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NILPOTENTS IN SEMIGROUPS OF PARTIAL ORDER-PRESERVING TRANSFORMATIONS

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In this paper we extend the results of Garba [1] on IO_n, the semigroup of all partial one–one order-preserving maps on $X_n = \{1,\dots,n\}$, to PO_n, the semigroup of all partial order-preserving maps on X_n. A description of the subsemigroup of PO_n generated by the set N of all its nilpotent elements is given. The set $\{\alpha \in PO_n: |\operatorname{im}\alpha| \leqq r$ and $|X_n \backslash \operatorname{dom}\alpha| \geqq r\}$ is shown to be contained in $\langle N \rangle$ if and only if $r \leqq \frac{1}{2}n$. The depth of $\langle N \rangle$, which is the unique k for which $\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k$ and $\langle N \rangle \neq N \cup N^2 \cup \cdots \cup N^{k-1}$, is shown to be equal to 3 for all $n \geqq 3$. The rank of the subsemigroup $\{\alpha \in PO_n: |\operatorname{im}\alpha| \leqq /n-2$ and $\alpha \in \langle N \rangle\}$ is shown to be equal to $6(n-2)$, and its nilpotent rank to be equal to $7n-15$.

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1. Introduction

In 1987, Gomes and Howie [3], and Sullivan [7] independently initiated the study of nilpotent generated subsemigroups of transformation semigroups on the set $X_n = \{1,\dots,n\}$, by considering I_n, the symmetric inverse semigroup and P_n, the semigroup of all partial transformations on X_n respectively. In [1] Garba considered IO_n, the semigroup of all partial one–one order-preserving maps on X_n. We shall now consider the larger semigroup PO_n of all partial order-preserving transformations on X_n.

Let N be the set of all nilpotent elements in PO_n, and $\langle N \rangle$ the sub-semigroup of PO_n generated by N. In Section 2 a description of the elements in $\langle N \rangle$ is given. We show also that the set $\{\alpha \in PO_n: |\operatorname{im}\alpha| \leqq r$ and $|X_n \backslash \operatorname{dom}\alpha| \geqq r\}$ is contained in $\langle N \rangle$ if and only if $r \leqq \frac{1}{2}n$.

Define the *depth* of $\langle N \rangle$, denoted by $\Delta(\langle N \rangle)$, to be the unique k for which

$$\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$

In Section 3 we show that $\Delta(\langle N \rangle) = 3$ for all $n \geqq 3$.

By the *rank* of a semigroup S we shall mean the cardinality of any subset A of minimal order in S such that $\langle A \rangle = S$. If S has a zero and is generated by nilpotents then the cardinality of the smallest subset A *consisting of nilpotents* for which $\langle A \rangle = S$ is called the *nilpotent rank* of S. In Section 4 we show that the rank of the subsemigroup $\{\alpha \in PO_n: |\operatorname{im}\alpha| \leqq n-2$ and $\alpha \in \langle N \rangle\}$ is equal to $6(n-2)$, and its nilpotent rank is equal to $7n-15$.

2. The nilpotent generated subsemigroup

We will denote an element α in PO_n by

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where for each $a_i \in A_i$, $a_i < a_{i+1}$ $(i=1,\dots,r)$ and $b_1 < b_2 < \cdots < b_r$. Let $x_i = \min\{x: x \in A_i\}$ and $y_i = \max\{x: x \in A_i\}$. For $i=1,\dots,r$, let $S_i = \{x \in X_n: x_i \leqq x \leqq y_i\}$, and for $i=1,\dots,r-1$, $T_i = \{x \in X_n: y_i < x < x_{i+1}\}$. Let $T_0 = \{x \in X_n: x < x_1\}$ and $T_r = \{x \in X_n: x > y_r\}$.

Following [1], we define $j_i(\alpha)$, the *length of the ith lower jump of* α, by

$$j_i(\alpha) = b_{i+1} - b_i - 1,\ (i=1,\dots,r-1),\ j_0(\alpha) = b_i - 1.$$

Then let

$$j_*(\alpha) = \sum_{i=0}^{r-1} j_i(\alpha).$$

An element α in PO_n is called nilpotent if $\alpha^k = 0$ for some $k \geqq 1$. We begin with a generalisation of Lemma 2.1 in [**1**].

Lemma 2.1. *An element α in PO_n is nilpotent if and only if for all $x \in \operatorname{dom}\alpha$, $x\alpha \neq x$.*

Proof. If $\alpha = 0$ (the empty map) the result is trivial. We may therefore suppose that $\operatorname{dom}\alpha \neq 0$. It is clear that if $x\alpha = x$ for some $x \in \operatorname{dom}\alpha$, then α cannot be nilpotent; for we would have

$$x = x\alpha = x\alpha^2 = \cdots.$$

Conversely, suppose that $x\alpha \neq x$ for all $x \in \operatorname{dom}\alpha$. We first show that if $\operatorname{dom}\alpha^k \neq \emptyset$ $k \geqq 2)$ then $x\alpha^k \neq x$ for all $x \in \operatorname{dom}\alpha^k$. (Note that if $\operatorname{dom}\alpha^k = \emptyset$ for some k then α is nilpotent.) Let $x \in \operatorname{dom}\alpha^k$. Then $x \in \operatorname{dom}\alpha^t$ for all t such that $1 \leqq t \leqq k$. In particular $x \in \operatorname{dom}\alpha$, and thus $x\alpha \neq x$. We therefore have $x\alpha < x$ or $x\alpha > x$. By the order-preserving property we have $x\alpha^k < x$ or $x\alpha^k > x$. Thus $x\alpha^k \neq x$.

Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

where $r = |\operatorname{im}\alpha|$. Now, if $b_r \in \operatorname{dom}\alpha$ then (since $b_r\alpha \neq b_r$) we must have $b_r < x_r$ $(=\min\{x: x \in A_r\})$, and by the order-preserving property we must have $\operatorname{im}\alpha \cap A_r = \emptyset$. Thus $b_r \in \operatorname{im}\alpha^2$, and so $\operatorname{im}\alpha^2 \subset \alpha$ (properly). If $b_r \notin \operatorname{dom}\alpha$ then $|\operatorname{dom}\alpha \cap \operatorname{im}\alpha| < r$, and so $|\operatorname{im}\alpha^2| < r = |\operatorname{im}\alpha|$, which shows that $\operatorname{im}\alpha^2 \subset \operatorname{im}\alpha$.

If we now denote by s the cardinality of $\operatorname{im}\alpha^2$, then α^2 can be written as

$$\begin{pmatrix} A'_1 & A'_2 & \dots & A'_s \\ b'_1 & b'_2 & \dots & b'_s \end{pmatrix}.$$

Since $x\alpha^2 \neq x$ for all $x \in \operatorname{dom}\alpha^2$, repeating the same argument as above with α^2 replacing α we obtain $\operatorname{im}\alpha^4 \subset \operatorname{im}\alpha^2$. If this process is to continue we will obtain a strict descent

$$\operatorname{im}\alpha \supset \operatorname{im}\alpha^2 \supset \operatorname{im}\alpha^4 \supset \cdots$$

and since $|\operatorname{im}\alpha|$ is finite there exists m such that $\operatorname{im}\alpha^{2^m} = \emptyset$, that is such that $\alpha^{2^m} = 0$. □

This result will be used below without comment.

By analogy with Theorem 2.7 in [1], we have:

Theorem 2.2. *An element*

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

in PO_n is not a product of nilpotents if and only if α satisfies one or both of the following:

(*i*) $1 \in A_1$, $n \in A_r$ *and* (*for all* i) $A_i = S_i$ *and* $|T_i| \leqq 1$,

(*ii*) $b_1 = 1$, $b_r = n$ *and all lower jumps of* α *are of length* 1 *at most.*

Proof. Suppose that α satisfies neither (i) and (ii). We distinguish four cases.

Case 1. $1 \notin A_1$, $b_1 \neq 1$. Here

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ 1 & 2 & \dots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

a product of two nilpotents.

Case 2. $1 \in A_1$, $b_1 \neq 1$.

(*a*) if $n \notin A_r$, then

$$\alpha = n_1 n_2 n_3,$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \dots & A_{r-1} & A_r \\ n-r+1 & \dots & n-1 & n \end{pmatrix},$$

$$n_2 = \begin{pmatrix} n-r+1 & \dots & n-1 & n \\ 1 & \dots & r-1 & r \end{pmatrix} \text{ and } n_3 = \begin{pmatrix} 1 & \dots & r-1 & r \\ b_1 & \dots & b_{r-1} & b_r \end{pmatrix}.$$

(b) $n \in A_r$ and $A_i \neq S_i$ for some i. Then there exists $c \in S_i \backslash A_i$ (such that $x_i < c < y_i$) and

$$\alpha = n_1 n_2 n_3,$$

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where

$$n_1=\begin{pmatrix} A_1 & \dots & A_{i-2} & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ x_2 & \dots & x_{i-1} & x_i & c & y_i & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2=\begin{pmatrix} x_2 & \dots & x_i & c & y_i & \dots & y_{r-1} \\ 1 & \dots & i-1 & i & i+1 & \dots & r \end{pmatrix}$$

and

$$n_3=\begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & r \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

(c) $n\in A_r$ and $|T_i|\geqq 2$ for some i. Then there exists $c, d\in T_i$ with $c<d$, and

$$\alpha=n_1n_2n_3,$$

a product of three nilpotents, where

$$n_1=\begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2=\begin{pmatrix} y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r \end{pmatrix}$$

and

$$n_3=\begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & r \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

Case 3. $1\notin A_1$, $b_1=1$.

(a) $b_r\neq n$. Define $c_i=b_i+1$. Then

$$\alpha=\begin{pmatrix} A_1 & A_2 & \dots & A_r \\ 1 & 2 & \dots & r \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & r \\ c_1 & c_2 & \dots & c_r \end{pmatrix}\begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

a product of three nilpotents.

(b) $b_r=n$. Then α must have at least one lower jump of length greater than 1. Since $b_1=1$ we may suppose that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}. Define

$$c_i=\begin{cases} b_i+1 & \text{if } i\leqq k, \\ b_i-1 & \text{if } i>k. \end{cases}$$

Note that $c_{k+1}=b_{k+1}-1\geqq(b_k+3)-1=b_k+2>c_k$. Hence $c_i<c_{i+1}$ for all i, and

$$\alpha=\begin{pmatrix}A_1 & A_2 & \dots & A_t\\ 1 & 2 & \dots & r\end{pmatrix}\begin{pmatrix}1 & 2 & \dots & r\\ c_1 & c_2 & \dots & c_r\end{pmatrix}\begin{pmatrix}c_1 & c_2 & \dots & c_r\\ b_1 & b_2 & \dots & b_r\end{pmatrix},$$

a product of three nilpotents.

Case 4. $1\in A_1$, $b_1=1$.

(*a*) $n\notin A_r$, $b_r\neq n$. Define $c_i=\max\{y_i,b_i\}+1$ for all i. Then

$$\alpha=\begin{pmatrix}A_1 & A_2 & \dots & A_r\\ c_1 & c_2 & \dots & c_r\end{pmatrix}\begin{pmatrix}c_1 & c_2 & \dots & c_r\\ b_1 & b_2 & \dots & b_r\end{pmatrix},$$

a product of two nilpotents.

(b) $n\notin A_r$, $b_r=n$. Then α must have at least one lower jump of length greater than 1. We may suppose that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}. Define

$$c_i=\begin{cases}b_i+1 & \text{if } 1\leqq i\leqq k,\\ b_i-1 & \text{if } i>k.\end{cases}$$

Then

$$\alpha=\begin{pmatrix}A_1 & \dots & A_r\\ n-r+1 & \dots & n\end{pmatrix}\begin{pmatrix}n-r+1 & \dots & n\\ c_1 & \dots & c_r\end{pmatrix}\begin{pmatrix}c_1 & \dots & c_r\\ b_1 & \dots & b_r\end{pmatrix},$$

a product of three nilpotents.

(c) $n\in A_r$, $b_r\neq n$.

(i) $A_i\neq S_i$ for some i. Then there exists c in $S_i\backslash A_i$ (such that $x_i<c<y_i$), and

$$\alpha=n_1n_2n_3,$$

a product of three nilpotents, where

$$n_1=\begin{pmatrix}A_1 & \dots & A_{i-1} & A_i & A_{i+1} & \dots & A_r\\ x_2 & \dots & x_i & c & y_i & \dots & y_{r-1}\end{pmatrix},$$

$$n_2=\begin{pmatrix}x_2 & \dots & x_1 & c & y_i & \dots & y_{r-1}\\ c_1 & \dots & c_{i-1} & c_i & c_{i+1} & \dots & c_r\end{pmatrix},\quad n_3=\begin{pmatrix}c_1 & c_2 & \dots & c_r\\ b_1 & b_2 & \dots & b_r\end{pmatrix}$$

and

$$c_j=\begin{cases}\max\{x_{j+1},b_j\}+1 & \text{if } 1\leqq j\leqq i-1\\ \max\{c,b_j\}+1 & \text{if } j=i,\\ \max\{y_{j-1},b_j\}+1 & \text{if } j>i.\end{cases}$$

(ii) $|T_i| \geqq 2$ for some i. Then there exists $c, d \in T_i$ with $c < d$ and

$$\alpha = n_1 n_2 n_3,$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \\ c_1 & \dots & c_{i-1} & c_i & c_{i+1} & c_{i+2} & \dots & c_r \end{pmatrix},$$

$$n_3 = \begin{pmatrix} c_1 & \dots & c_{i-1} & c_i & c_{i+1} & \dots & c_r \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}$$

and

$$c^j = \begin{cases} \max\{x_{j+1}, b_j\} + 1 & \text{if } 1 \leqq j \leqq i-1, \\ \max\{c, b_j\} + 1 & \text{if } j = i, \\ \max\{d, b_j\} + 1 & \text{if } j = i+1, \\ \max\{y_{j-1}, b_j\} + 1 & \text{if } j > i+1. \end{cases}$$

(d) $n \in A_r$, $b_r = n$. Then α has at least one lower jump of length greater than 1, and either $A_i \neq S_i$ for some i or $|T_i| \geqq 2$ for some i. We may assume that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}. Define

$$c_j = \begin{cases} b_j + 1 & \text{if } 1 \leqq j \leqq k, \\ b_j - 1 & \text{if } j > k. \end{cases}$$

Then

$$\alpha = n_1 n_2 n_3 n_4,$$

where

$$n_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r \end{pmatrix},$$

$$n_3 = \begin{pmatrix} 1 & 2 & \dots & r \\ c_1 & c_2 & \dots & c_r \end{pmatrix}, \quad n_4 = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

$c \in S_i \backslash A_i$ and $d = y_i$ if $A_i \neq S_i$ for some i, or $c, d \in T_i$ if $|T_i| \geqq 2$ for some i (with $c < d$).

Conversely, suppose that α satisfies condition (i). Without loss of generality we may assume that α is expressible as a product

$$\alpha = n_1 n_2 \dots n_k$$

of k nilpotents with

$$n_1 = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix}.$$

We must first show by induction that $c_i > y_i$ for all i. The result is clearly true for $i = 1$. So suppose that it is true for all $i \leqq k$ and that $c_{k+1} < y_{k+1}$. Then since $A_{k+1} = S_{k+1}$ we must have $c_{k+1} < x_{k+1}$. Thus $y_k < c_k < c_{k+1} < x_{k+1}$. But this will mean $|T_k| \geqq 2$, which is a contradiction. So $c_i > y_i$ for all i. In particular we have $c_r > y_r = n$, and so c_r does not exist. Hence α is not a product of nilpotents.

Suppose that α satisfies (ii) and α is expressible as a product $\alpha = n_1 n_2 \dots n_k$ of k nilpotents. We may then assume that

$$n_k = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

where $\{c_1, \dots, c_r\} = \operatorname{im} n_{k-1}$. We will begin by showing inductively that $c_i \geqq b_i + 1$ for all i. The result is clearly true for $i = 1$. So suppose that it is true for all $i \leqq k$ and that $c_{k+1} \leqq b_{k+1} - 1$. Then since all the lower jumps of α are of length 1 at most, we have $b_{k+1} \leqq b_k + 2$. Thus $c_{k+1} \leqq b_{k+1} - 1 \leqq b_k + 1 \leqq c_k$. This is impossible. So $c_i \geqq b_i + 1$ for all i. In particular we have $c_r \geqq b_r + 1 = n + 1$, and so c_r does not exist. Hence α is not a product of nilpotents.

The next result is analogous to Theorem 2.8 in [**1**].

Theorem 2.3 *The set*

$$A = \{\alpha \in PO_n : |\operatorname{im} \alpha| \leqq p \text{ and } |X_n \backslash \operatorname{dom} \alpha| \geqq p\}$$

is contained in $\langle N \rangle$ *if and only if* $p \leqq \frac{1}{2}n$.

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \in A,$$

and suppose that $p \leqq \frac{1}{2}n$. Then by Theorem 2.2, to show that $\alpha \in \langle N \rangle$ we are required to prove the following:

(i) If $1 \in A_1$, $n \in A_r$, then for some i it is the case that $A_i \neq S_i$ or $|T_i| \geqq 2$.

(ii) If $b_1 = 1$, $b_r = n$, then α has a lower jump of length greater than 1.

So suppose by way of contradiction that $1 \in A_1$, $n \in A_r$ and that there exists no i for which $A_i \neq S_i$ or $|T_i| \geqq 2$. Then $X_n \backslash \operatorname{dom} \alpha = \bigcup_{i=1}^{r-1} T_i$, and

$$r \leqq |X_n \backslash \operatorname{dom} \alpha| = \sum_{i-1}^{r-1} |T_i| \leqq r-1 \leqq p-1.$$

This is a contradiction; thus α satisfies (i).

Now, suppose that $b_1 = 1$, $b_r = n$ and that all lower jumps of α are of length at most 1. Then $j_*(\alpha) \leqq r-1 \leqq p-1$. Also $n = b_r = r + j_*(\alpha)$ and so

$$j_*(\alpha) = n - r \geqq n - p \geqq p \text{ (since } p \leqq \tfrac{1}{2}n).$$

This is also a contradiction; thus α satisfies (ii).

To complete the proof of the theorem, we now show that if $r > n/2$, then there exists $\alpha \in A$ such that $\alpha \notin \langle N \rangle$.

Consider an element α for which $|\operatorname{im} \alpha| = r \geqq n/2 + 1$ and $X_n \backslash \operatorname{im} \alpha = \{2, 4, \ldots, 2s\}$, where $s = n - r$. Then we have

$$2s = 2(n-r) \leqq 2n - (n+2) = n - 2,$$

from which we can conclude that $n \in \operatorname{im} \alpha$, and thus $b_r = n$. It is clear that $b_1 = 1$ and that all lower jumps of α are of length 1. Hence α satisfies condition (ii) in Theorem 3.2. So α is not a product of nilpotents. □

3. The depth of the nilpotent-generated subsemigroup

By the proof of Theorem 2.2 we can express α in $\langle N \rangle$ as a product of at most four nilpotents, with elements having $1 \in A_1$, $n \in A_r$, $b_1 = 1$, $b_r = n$ expressible as a product of exactly four nilpotents. As in [1] we now show that even such elements can be expressed as a product of two or three nilpotents.

Proposition 3.1. *Let α in $\langle N \rangle$ be such that $1 \in A_1$, $n \in A_r$, $b_1 = 1$ and $b_r = n$. Then α is expressible as a product of at most three nilpotents.*

Proof. By Theorem 2.2 there exists i for which $A_i \neq S_i$ or $|T_i| \geqq 2$, and α has a lower jump of length greater than 1. We will assume that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}.

Let $c \in S_i \backslash A_i$ or $c = \min\{x : x \in T_i\}$, and $d \in T_i$ with $d \neq c$. We first show inductively that $c - i + j > y_i$ if $1 \leqq j \leqq i-1$ and $c - i + j < x_j$ if $j > i$. The results are true respectively for $j = i-1$ and $j = i+1$, since $y_{i-1} < x_i \leqq c-1$ and $c+1 \leqq (y_i \text{ or } d) < x_{i+1}$. Suppose that they are true (respectively) for $j = s \leqq i-1$ and $j = t > i$; that is, $y_s < c - i + s$ and $x_t > c - i + t$.

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Then $y_{s-1} \leqq y_s - 1 < c-i+s-1$ and $c-i+t+1 < x_t + 1 \leqq x_{t+1}$, as required. Next we show that $b_k - k + j + 1 > b_j$ if $1 \leqq j \leqq k$ and $b_k - k + j + 1 < b_j$ if $j > k$. For $j = k$ and $k+1$ we have $b_k + 1 > b_k$ and $b_k + 2 < b_{k+1}$. So suppose that the results are true for $j = s \leqq k$ and $j = t \geqq k+1$, that is $b_k - k + s + 1 > b_s$ and $b_k - k + t + 1 < b_t$. Then $b_k - k + s > b_s - 1 \geqq b_{s-1}$ and $b_k - k + t + 2 < b_t + 1 \leqq b_{t+1}$.

We now distinguish two cases.

Case 1. $c-i+k = b_k + 1$. Then $c-i+j = b_k - k + j + 1$ for all $j = 1, \ldots, r$ and

$$\alpha = n_1 n_2,$$

a product of two nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \ldots & A_k & A_{k+1} & \ldots & A_r \\ b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r + 1 \end{pmatrix},$$

and

$$n_2 = \begin{pmatrix} b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r + 1 \\ b_1 & \ldots & b_k & b_{k+1} & \ldots & b_r \end{pmatrix}.$$

Case 2. $c-i+k \neq b_k + 1$. Then $c-i+j \neq b_k - k + j + 1$ for all $j = 1, \ldots, r$ and

$$\alpha = n_1 n_2 n_3,$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \ldots & A_k & A_{k+1} & \ldots & A_r \\ c-i+1 & \ldots & c-i+k & c-i+k+1 & \ldots & c-i+r \end{pmatrix},$$

$$n_2 = \begin{pmatrix} c-i+1 & \ldots & c-i+k & c-i+k+1 & \ldots & c-i+r \\ b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r + 1 \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r + 1 \\ b_1 & \ldots & b_k & b_{k+1} & \ldots & b_r \end{pmatrix} \qquad \square$$

The following Theorem now follows from Proposition 3.1 above and Theorem 3.3 in **[1]**.

Theorem 3.2. *Let N be the set of all nilpotents in PO_n, $\langle N \rangle$ the subsemigroup of PO_n generated by the nilpotent elements, and $\Delta(\langle N \rangle)$ the unique k for which*

$$\langle N\rangle = N\cup N^2\cup\cdots\cup N^k,\quad \langle N\rangle \neq N\cup N^2\cup\cdots\cup N^{k-1}.$$

Then $\Delta(\langle N\rangle)=3$ *for all* $n\geqq 3$. □

4. The nilpotent rank

An element α in PO_n, and indeed in the larger semigroup P_n of all partial transformations of X_n, is said to have *projection characteristic* (k,r) or to belong to the set $[k,r]$ if $|\mathrm{dom}\,\alpha|=k$ and $|\mathrm{im}\,\alpha|=r$. We use the standard notation

$$J_r=\{\alpha\colon |\mathrm{im}\,\alpha|=r\}=\bigcup_{r\leqq k\leqq n}[k,r].$$

Lemma 4.1. *Every element* $\alpha\in\langle N\rangle\cap J_r$, where $r\leqq n-3$, *is expressible as a product of elements in* $\langle N\rangle\cap J_{r+1}$.

Proof. Let

$$\alpha=\begin{pmatrix} A_1 & A_2 & \dots & A_r\\ b_1 & b_2 & \dots & b_r\end{pmatrix}$$

be an element in $\langle N\rangle$ such that $|\mathrm{im}\,\alpha|=r\leqq n-3$. From Proposition 4.1 in [**1**], if $\alpha\in\langle N\rangle\cap[r,r]$ then α can be expressed as a product of two elements in $\langle N\rangle\cap[r+1,r+1]$. We will therefore assume that $\alpha\in\langle N\rangle\cap[k,r]$, $r+1\leqq k\leqq n-1$.

By Theorem 2.2, since $\alpha\in\langle N\rangle$ then at least one of the following holds:

(i) $1\notin A_1$ (that is, $|T_0|\geqq 1$);

(ii) $n\notin A_r$ (that is, $|T_r|\geqq 1$);

(iii) $A_i\neq S_i$ for some i such that $1\leqq i\leqq r-1$;

(iv) $|T_2|\geqq 2$ for some i such that $1\leqq i\leqq r-1$.

Suppose that (i) or (ii) or (iv) holds. Then

$$\alpha=\gamma_1\gamma_2\gamma_3,$$

where

$$\gamma_1=\begin{pmatrix} A_1 & \dots & A_{j-1} & x_j & A_j\backslash\{x_j\} & A_{j+1} & \dots & A_r\\ 1 & \dots & j-1 & j & j+1 & j+2 & \dots & r+1\end{pmatrix},$$

$$\gamma_2=\begin{pmatrix} 1 & \dots & j-1 & \{j,j+1\} & j+2 & \dots & r+1 & r+2\\ 2 & \dots & j & j+2 & j+3 & \dots & r+2 & r+3\end{pmatrix},$$

$$\gamma_3=\begin{pmatrix} 2 & \dots & j & j+2 & j+3 & \dots & r+2 \\ b_1 & \dots & b_{j-1} & b_j & b_{j+1} & \dots & b_r \end{pmatrix},$$

and it is assumed that $|A_j|\geqq 2$, $x_j=\min\{x: x\in A_j\}$. Observe that $\gamma_3\in\langle N\rangle$ by Theorem 2.7 in [1], and that γ_2 is nilpotent by Lemma 2.1. Further,since (i) or (ii) or (iv) holds and $r+1\neq n$, it follows from Theorem 2.2 that $\gamma_1\in\langle N\rangle$. Finally, since $\gamma_3\in\langle N\rangle\cap[r,r]$, γ_3 can be expressed as a product of two elements in $\langle N\rangle\cap[r+1, r+1]$, by [1, Proposition 4.1]. Thus α is expressible as a product of (four) elements in $\langle N\rangle\cap J_{r+1}$.

Now suppose that (iii) holds: that is, $A_i\neq S_i$ for some i. Consider first the case where $k<n-1$. Then we may assume that there exists $x\in X_n\backslash\operatorname{dom}\alpha$ such that $y_j<x<y_{j+1}$ for some j, where $y_t=\max\{x: x\in A_t\}$. Here we have

$$\alpha=\beta_1\beta_2,$$

where

$$\beta_1=\begin{pmatrix} A_1 & \dots & A_j & x & A_{j+1} & \dots & A_r \\ 1 & \dots & j & j+1 & j+3 & \dots & r+2 \end{pmatrix},$$

$$\beta_2=\begin{pmatrix} 1 & \dots & j & j+3 & \dots & r+2 \\ b_1 & \dots & b_j & b_{j+1} & \dots & b_r \end{pmatrix}.$$

Observe here too, that β_2 belongs to $\langle N\rangle$ and can be expressed as a product of two elements of $\langle N\rangle\cap[r+1, r+1]$ by [1, Proposition 4.1]. Also, since $A_i\neq S_i$ for some i and $r+2\neq n$, we have $\beta_1\in\langle N\rangle$ by Theorem 2.2.

Now consider the case where $k=n-1$. Then it is clear that $|A_i|\geqq 2$. If $|A_i|=2$ then there exists another block, say A_k, such that $|A_k|\geqq 2$ (since $r\leqq n-3$ by hypothesis), and

$$\alpha=\delta_1\delta_2\delta_3,$$

where

$$\delta_1=\begin{pmatrix} A_1 & \dots & A_{k-1} & x_k & A_k\backslash\{x_k\} & A_{k+1} & \dots & A_r \\ 1 & \dots & k-1 & k & k+1 & k+2 & \dots & r+1 \end{pmatrix},$$

$$\delta_2=\begin{pmatrix} 1 & \dots & k-1 & \{k,k+1\} & k+2 & \dots & r+2 \\ 2 & \dots & k & k+2 & k+3 & \dots & r+3 \end{pmatrix}$$

and

$$\delta_3=\begin{pmatrix} 2 & \dots & k & k+2 & k+3 & \dots & r+2 \\ b_1 & \dots & b_{k-1} & b_k & b_{k+1} & \dots & b_r \end{pmatrix}.$$

Note that $\delta_1\in\langle N\rangle$ by Theorem 2.2. Also $\delta_2\in\langle N\rangle$ by Lemma 2.1, and δ_3 is expressible as the product of two elements in $\langle N\rangle\cap[r+1, r+1]$, by [1, Proposition 4.1]. If $|A_i|>2$

then there exists $a_i \in A_i$ and $s_i \in S_i \setminus A_i$ such that either $x_i < a_i < s_i < y_i$ or $x_i < s_i < a_i < y_i$. If $x_i < a_i < s_i < y_i$ then

$$\alpha = \lambda_1 \lambda_2 \lambda_3,$$

where

$$\lambda_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & x_i & A_i \setminus \{x_i\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r+1 \end{pmatrix},$$

$$\lambda_2 = \begin{pmatrix} 1 & \dots & i-1 & \{i, i+1\} & i+2 & \dots & r+2 \\ 2 & \dots & i & i+2 & i+3 & \dots & r+3 \end{pmatrix}$$

and

$$\lambda_3 = \begin{pmatrix} 2 & \dots & i & i+2 & i+3 & \dots & r+2 \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

If $x_i < s_i < a_i < y_i$ then

$$\alpha = \lambda_1 \lambda_2 \lambda_3,$$

where

$$\lambda_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i \setminus \{y_i\} & y_i & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r+1 \end{pmatrix},$$

and where λ_2 and λ_3 are defined as before. Note that by the same argument as in previous cases, λ_1, λ_2, $\lambda_3 \in \langle N \rangle$ and λ_3 can be expressed as a product of two elements of $\langle N \rangle \cap [r+1, r+1]$. □

Let N_1 and N_2 be the set of all nilpotent elements in PO_n in J_{n-1} and in J_{n-2} respectively. Then, since all the elements in N_1 are one–one maps, we have by Proposition 4.2 in [**1**] that N_1 does not generate $\langle N \rangle$. However, by Lemma 4.1 above we do have

$$\langle N_2 \rangle = \langle N \rangle \setminus J_{n-1}.$$

Our aim here is to determine the rank and the nilpotent rank of $\langle N_2 \rangle$.

First, notice that from Theorem 2.2 it is easy to verify that $\langle N \rangle$ is regular. Hence by [**6**, Proposition II.4.5] two elements of $\langle N \rangle$ are $\mathscr{L}$-equivalent in $\langle N \rangle$ if and only if they have the same image, and are $\mathscr{R}$-equivalent in $\langle N \rangle$ if and only if they have the same kernel. This applies also to $\langle N_2 \rangle = \langle N \rangle \setminus J_{n-1}$, since every element of $\langle N \rangle \setminus J_{n-1}$ has an inverse in $\langle N \rangle \setminus J_{n-1}$, and so $\langle N_2 \rangle$ is again regular.

Now recall from [**1**, Section 4] that the number of $\mathscr{R}$-classes and that of $\mathscr{L}$-classes containing nilpotents, or elements that are expressible as products of nilpotents, in a $\mathscr{J}$-class, J_r of IO_n, where $n/2 < r \leqq n-2$ (notice in passing that $n/2 < n-2$ if and only if

$n \geqq 5$) are both equal to $\binom{n}{r}-\binom{r-1}{n-r}$. It therefore follows that the number of $\mathscr{R}$-classes in $\langle N_2\rangle \cap [n-2,\ n-2]$ is equal to the number of $\mathscr{L}$-classes in $\langle N_2\rangle \cap J_{n-2}$ and is $\binom{n}{n-2}-\binom{n-3}{2}=3(n-2)$.

Following [5], we shall refer to an equivalence ρ on the set X_n as *convex* if its classes are convex subsets A of X_n, where a convex subset of X_n means a subset A for which

$$x, y \in A \text{ and } x \leqq z \leqq y \Rightarrow z \in A.$$

By Theorem 2.2 any convex equivalence having $n-2$ classes on the subset $\{1,\ldots,n-1\}$ or $\{2,\ldots,n\}$ determines an $\mathscr{R}$-classes in $\langle N_2\rangle \cap [n-1,\ n-2]$. Thus the number of $\mathscr{R}$-classes in $\langle N_2\rangle \cap [n-1, n-2]$ determined by these convex equivalences is $2(n-2)$. On the other hand any convex equivalence having $n-2$ classes on a subset containing 1 and n represents an $\mathscr{R}$-class in $\langle N_2\rangle \cap [n-1, n-2]$ if and only if i and $i+2$ belong to the same equivalence class for some i in $\{1,\ldots,n-2\}$. This follows from Theorem 2.2, because $|T_i| \geqq 2$ is not possible for an element of $[n-1, n-2]$ and so the only possibility for such an α to be in $\langle N\rangle$ is for some A_i to be distinct from S_i. Thus the number of such convex equivalences is $n-2$. Hence the number of $\mathscr{R}$-classes in $\langle N_2\rangle \cap [n-1, n-2]$ is $3(n-2)$. We therefore have $6(n-2)$ as the number of $\mathscr{R}$-classes in $\langle N_2\rangle \cap J_{n-2}$.

We now show that every element $\alpha \in \langle N_2\rangle \cap [n-1, n-2]$ is expressible in terms of a fixed element in its own $\mathscr{R}$-class and an element in $\langle N_2\rangle \cap [n-2, n-2]$. More generally we shall show:

Lemma 4.2. *Every element $\alpha \in \langle N_2\rangle \cap [k,r]$, $r<k\leqq n-1$ is expressible as a product of a nilpotent in $\langle N_2\rangle \cap [k,r]$ and an element in $\langle N_2\rangle \cap [r,r]$.*

Proof. Let $\alpha \in \langle N_2\rangle \cap [k,r]$ and suppose that

$$\alpha=\begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}.$$

We shall distinguish four cases.

Case 1. $1 \notin A_1$. Then

$$\alpha=\begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ 1 & 2 & \ldots & r \end{pmatrix}\begin{pmatrix} 1 & 2\ldots & r \\ b_1 & b_2\ldots & b_r \end{pmatrix}.$$

Case 2. $n \notin A_r$. Then

$$\alpha=\beta\gamma$$

where

$$\beta=\begin{pmatrix} A_1 & A_2 & \ldots & A_{r-1} & A_r \\ n-r+1 & n-r+2 & \ldots & n-1 & n \end{pmatrix},$$

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$$\gamma=\begin{pmatrix} n-r+1 & n-r+2 & \dots & n-1 & n \\ b_1 & b_2 & \dots & b_{r-1} & b_r \end{pmatrix}.$$

That $\gamma\in\langle N\rangle$ follows from [**1**, Theorem 2.6].

Case 3. $1\in A_1$, $n\in A_r$ and $A_i\neq S_i$ for some i. Let c be a fixed element in $S_i\backslash A_i$. Then

$$\alpha=\lambda\mu$$

where

$$\lambda=\begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & \dots & A_r \\ x_2 & \dots & x_i & c & y_i & \dots & y_{r-1} \end{pmatrix},$$

$$\mu=\begin{pmatrix} x_2 & \dots & x_i & c & y_i & \dots & y_{r-1} \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

The latter element is in $\langle N\rangle$ by [**1**, Theorem 2.6].

Case 4. $1\in A_1$, $n\in A_r$, $A_i=S_i$ for all i and $|T_i|\geqq 2$ for some i. Let c, d be two fixed elements in T_i with $c<d$. Then

$$\alpha=\zeta\xi$$

where

$$\zeta=\begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_t \\ y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$\xi=\begin{pmatrix} y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & b_{i+2} & \dots & b_r \end{pmatrix}.$$

□

Theorem 4.3. *Let $n\geqq 5$. Then rank $(\langle N_2\rangle)=6(n-2)$.*

Proof. Since $\langle N_2\rangle\cap J_{n-2}$ has $6(n-2)$ $\mathscr{R}$-classes we have

$$\text{rank }(\langle N_2\rangle)\geqq 6(n-2).$$

By Proposition 2.4 in [**2**], $[n-2,n-2]\cap\langle N_2\rangle$ is generated by a set of $3(n-2)$ elements. If we now choose a set of $3(n-2)$ elements to cover the $\mathscr{R}$-classes in $[n-1,n-2]$ as in Lemma 4.2, we obtain a generating set of $\langle N_2\rangle$ consisting of $6(n-2)$ elements. The result follows. □

Lemma 4.4. *Every $\mathscr{L}$-class in J_{n-2} whose elements have image*

$$\{1,2,\ldots,i-1,\ i+2,\ldots,n\}$$

for $i=2,\ldots,n-2$ *contains a single nilpotent. Thus there are at least* $n-3$ $\mathscr{L}$*-classes in* J_{n-2} *containing only one nilpotent.*

Proof. Let α be an element whose $\mathscr{L}$-class is represented by $\{1,\ldots,i-1,i+2,\ldots,n\}$. Then the only domain for which α is nilpotent is that represented by the set $\{2,\ldots,n-1\}$. $\square$

Theorem 4.5. $\operatorname{nilrank}(\langle N_2\rangle)=7n-15$.

Proof. Since any generating set of $\langle N_2\rangle$ must cover the $\mathscr{L}$-classes in $\langle N_2\rangle\cap J_{n-2}$, the $n-3$ nilpotents whose image set is $\{1,\ldots,i-1,i+2,\ldots,n\}$ for $i=2,\ldots,n-2$ must be contained in a generating set consisting of only nilpotent elements (see Lemma 4.4). By the same Lemma 4.4. (proof) all the $n-3$ nilpotents belong to the same $\mathscr{R}$-class, determined by the set $\{2,\ldots,n-1\}$. For the generating set to cover all the $\mathscr{R}$-classes we must now choose $6(n-2)-1$ nilpotents from the remaining $\mathscr{R}$-classes, making a total of $7n-16$ nilpotents. However the $7n-16$ nilpotents cannot generate $\langle N_2\rangle$. For if α is an element in the same $\mathscr{R}$-class as the $n-3$ nilpotents (that is the $\mathscr{R}$-class represented by the set $\{2,\ldots,n-1\}$) and if we suppose that

$$\alpha=n_1n_2\cdots n_k$$

is the decomposition of α in terms of nilpotents from the chosen $7n-16$ nilpotents, then we must have

$$n_1=\begin{pmatrix}2 & 3 & \ldots & i & i+1 & \ldots & n-1\\ 1 & 2 & \ldots & i-1 & i+2 & \ldots & n\end{pmatrix},$$

$$n_2=\begin{pmatrix}1 & 2 & \ldots & i-1 & i+2 & \ldots & n\\ 2 & 3 & \ldots & i & i+1 & \ldots & n-1\end{pmatrix}$$

and

$$n_3=\begin{pmatrix}2 & 3 & \ldots & j & j+1 & \ldots & n-1\\ 1 & 2 & \ldots & j-1 & j+2 & \ldots & n\end{pmatrix}$$

for some i, $j=2,\ldots,n-2$. But then n_1n_2 is a left identity for n_3, and so

$$\alpha=n_3n_4\cdots n_k.$$

By the same reasoning we must also have

$$n_4=\begin{pmatrix}1 & 2 & \ldots & j-1 & j+2 & \ldots & n\\ 2 & 3 & \ldots & j & j+1 & \ldots & n-1\end{pmatrix}$$

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and

$$n_5 = \begin{pmatrix} 2 & 3 & \dots & l & l+1 & \dots & n-1 \\ 1 & 2 & \dots & l-1 & l+2 & \dots & n \end{pmatrix}.$$

But again $n_3 n_4$ is then a left identity for n_5, and

$$\alpha = n_5 \cdots n_k.$$

Continuing this way we obtain

$$\alpha = \begin{cases} n_k & \text{if } k \text{ is odd,} \\ \begin{pmatrix} 2 & 3 & \dots & n-1 \\ 3 & 3 & \dots & n-1 \end{pmatrix} & \text{if } k \text{ is even.} \end{cases}$$

Thus if α is not any of the $n-3$ nilpotents in its $\mathscr{R}$-class, and is not the left identity in the $\mathscr{R}$-class, then α cannot be expressed as a product of nilpotents from the chosen $7n-16$ nilpotents. We therefore have

$$\operatorname{nilrank}(\langle N_2 \rangle) \geqq 7n - 15.$$

We now show that we can choose $7n-15$ nilpotents in N_2 that can generate $\langle N_2 \rangle$. Denote by $A_{i,j}$ the subset $X_n \backslash \{i, j\}$ of cardinality $n-2$, and by $\alpha_{s,t}^{i,j}$ the element whose domain is $A_{i,j}$ and image $A_{s,t}$. Then arrange the $3(n-2)$ subsets of X_n of cardinality $n-2$, representing the $\mathscr{L}$- and the $\mathscr{R}$-classes in $\langle N_2 \rangle \cap [n-2, n-2]$ as follows:

$$A_{2,n}, A_{1,3}, A_{3,n}, \dots, A_{1,i}, A_{i,n}, \dots, A_{1,n-1}, A_{n-1,n}, A_{1,n}, A_{2,3}, A_{3,4}, \dots, A_{n-2,n-1}, A_{1,2}.$$

By [2, Proposition 2.4], $\langle N_2 \rangle \cap [n-2, n-2]$ is generated by the set

$$B = \{\alpha_{1,3}^{2,n}, \alpha_{3,n}^{1,3}, \alpha_{1,4}^{3,n}, \dots, \alpha_{i,n}^{1,i}, \alpha_{1,i+1}^{i,n}, \dots, \alpha_{n-1,n}^{1,n-1}, \alpha_{1,n}^{n-1,n},$$
$$\alpha_{2,3}^{1,n}, \alpha_{3,4}^{2,3}, \dots, \alpha_{n-2,n-1}^{n-3,n-2}, \alpha_{1,2}^{n-2,n-1}, \alpha_{2,n}^{1,2}\}.$$

It is easy to see that $\alpha_{i,n}^{1,i}$, $\alpha_{1,i+1}^{i,n}$ (for $i = 3, \dots, n-1$), $\alpha_{1,3}^{2,n}$, $\alpha_{2,3}^{1,n}$ and $\alpha_{2,n}^{1,2}$ are all nilpotents. It is also not difficult to see that

$$\alpha_{3,4}^{2,3}, \dots, \alpha_{n-2,n-1}^{n-3,n-2}, \alpha_{1,2}^{n-2,n-1} \tag{4.6}$$

are all non-nilpotent. In fact n is fixed by all of these elements. Let us denote by B' the set of all nilpotent elements in B. Let T be the set of $4(n-2)-1$ elements given by

$$T = B' \cup \{\alpha_{3,4}^{1,n}, \dots, \alpha_{n-2,n-1}^{1,n}, \alpha_{1,2}^{1,n}, \alpha_{1,n}^{2,3}, \dots, \alpha_{1,n}^{n-2,n-1}\}.$$

It is easy here too, to see that all the elements in T are nilpotents. Next we observe that the non-nilpotent elements in B, given by (4.6) are expressible as products of elements in T. In fact we have

$$\alpha^{i,i+1}_{i+1,i+2}=\alpha^{i,i+1}_{1,n}\alpha^{1,n}_{i+1,i+2} \quad \text{for } i=2,\dots,n-3$$

and

$$\alpha^{n-2,n-1}_{1,2}=\alpha^{n-2,n-1}_{1,n}\alpha^{1,n}_{1,2}.$$

Thus

$$\langle B\rangle=\langle T\rangle.$$

If we now choose a set H of $3(n-2)$ nilpotents to cover the $\mathscr{R}$-classes in $\langle N_2\rangle\cap[n-1,n-2]$ as in Lemma 4.2 we obtain a generating set $H\cup T$ of $\langle N_2\rangle$ consisting of nilpotent elements. Since $|H\cup T|=7n-15$ the proof is complete. $\square$

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REFERENCES

1. G. U. Garba, Nilpotents in semigroups of partial one–one order-preserving mappings, *Semigroup Forum* **48** (1994), 37–49.

2. G. U. Garba, On the nilpotent rank of partial transformation semigroups, *Portugaliae Math.* **51** (1994), 163–172.

3. Gracinda M. S. Gomes and John M. Howie, Nilpotents in finite symmetric inverse semigroups, *Proc. Edinburgh Math. Soc.* **30** (1987), 383–395.

4. Gracinda M. S. Gomes, and John M. Howie, On the ranks of certain finite semigroups of transformations, *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 395–403.

5. Gracinda M. S. Gomes and John M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* **45** (1992), 272–282.

6. John M. Howie, *An introduction to semigroup theory* (Academic Press, London, 1976).

7. R. P. Sullivan, Semigroups generated by nilpotent transformations, *J. Algebra* **110** (1987), 324–345.

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