a certain neighbourhood of this point, by a Taylor series. This series will converge and represent the function throughout the largest circle, about $z = z_0$ as centre, which contains in its interior no point of E.

We determine the nature of the function

$$\phi(z) = \int_E \frac{du}{P - z}$$

in the neighbourhood of the point $z = \infty$. Let us begin by making the transformation $z' = 1/z$, writing $\Psi(z') = \phi(1/z')$ and examining the transformed function $\Psi(z')$ for values in the neighbourhood of $z' = 0$. First, we have

$$\Psi(z') = \int_E \frac{z'du}{Pz' - 1},$$

and $\Psi(0) = 0$. We next take successive derivatives of $\Psi(z')$, then place $z' = 0$, obtaining

$$\Psi'(z') = -\int_E \frac{du}{(Pz' - 1)^2}, \quad \Psi'(0) = -\int_E du,$$

$$\Psi''(z') = 2!\int_E \frac{Pdu}{(Pz' - 1)^3}, \quad \Psi''(0) = -2!\int_E Pdu,$$

and, in general,

$$\Psi^{(n)}(z') = (-1)^n n!\int_E \frac{P^{n-1}du}{(Pz' - 1)^{n+1}}, \quad \Psi^{(n)}(0) = -n!\int_E P^{n-1}du$$
$$(n = 1, 2, 3, \ldots).$$

The series

$$\Psi(z') = \Psi(0) + \Psi'(0)z' + \Psi''(0)\frac{z'^2}{2!} + \ldots + \Psi^{(n)}(0)\frac{z'^n}{n!} + \ldots$$

becomes, by expressing Ψ and its derivatives in terms of integrals,

$$\Psi(z') = -\left(\int_E z'du + \int_E Pz'^2du + \ldots + \int_E P^{n-1}z'^n du + \ldots\right).$$

Then

$$\phi(z) = -\left(\int_E z^{-1}du + \int_E Pz^{-2}du + \ldots + \int_E P^{n-1}z^{-n}du + \ldots\right).$$

The appearance of negative powers of z in the right member indicates that $\phi(z)$ is analytic in the neighbourhood of the point $z = \infty$, and the absence of

https://doi.org/10.4153/CJM-1952-038-9 Published online by Cambridge University Press

the constant term shows that the function has a root at infinity. We have proved

THEOREM 2.

$$\phi(z) = \int_E \frac{du}{P - z}$$

is analytic at $z = \infty$, *and* $\phi(\infty) = 0$.

An examination of the single-valued character of $\phi(z)$ discloses a question: Does $\phi(z)$ return to its original value when z describes a continuous closed path around E? Let us enclose E in the smallest rectangle R which contains the point set in its interior. The domain exterior to R we designate by S; $\phi(z)$ is analytic in S. Moreover, S is simply connected. Therefore, we have an analytic function in a simply connected domain; a fact which proves, according to the monodromy theorem, that $\phi(z)$ is single-valued for any closed path described by z around E.

A second question confronts us: Can $\phi(z)$ be analytically continued through E? Let us consider a path K in $C(E)$ which begins at a point z_0 and separates E into two distinct proper subsets E_1 and E_2, each of which has positive linear measure. The path K, which satisfies this condition, is known to exist because of the density classification by which irregular sets are defined.[1] By Corollary 1, $\phi(z)$ can be represented in a certain neighbourhood of z_0 by a power series which will converge and represent a functional element within the largest circle, centre z_0, which contains in its interior no point of E. We continue this element of $\phi(z)$ along K through power series expansions by choosing points, as centres of circles of convergence, along the path in such a way that the circles form a chain. Continuation by this means is possible through repeated application of the corollary and the use of the identity theorem for analytic functions. We observe from the foregoing remarks that the analytic continuation of $\phi(z)$ along K is everywhere feasible.

We form the function $\phi(z) = \phi_1(z) + \phi_2(z)$, where

$$\phi_1(z) = \int_{E_1} \frac{du_1}{P - z}, \quad \phi_2(z) = \int_{E_2} \frac{du_2}{P - z},$$

and choose for z a path which includes k in the following manner: from z_0, z passes along k, encircles E_2 by a counterclockwise movement, continues a circuit around E_1 to k, and finally reverses its direction along this path to z_0. The path thus described by z is equivalent to a single continuous closed circuit around E, a case for which we have shown $\phi(z)$ to be single-valued. Hence

$$\phi(z) = \int_{E_1} \frac{du_1}{P - z} + \int_{E_2} \frac{du_2}{P - z} = \int_E \frac{du}{P - z},$$

and we have proved

THEOREM 3. *The analytic function*

$$\phi(z) = \int_E \frac{du}{P - z}$$

is single-valued in $C(E)$ *if* E *is an irregular point-set.*

We shall prove [**6**, pp. 128-129]

THEOREM 4. *The point-set E is a singular set for $\phi(z)$.*

Let us assume that $\phi(z)$ is constant. In $C(E)$, $\phi(z)$ is a single-valued analytic function, which is attested by Theorem 3. According to Theorem 2, $\phi(z) = 0$ for $z = \infty$. Therefore, if $\phi(z)$ is constant, $\phi(z) \equiv 0$ and $z\phi(z) \equiv 0$. However,

$$\lim_{z\to\infty} z\phi(z) = \lim_{z\to\infty} \int_E \frac{du}{(P/z) - 1} = - \int_E du = - L_2(f).$$

We thus have a contradiction since the measure $L_2(f)$ is known to be non-negative. We conclude that at least some points of E are singular for $\phi(z)$. Let us denote these singular points by E_1, and represent E as the sum

$$E = E_1 + E_{11}.$$

Then

$$\phi(z) = \phi_1(z) + \phi_{11}(z),$$

where $\phi_1(z)$ is the function which has E_1 as its singular set. Applying the foregoing procedure to $\phi_{11}(z)$, we show that at least some points of E_{11} are singular for $\phi_{11}(z)$, for example, E_2. We now represent E_{11} as the sum

$$E_{11} = E_2 + E_{12},$$

and

$$\phi_{11}(z) = \phi_2(z) + \phi_{12}(z),$$

where $\phi_2(z)$ is the function which has E_2 as its singular set. We continue this process and obtain

$$E = E_1 + E_2 + E_3 + \ldots$$

singular sets for

$$\phi(z) = \phi_1(z) + \phi_2(z) + \phi_3(z) + \ldots$$

respectively, remembering that sets of measure zero correspond to isolated singular points. We mean, by this, that the characteristics of functions having a set of singular points of measure zero compare very closely to those having isolated essential singular points.[9] This completes the proof.

3. The curvilinear integral. We come to the third property, namely, to determine whether or not the function $\phi(z)$ has a bounded imaginary part. We consider the construction of an expression analogous to a curvilinear integral of $f(P)$ along E.

According to the researches of Besicovitch [**1**, p. 455], there exists a finite or denumerable set G of Jordan curves which contain almost all points of E. We denote by J_1 the curve of G which contains a subset $'E$ of E such that the linear measure of the plane set $'E$ satisfies the inequality $L('E) > L(E) - \epsilon$, ϵ being a positive number.

The set $'E$ is a closed subset of E. Therefore, the $C('E)$ complementary to

[9]For details concerning this matter, cf. [**6**, pp. 126-127].

$'E$ is open and consist of denumerably many open arcs, with no two arcs having any point in common. Let the diameters of these arcs be arranged in a denumerable order, that of increasing magnitude. We thus have a denumerably increasing sequence of diameters which we designate by $h_1, h_2, h_3, \ldots, h_n, \ldots$. Generalizing the method of Golubev, [**6**, p. 122] we obtain $'E$ by removing a denumerable sequence of arcs whose diameters are given above, in the following manner: we remove first from $C('E)$ the complementary arcs whose diameter is h_1, which decomposes $'E$ into a denumerable sequence of sets

$$'E_1^1, 'E_2^1, \ldots, 'E_m^1, \ldots.$$

We next remove the complementary arcs having a diameter h_2 which decomposes the above sequence $('E_m^1)$ into a new sequence

$$'E_1^2, 'E_2^2, \ldots, 'E_m^2, \ldots.$$

The set $'E_m^2$, for each m, is the union of the decompositions which result from the removal of a complementary arc of diameter h_2 from each set of the sequence $('E_m^1)$. In general, at any step n in the process we obtain, in a similar manner, a new sequence

$$'E_1^n, 'E_2^n, \ldots, 'E_m^n, \ldots,$$

and likewise $'E_m^n$ is the union of decompositions resulting from the removal of complementary arc of diameter h_n from each set of the sequence $('E_m^{n-1})$. The operation continues and we obtain the infinite double sequence of point-sets[10]

$$\begin{array}{l} 'E_1^1, 'E_1^2, \ldots, 'E_1^n, \ldots \\ 'E_2^1, 'E_2^2, \ldots, 'E_2^n, \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 'E_m^1, 'E_m^2, \ldots, 'E_m^n, \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

Let $f(P)$ be a continuous function of P along $'E$. We enclose each set $'E_m^n$ in a convex point-set u_m^n which satisfies conditions (a) and (b) of §1, the diameter, in this case, being the real number[11]

$$|e^{i\theta^n{}_m} du_m^n|.$$

Let $f(P_m^n)$ be the value of the function at any point P_m^n on E_m^n. We consider the summation by rows of the double series

[10]This double infinite sequence is one in which the rows appear, in the construction of the sequence, as columns.

[11]The convex point-set u_m^n is orthogonally projected on an arbitrary straight line. There exists, as a result, a distance which depends upon the direction of this line in the plane. The upper bound of the lengths of distances for all possible directions of the straight line is called the one-dimensional Carathéodory diameter of u_m^n. Cf. [**4**, p. 426].

$$\begin{array}{l} f(P_1^1)|e^{i\theta^1{}_1}du_1^1| + f(P_1^2)|e^{i\theta^2{}_1}du_1^2| + \ldots + f(P_1^n)|e^{i\theta^n{}_1}du_1^n| + \ldots \\ + f(P_2^1)|e^{i\theta^1{}_2}du_2^1| + f(P_2^2)|e^{i\theta^2{}_2}du_2^2| + \ldots + f(P_2^n)|e^{i\theta^n{}_2}du_2^n| + \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + f(P_m^1)|e^{i\theta^1{}_m}du_m^1| + f(P_m^2)|e^{i\theta^2{}_m}du_m^2| + \ldots + f(P_m^n)|e^{i\theta^n{}_m}du_m^n| + \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

and define the curvilinear integral of $f(P)$ along $'E$ in the following manner:

$$\int_{'E} f(P)|e^{i\theta}du| = \liminf_{\rho\to 0} \sum_m \sum_n f(P_m^n)|e^{i\theta^n{}_m}du_m^n|.$$

The limit exists because of the continuity of $f(P)$ on $'E$.

4. Characterization of the curvilinear integral and $\mathfrak{T}\phi(z)$. We shall show some of the interesting properties of the curvilinear integral. We begin with an analogue of a well-known integral theorem.

THEOREM 4. *If M denotes the maximum value of $|f(P)|$ for any P on $'E$, and $L_1('E)$ the linear measure of $'E$,*

$$\left|\int_{'E} f(P)|e^{i\theta}du|\right| \leqslant ML_1('E).$$

From §3,

$$\left|\int_{'E} f(P)|e^{i\theta}du|\right| \leqslant \int_{'E} |f(P)||e^{i\theta}du| \leqslant M \liminf_{\rho\to 0} \sum_m \sum_n |e^{i\theta^n{}_m}du_m^n|.$$

The double summation represents, for each n and every m, the summation by rows of the diameters of convex sets (u_m^n) which contain sets $('E_m^n)$ respectively, and is equal, as $\rho \to 0$, to the linear measure of $'E$, $L_1('E)$, which by hypothesis is greater than zero.

We decompose $'E$ into two distinct proper subsets $'E_1$ and $'E_2$, each of which has positive linear measure, and prove

THEOREM 5.

$$\int_{'E} |f(P)||e^{i\theta}du| \leqslant \int_{'E_1} |f(P)||e^{i\theta_1}du_1| + \int_{'E_2} |f(P)||e^{i\theta_2}du_2|.$$

We have

$$\int_{'E} |f(P)||e^{i\theta}du| \leqslant M \liminf_{\rho\to 0} \sum_m \sum_n |e^{i\theta^n{}_m}du_m^n|.$$

In like manner,

$$\int_{'E_1} |f(P)||e^{i\theta_1}du_1| \leqslant M \liminf_{\rho\to 0} \sum_m \sum_n |e^{i\theta^n{}_{1m}}du_{1m}^n|,$$

and

$$\int_{'E_2} |f(P)||e^{i\theta_2}du_2| \leqslant M \liminf_{\rho\to 0} \sum_m \sum_n |e^{i\theta^n{}_{2m}}du_{2m}^n|.$$

In accordance with the meaning of the double sums of the right members, and by making use of the triangle theorem, we have

$$M \liminf_{\rho\to 0} \sum_m \sum_n |e^{i\theta^n{}_m} du_m^n| < M \liminf_{\rho\to 0} \sum_m \sum_n (|e^{i\theta^n{}_{1m}} du_{1m}^n| + |e^{i\theta^n{}_{2m}} du_{2m}^n|),$$

if $\theta \neq \theta_1 \neq \theta_2 \neq 0$. If $\theta = \theta_1 = 0$, the equality sign holds for the same reasons.

THEOREM 6. *If $_+'E$ and $_-'E$ denote two opposite directions in which the integral is taken along $'E$,*

$$\int_{_+'E} f(P)|e^{i\theta} du| = \int_{_-'E} f(P)|e^{i\theta} du|,$$

i.e., the value of the integral is independent of the direction of integration.

The proof of this Theorem follows immediately from the definition of the integral.

THEOREM 7. *The curvilinear integral is dependent upon the particular subset $^{(n)}E$ of E through which a Jordan curve passes.*

According to a theorem of Besicovitch [**1**, p. 455], there exists a finite or denumerable set of Jordan curves that can be passed through the points of E. The intersection of each Jordan curve with E is of positive measure. The curve J_1, as we have indicated in §3, contains a subset $'E$ of E. We denote by J_2, $J_3, \ldots, J_n, \ldots$ those Jordan curves that contain subsets $''E, '''E, \ldots, {}^{(n)}E, \ldots$ of E respectively. Through the application of procedures and operations used in §3, the curvilinear integral of $f(P)$ along each $^{(n)}E$ $(n = 2, 3, \ldots)$ can be easily shown to exist. The Jordan curves J_n are assumed to be distinct. This proves the theorem.

We resolve

$$\int_{'E} f(P)|e^{i\theta} du|$$

into real and imaginary parts. We have shown that

$$\int_{'E} f(P)|e^{i\theta} du| = \liminf_{\rho\to 0} \sum_m \sum_n f(P_m^n)|e^{i\theta^n{}_m} du_m^n|.$$

Now let[12]

$$f(P) = f_1(P_1) + if_2(P_1).$$

Then

$$\liminf_{\rho\to 0} \sum_m \sum_n f(P_m^n)|e^{i\theta^n{}_m} du_m^n| = \liminf_{\rho\to 0} \sum_m \sum_n f_1(P_{1m}^n)|e^{i\theta^n{}_m} du_m^n|$$
$$+ \liminf_{\rho\to 0} \sum_m \sum_n f_2(P_{1m}^n)|e^{i\theta^n{}_m} du_m^n|.$$

[12] $f_1(P_1)$ and $f_2(P_1)$ are real functions of the point P_1.

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$f(P)$ is continuous along $'E$. Therefore,

$$\int_{'E} f(P)\,|e^{i\theta}du| = \int_{'E} f_1(P_1)\,|e^{i\theta}du| + i\int_{'E} f_2(P_1)\,|e^{i\theta}du|.$$

The two integrals on the right, consistent with their meaning, represent a generalization of curvilinear integrals of functions of real variables.

We form the function

$$\phi(z) = \int_{'E} \frac{|e^{i\theta}du|}{P - z},$$

where z is in $C('E)$, and investigate the character of $\mathfrak{T}\phi(z)$. For this purpose, we let $P = a + ib$. Then

$$\phi(z) = \int_{'E} \frac{|e^{i\theta}du|}{P - z} = \int_{'E} \frac{(a - x)|e^{i\theta}du| - i(b - y)|e^{i\theta}du|}{(a - x)^2 + (b - y)^2}$$

$$\mathfrak{J}\phi(z) = \int_{'E} \frac{(y - b)|e^{i\theta}du|}{(a - x)^2 + (b - y)^2}.$$

There exists no positive number M which is not exceeded[13] by $|\mathfrak{T}\phi(z)|$ for any z in $C('E)$. We conclude that E is not a removable point-set.

With reference to the classical Riemann theorem on removable singularities of an analytic function, Besicovitch [**2**, p. 2] has shown that a removable set of singularities cannot be an arbitrary set of positive linear measure. The results which we have obtained indicate that removable sets cannot be irregular sets. This means that removable sets are restricted to the class of regular sets of positive linear measure.

We have demonstrated the possibility of inspecting an analytic function as a function of its set of singular points. If we take this point of view, the core of the study of analytic functions may well shift from the domain in which the function is analytic to the set of its singular points. The question as to whether or not it was possible to examine an analytic function in this manner was raised by Golubev [**6**, pp. 156-157] at the close of his study.

In our extension of the study of Golubev to the case of irregular sets of positive measure, we have constructed an analytic function satisfying two of the three properties which we proposed for investigation. We summarize our work with the following theorem:

THEOREM 8. *Let E be a bounded and closed point-set in the complex plane. If E is an irregular, linearly measurable point-set, there exists in the neighbourhood of E a function $\phi(z)$, single-valued and analytic in the extended plane, with E as its singular set.*

[13]The assertion in the text can be easily verified by replacing $|e^{i\theta}du|$ by its analogue ds.

https://doi.org/10.4153/CJM-1952-038-9 Published online by Cambridge University Press

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