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THE NATURAL PARTIAL ORDER ON AN ABUNDANT SEMIGROUP

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In this paper we will study the properties of a natural partial order which may be defined on an arbitrary abundant semigroup: in the case of regular semigroups we recapture the order introduced by Nambooripad [24]. For abelian *PP* rings our order coincides with a relation introduced by Sussman [25], Abian [1, 2] and further studied by Chacron [7]. Burmistrovič [6] investigated Sussman's order on separative semigroups. In the abundant case his order coincides with ours: some order theoretic properties of such semigroups may be found in a paper by Burgess [5].

Many properties and constructions on abundant semigroups may be described in terms of its natural partial order: one of the main results of Section 2 is the connection we establish between idempotent connectedness and the partial order being, in some sense, self-dual. In Section 3 we extend Nambooripad's Theorem 3.3 [24] and show that the order is compatible with the multiplication on a concordant semigroup just when the semigroup is locally type A. In Section 4 we obtain a description of the finest 0-restricted primitive good congruence on a concordant semigroup. Section 1 is a preliminary section in which we consider, in particular, the behaviour of good homomorphisms between abundant semigroups and obtain a generalisation of Lallement's Lemma.

1. Preliminaries

We will assume some familiarity with the contents of [11] and [15]: the only divergence from the terminology established there is that we prefer to call *-ideals good ideals, bringing them in line with the good homomorphisms.

The class of *idempotent connected* (IC) abundant semigroups was introduced by El-Qallali and Fountain [11]. We begin this section with an alternative characterisation of these semigroups due to Fountain, which, for the purposes of this paper, may serve as an alternative definition.

Proposition 1.1. *Let S be abundant, then the following properties are equivalent:*

(a) *S is IC.*

(b) *For each element a of S two conditions hold:*

  (i) *For some a^* and $e \in \omega(a^*)$ there exists an element $b \in S$ such that $ae = ba$.*

  (ii) *For some a^+ and $f \in \omega(a^+)$ there exists an element $c \in S$ such that $fa = ac$.*

(c) *For each element a of S two conditions hold:*

(i) *For some* a^* *and* $e \in \omega(a^*)$ *there exists an idempotent* $f \in \omega(a^+)$ *such that* $ae = fa$.

(ii) *For some* a^+ *and* $h \in \omega(a^+)$ *there exists an idempotent* $g \in \omega(a^+)$ *such that* $ha = ag$.

Proof. (a) implies (b). Take $b = e\alpha^{-1}$ and $c = f\alpha$.

(b) implies (c). Choose idempotents a^* and a^+ and let $f\,\omega\,a^+$. Then by (b) there exists an element c, with $fa = ac$. Now,

$$ac = fa = f^2 a = f(fa) = f(ac) = (fa)c = acc = ac^2,$$

so $a^* = a^* c^2$. Also,

$$ac = fa = faa^* = aca^* \quad \text{so} \quad a^*c = a^*ca^*.$$

So $a^*c^2 = a^*c \cdot a^*c$. Hence $a^*c = (a^*c)^2$ is an idempotent. Now a^*ca^* is an idempotent with $a^*ca^*\,\omega\,a^*$ thus $a^*ca^* \in \langle a^* \rangle$. Since $fa = ac = aca^* = aa^*ca^*$, by putting $e = a^*ca^*$ we have that there exists an element e with $e\,\omega\,a^*$ and $fa = ae$.

(c) implies (a). Let x be an element of $\langle a^+ \rangle$. Then x takes the form, $x = f_k \dots f_1$ where each f_i is an idempotent with $f_i\,\omega\,a^+$. Therefore, $xa = f_k \dots f_1 a = f_k \dots f_2 a e_1 = a e_k \dots e_1$ where each $e_i^2 = e_i\,\omega\,a^*$. If $xa = ay_1 = ay_2$ where $y_1, y_2 \in \langle a^* \rangle$ then $a^*y_1 = a^*y_2$ so $y_1 = y_2$. And if $x_1 a = x_2 a$ where x_1 and x_2 belong to $\langle a^+ \rangle$ then $x_1 a^+ = x_2 a^+$ so $x_1 = x_2$. This means that there is a one-to-one map $\alpha: \langle a^+ \rangle \to \langle a^* \rangle$ with $xa = a(x\alpha)$. Similarly there is a one-to-one map $\beta: \langle a^* \rangle \to \langle a^+ \rangle$ with $ay = (y\beta)a$. Now for any $y \in \langle a^* \rangle$, $ay = (y\beta)a = a(y\beta)\alpha$. Therefore $y = y\beta\alpha$, which entails that α is onto.

The next result, which is Result 2 of Hall **[16]**, establishes a useful link between the regularity of a product of arbitrary regular elements and that of idempotents. Note that we denote the set of regular elements of a semigroup S by Reg(S).

Proposition 1.2. *Let S be an arbitrary semigroup. Then the following are equivalent:*

(i) *For all idempotents e and f of S the element ef is regular.*

(ii) $\langle E(S) \rangle$ *is a regular subsemigroup.*

(iii) Reg(S) *is a regular subsemigroup.*

Any semigroup satisfying this proposition will be said to satisfy the *regularity condition*: not all abundant semigroups satisfy this condition. Example 3.1 of **[14]** exhibits an abundant and non-regular semigroup generated by its idempotents. In a semigroup satisfying the regularity condition, Nambooripad showed that the *sandwich set of e* and f, where e and f are idempotents takes the following form:

$$S(e,f) = \{h \in E(S) : he = h = fh \text{ and } ehf = ef\}.$$

We refer the reader to **[23**, Theorem 1.1 (a3)] for a proof. Our results will lean very heavily on the properties of the sandwich sets: many of which carry over to abundant semigroups satisfying the regularity condition without too much difficulty, due to the fact that for regular elements $\mathscr{L}^* = \mathscr{L}$ and $\mathscr{R}^* = \mathscr{R}$.

Proposition 1.3. *Let S be an abundant semigroup satisfying the regularity condition.*

(i) *Let $x, y \in S$ and choose idempotents x^* and y^+ and let h be an element of $S(x^*, y^+)$ then, $xy = (xh)(hy)$ and $xh\mathcal{L}^*h$ and $hy\mathcal{R}^*h$.*

(ii) *If $e\,\omega^r f$ then*

$$S(e,f) = \omega^r(f) \cap E(L_e^*) \quad \text{and} \quad S(f,e) = \omega(f) \cap E(R_e^*);$$

and if $e\,\omega^1 f$ then

$$S(e,f) = \omega(f) \cap E(L_e^*) \quad \text{and} \quad S(f,e) = \omega^1(f) \cap E(R_e^*).$$

(iii) *Let S and T be abundant semigroups satisfying the regularity condition and let $\theta: S \to T$ be a homomorphism. Then if $e, f \in E(S)$ and $h \in S(e,f)$ then $h\theta \in S(e\theta, f\theta)$ in T.*

Proof. We will prove (i) and leave (ii) and (iii) for the interested reader to verify. If $h \in S(x^*, y^+)$, then $x^*hy^+ = x^*y^+$ so that,

$$xy = (xx^*)(y^+y) = x(x^*y^+)y = x(x^*hy^+)y = (xx^*)h(y^+y) = xhy.$$

Now $xh\mathcal{L}^*x^*h$, since $\mathcal{L}^*$ is a right congruence, furthermore $h \in \omega^1(x^*)$ therefore $x^*h\mathcal{L}h$. This gives us $xh\mathcal{L}^*h$. We may similarly show that $hy\mathcal{R}^*h$.

To date, the most important class of abundant semigroups arises when we combine the regularity condition with idempotent connectedness. These are the *concordant semigroups*: those with a semilattice of idempotents are just the type A semigroups [**14**].

Lemma 1.4. *Let $\alpha: \langle a^+ \rangle \to \langle a^* \rangle$ be a connecting isomorphism in a concordant semigroup S. Let $a, b \in S$ and $h \in S(b^*, a^+)$ then $(ha^+)\alpha \in E(L_{ba}^*) \cap \omega(a^*)$.*

Proof. By Proposition 1.3, $h \in S(b^*, a^+)$ implies that $bh\mathcal{L}^*h$. Since $\mathcal{L}^*$ is a right congruence $ba\mathcal{L}^*ha$. Now $ha^+ \in \omega(a^+)$ which means that the element $(ha^+)\alpha$ is well-defined. By Lemma 3.3 of El-Qallali and Fountain [**11**], $(ha^+)a\mathcal{L}^*(ha^+)\alpha$ thus $ha\mathcal{L}^*(ha^+)\alpha$, giving $(ha^+)\alpha\mathcal{L}^*ba$. The result follows from the fact that $(ha^+)\alpha \in \omega(a^*)$.

It will be useful at this point to recall some notation and definitions concerning ordered sets we will need. If $(X, \leqq)$ is a poset, then a subset A of X is said to be *dense in X*, if for each element x of X there exists an element a of A such that $a \leqq x$. The subset A is said to be *directed* if for all elements x and y in A there exists an element z belonging to A such that $z \leqq x$ and $z \leqq y$. The set A is said to be an *order ideal* if for each element of a of A and any x with $x \leqq a$ then x belongs to A. For each element a of X put $[a] = \{x \in X : x \leqq a\}$, called the *principal order ideal generated by a*.

In any semigroup S subsemigroups of the form eSe where e is an idempotent are called *local submonoids*. If each local submonoid is adequate (type A, inverse) then S is said to be *locally adequate* (*type A, inverse*): local submonoids may inherit properties from the oversemigroup. In the following note that $E(eSe)$ is an order ideal of $E(S)$.

Proposition 1.5. *Let S be an abundant semigroup.*

(i) *Each local submonoid is abundant.*

(ii) *If S is IC then each local submonoid is IC.*

(iii) *If S satisfies the regularity condition then each local submonoid likewise satisfies the regularity condition.*

Proof. (i) Let a be an element of the local submonoid eSe and let f be an idempotent of S with $f\mathscr{L}^*(S)a$. Certainly, $ae=a$ so that $fe=f$. Now, $f\omega^1 e$ so that $ef\in E(S)$, $ef\omega e$ and $ef\mathscr{L}f$. Since $E(eSe)$ is an order ideal of $E(S)$, the element ef belongs to $E(eSe)$. Since $ef\mathscr{L}^*(S)a$, then $ef\mathscr{L}^*(eSe)a$. This implies that each element of eSe is $\mathscr{L}^*$-related in eSe to an idempotent likewise belonging to eSe. A similar result for $\mathscr{R}^*$ gives us the required abundancy.

(ii) This is straightforward.

(iii) Note that if f and g belong to $E(eSe)$ then $S(f,g)\subseteq eSe$ and apply Proposition 1.2.

The next sequence of results considers the behaviour of various classes of abundant semigroups under good homomorphisms. We begin with a generalisation of Lallement's Lemma due to Fountain (unpublished).

Theorem 1.6. *Let S be an abundant semigroup satisfying the regularity condition. Let $\alpha: S\to T$ be a good homomorphism into a semigroup T and let $a\alpha$ be an idempotent of T for some element a of S. Then there is an idempotent $h\in S$ such that $h\alpha=a\alpha$.*

Proof. From the fact that S is abundant we can find idempotents e and f in $E(S)$ with $f\mathscr{R}^*a\mathscr{L}^*e$. The mapping α is good so $f\alpha\mathscr{R}^*a\alpha\mathscr{L}^*a\alpha$. All the elements $f\alpha$, $a\alpha$ and $e\alpha$ are idempotents in T, meaning that we may remove the stars from the above expression, to obtain $f\alpha\mathscr{R}a\alpha\mathscr{L}e\alpha$. In particular, $(a\alpha)(f\alpha)=f\alpha$. By assumption, $S(e,f)$ is non-empty, choose an element h belonging to it. By Proposition 1.3(i),

$$a\alpha=(a^2)\alpha=(a\cdot a)\alpha=(ah)\alpha\cdot(ha)\alpha.$$

But,

$$(ah)\alpha=(a(fh))\alpha=(a\alpha)(f\alpha)(h\alpha)=(f\alpha)(h\alpha)=h\alpha.$$

We can similarly show that $(ha)\alpha=h\alpha$. Therefore,

$$a\alpha=(ah)\alpha\cdot(ah)\alpha=h\alpha\cdot h\alpha=h^2\alpha=h\alpha.$$

Unfortunately the regularity condition cannot easily be relaxed.

Example 1.7. From Corollary 3.5 of [15], $\mathscr{H}^*$ is a good congruence on primitive abundant semigroups. Hence $S/\mathscr{H}^*$ is abundant. Let $\alpha: S\to S/\mathscr{H}^*$ be the corresponding natural map. We will construct a primitive abundant semigroup S with an $\mathscr{H}^*$-class, denoted by H^* having the following properties:

(i) $H^* \cap E(S) \neq \emptyset$.

(ii) $H^*H^* \subseteq H^*$.

In fact by Corollary 3.6 of [15], (ii) is implied by the condition that $H^*H^* \neq \{0\}$. In this case, if a is an element of H^* then $a\alpha = H^*$, condition (ii) guarantees that $a\alpha$ is an idempotent: on the other hand by property (i) there is no corresponding idempotent $h \in S$ with $h\alpha = a\alpha$.

We use Example 2.6 of [15]. Let $T = \{a^k : k \geqq 0\}$ be the cyclic monoid generated by the symbol a. Let $I = J = \{a^t : t \geqq 1\}$. Put,

$$S = \{(x)_{ij} : i, j \in \{1, 2\}, x \in T \text{ if } i = j, x \in I \text{ if } i \neq j\}$$

with multiplication $(x)_{ij} \cdot (y)_{hk} = (xp_{jh}y)_{ik}$ where

$$P = (p_{ij}) = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}.$$

Then

$$H^* = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in I \right\}$$

is an $\mathscr{H}^*$-class not containing an idempotent and $H^*H^* \neq \{0\}$.

Proposition 1.8. *Let $\phi: S \to T$ be a good homomorphism with S abundant onto a semigroup T. Then if S satisfies the regularity condition then so does T.*

Proof. By Theorem 1.6, for each idempotent $a\phi$ of the image there exists an idempotent $e \in S$ with $e\phi = a\phi$. Let $a_1\phi \dots a_n\phi$ be a typical element of $\langle E(S\phi) \rangle$ where $a_i\phi \in E(S\phi)$ for each $i = 1, \dots, n$. Then we can find idempotents $e_i \in S$ for each i, with $e_i\phi = a_i\phi$. It follows that we have $a_1\phi \dots a_n\phi = (e_1 \dots e_n)\phi$. But $e_1 \dots e_n \in \langle E(S) \rangle$, therefore regular: hence $(e_1 \dots e_n)\phi$ is also regular.

Proposition 1.9. *Let $\phi: S \to T$ be a good homomorphism from an abundant semigroup S satisfying the regularity condition onto a semigroup T. Let $e', f' \in E(S\phi)$ and suppose that $e'\omega f'$ and that $f\phi = f'$ for some $f \in E(S)$. Then there exists an idempotent $k \in E(S)$ such that $k\phi = e'$ and $k\omega f$.*

Proof. By Theorem 1.6 there exist idempotents e and f in $E(S)$ with $e\phi = e'$ and $f\phi = f'$. Let $h \in S(e, f)$ and $g \in S(f, e)$, by Proposition 1.3(iii), $h\phi \in S(e', f')$ and $g\phi \in S(f', e')$. From the fact that $e'\omega f'$ and by Proposition 1.3(ii), $S(e', f') = \omega(f') \cap E(L^*_{e'})$. From this result $e'\mathscr{L}^*h\phi$, and so $e'f'\mathscr{L}^*h\phi f'$, because $\mathscr{L}^*$ is a right congruence. Therefore $e'\mathscr{L}^*h\phi f'$. But $h\phi \in \omega(f')$ so we have $e'\mathscr{L}^*h\phi\omega f'$, similarly $e'\mathscr{R}^*g\phi\omega f'$.

Again if $k \in S(hf, fg)$ then $k\phi \in S((hf)\phi, (fg)\phi) = S(h\phi, g\phi) = S(e', e')$ which is simply $\{e'\}$. Consequently $k\phi = e'$. But in addition $k \in S(hf, fg)$ implies that $k(hf) = k = (fg)k$ so $k\omega^l hf\omega f$ and $k\omega^r fg\omega f$, giving $k\omega f$ as required.

Theorem 1.10. *Let $\phi: S \to T$ be a good homomorphism from a concordant semigroup S onto a semigroup T then T is also concordant.*

Proof. It only remains to show that T is IC. Let $a\phi \in T$ and choose an idempotent a^* in S. Since ϕ is good then $a^*\phi \mathscr{L}^* a\phi$ in T. Let e' be an idempotent in $\omega(a^*\phi)$. By Proposition 1.9 there exists an idempotent $e \in \omega(a^*)$ with $e\phi = e'$. The semigroup S is IC and so by Proposition 1.1 there exists an element b of S with $ae = ba$. Hence $a\phi e' = b\phi \cdot a\phi$. In a similar way we can show that the second condition of Proposition 1.1(b) holds, and we are home.

Good homomorphisms and concordant semigroups form a category: a detailed study of which may be found in the thesis by Armstrong [**4**].

2. The natural partial order

We introduce a partial ordering on the L^* and R^* classes of a semigroup. For elements x and y of a semigroup S we say that $R^*_x \leqq R^*_y$ if and only if $R^*(x) \subseteq R^*(y)$ where $R^*(x)$ is the principal good ideal generated by the element x (consult Lemma 1.6 of [**15**] for the details). The ordering on the $\mathscr{L}^*$-classes is defined in the usual left–right dual fashion.

Lemma 2.1. *For any elements a and x of S, $R^*_{ax} \leqq R^*_a$.*

Proof. The product ax lies in aS^1, which is the smallest right ideal containing a. Since $R^*(a)$ is a right ideal containing the element a we must have, $aS^1 \subseteq R^*(a)$ from which it follows that $ax \in R^*(a)$. On the other hand, $R^*(ax)$ is the smallest good right ideal containing the product ax, since $R^*(a)$ is a good right ideal, $R^*(ax) \subseteq R^*(a)$.

In the case of regular elements, however, we have introduced nothing new.

Lemma 2.2. *Let S be a semigroup and let a and b be regular elements of S then $R^*_a \leqq R^*_b$ if and only if $R_a \leqq R_b$.*

Proof. Let x and y be the inverses of a and b respectively. Then $aRax$ and $bRby$ and so $R^*(a) = axS$ and $R^*(b) = byS$. From $axS \subseteq byS$ we have that $a \in b(yS) \subseteq bS$ and so $R_a \leqq R_b$.

Conversely, suppose that $R_a \leqq R_b$ then, $a \in bS^1$. But then $a = bx$ for some element x belonging to S^1. Applying Lemma 2.1 we have that $R^*_a = R^*_{bx} \leqq R^*_b$.

We must now turn to our main definition.

Proposition 2.3. *Let S be an abundant semigroup. Define two relations on S as follows: for all elements x and y of S*

$$x \leqq_r y \text{ if and only if } R^*_x \leqq R^*_y \text{ and there exists an idempotent } f \in R^*_x \text{ such that } x = fy.$$

$$x \leqq_l y \text{ if and only if } L^*_x \leqq L^*_y \text{ and there exists an idempotent } e \in L^*_x \text{ such that } x = ye.$$

Then $\leqq_r (\leqq_l)$ is a partial order on S which coincides with ω on $E(S)$.

Proof. Reflexivity follows from the fact that S is abundant. Suppose that $x \leqq_r y \leqq_r z$ where $x, y, z \in S$. Then straightaway $R^*_x \leqq R^*_z$. Now $x = fy$ and $y = gz$ for idempotents $f \in R^*_x$ and $g \in R^*_y$ hence $R^*_f \leqq R^*_g$. By application of Lemma 2.2 this means that $R_f \leqq R_g$, from which we have $fg \in E(S)$, $fg\omega g$ and $fg\mathscr{R}f$, in the usual way. But $x = (fg)z$ and $fg \in E(R^*_f) = E(R^*_x)$. Consequently $\leqq_r$ is transitive. If $x \leqq_r y \leqq_r x$, then $R^*_x = R^*_y$ and the equality $x = fy$ for some idempotent $f \in R^*_x$ implies that $x = y$. Together with a similar argument, we have shown that both $\leqq_r$ and $\leqq_1$ are partial orders on S. It is simple to verify that they both coincide with the order ω on $E(S)$.

We define *the natural partial order* to be $\leqq = \leqq_r \cap \leqq_1$. Note that Proposition 1.2 of [24] effectively shows that for S regular $\leqq_r = \leqq_1$. We may give an alternative description of the natural partial order entirely in terms of idempotents.

Proposition 2.4. *In an abundant semigroup S for elements x and y of S, $x \leqq y$ if and only if there are idempotents e and f such that $x = ey = yf$.*

Proof. Suppose that $x = ey = yf$. From $x = yf$ and Lemma 2.1 we have $R^*_x \leqq R^*_y$. Choosing an $x^+, x = x^+x = x^+ey$, however $ex = x$ so $ex^+ = x^+$. This means that x^+e is an idempotent, furthermore $x^+e \in R^*_x$, so that $x \leqq_r y$. A similar argument shows that $x \leqq_1 y$.
The converse is straightforward.

Proposition 2.5. *Let x and y be elements of an abundant semigroup S. Then $x \leqq_r y$ if and only if for each idempotent $y^+ \in R^*_y$ there exists an idempotent $x^+ \in R^*_x$ such that $x^+\omega y^+$ and $x = x^+y$. The dual result holds for $\leqq_1$.*

Proof. Suppose that $x \leqq_r y$. Then $R^*_x \leqq R^*_y$ and $x = ey$ for some idempotent $e \in R^*_x$. Let f be an idempotent in R^*_y. Then, $R^*_e = R^*_x \leqq R^*_y = R^*_f$ and so that $e\mathscr{R}^*e_1 = ef\omega f$ and $e_1y = efy = ey = x$.
Conversely, suppose that $x = ey$ where e is an idempotent in R^*_a and $e\omega f$ for some idempotent in R^*_y. Then $e = fe$ so that $R^*_e = R^*_{fe} \leqq R^*_f$ by Lemma 2.1.

For regular and type A semigroups $\leqq_1 = \leqq_r$ this latter result noted by Armstrong [3]. We will now show that for IC abundant semigroups in general the definition of the natural partial orders is likewise self-dual, and that this feature actually characterises this class of abundant semigroups.

Theorem 2.6. *Let S be an abundant semigroup. Then S is IC if and only if $\leqq_1 = \leqq_r$.*

Proof. Suppose that S is IC and that $x \leqq_r y$. By Proposition 2.5, having chosen an idempotent $y^+ \in R^*_y$ we may choose an idempotent $x^+ \in R^*_x$ such that $x^+\omega y^+$ and $x = x^+y$. Now $x^+\omega y^+$ implies that $x^+ \in \langle y^+ \rangle$, by the definition of the connecting homomorphism $\alpha: \langle y^+ \rangle \to \langle y^* \rangle$, $x = x^+y = y(x^+\alpha)$. The element $x^+\alpha$ is evidently an idempotent, so that by Proposition 2.4 we have, $x \leqq y$. Thus $\leqq_r = \leqq$, together with a similar result for $\leqq_1$ we have $\leqq_r = \leqq_1$.
Now let S be abundant with the property that $\leqq_r = \leqq_1$. Let a be an element of S and let $e \in \omega(a^*)$, we will produce an element $f \in \omega(a^+)$ such that $ae = fa$. Let $z = ae$. Since $ze = z$ we have $z^*e = z^*$. In the usual way ez^* is an idempotent, $ez^*\mathscr{L}z^*$ and $ez^*\omega e$. Now

$z=zz^*=aez^*=a(ez^*)=ag$ where $g=ez^*$ and $g\mathscr{L}^*z$. Also, $L_z^*=L_{ae}^*\leqq L_e^*$. But $e\omega a^*$ so $e=ea^*$ thus $L_e^*\leqq L_a^*$, so we have $L_z^*\leqq L_a^*$. This shows that $z\leqq_1 a$. By assumption, $z\leqq_r a$ an application of Proposition 2.5 then provides an idempotent $f\in\omega(a^+)$ with $z=fa$.

Proposition 2.7. *Let S be abundant then,*

(i) *If* $x\leqq_r e$ ($x\leqq_1 e$) *where* $e\in E(S)$ *then* $x\in E(S)$.

(ii) *If* $b\leqq_r a$ ($b\leqq_1 a$) *where a is regular then b is regular.*

(iii) *If* $x, y\in S$ *with* $x\mathscr{R}^*y$ ($x\mathscr{L}^*y$) *and* $x\leqq_r y$ ($x\leqq_1 y$) *then* $x=y$.

Proof. We will prove the results in the case of $\leqq_r$.

(i) If $x\leqq_r e$ then $x=fe$ for some idempotent $f\mathscr{R}^*x$. But then $R_x^*\leqq R_e^*$ is simply $R_f\leqq R_e$ which gives $ef=f$, so that fe is an idempotent.

(ii) Let $a\in\text{Reg}(S)$ and suppose that $b\leqq_r a$. Then $b=fa$ where $f\in R_b^*$ and $R_b^*\leqq R_a^*$. Let $x\in V(a)$, then $bxb=fa\cdot x\cdot fa=f(ax\cdot f)a$. However, $ax\mathscr{R}a$ so that $R_f^*=R_b^*\leqq R_a^*=R_{ax}^*$ hence $R_f^*\leqq R_{ax}^*$ giving $R_f\leqq R_{ax}$. From this it follows that $f\omega^r ax$ and so $ax\cdot f=f$. Applying the above, $bxb=f(ax\cdot f)a=f^2a=fa=b$ and the regularity of b is established.

(iii) We have $x\mathscr{R}^*y$ and $R_x^*\leqq R_y^*$ with $x=ey$ where e is an idempotent belonging to R_x^*. By assumption $e\mathscr{R}^*y$ therefore $ey=y=x$.

Property (iii) may be rephrased by saying that $\mathscr{L}^*$ ($\mathscr{R}^*$) are *strictly compatible* with respect to $\leqq$. Properties (i) and (ii) may be paraphrased by saying that $(E(S), \leqq_r)$ and $(\text{Reg}(S), \leqq_r)$ are *order ideals* of $(S, \leqq_r)$. Note also that result (ii) is closely related to Proposition 1.2(d) of [**24**].

Proposition 2.8. *The order* $\leqq_r$ *and* $\leqq_1$ *coincide on* Reg(*S*).

Proof. Let a and b be regular elements with $a\leqq_r b$. Pick an idempotent f with $f\mathscr{R}^*b$. By Proposition 2.5 there exists an idempotent e with $e\mathscr{R}^*a$, $e\omega f$ and $a=eb$. Choose an idempotent g with $g\mathscr{L}^*b$. Since b is regular $g\mathscr{L}b$. But then $f\mathscr{D}b$ and D_g is a regular $\mathscr{D}$-class, and so there exists $b^{-1}\in V(b)$ with $b^{-1}b=g$. Furthermore, $bb^{-1}\mathscr{R}b$ so that $bb^{-1}\mathscr{R}^*b$. By Proposition 2.5 there exists an idempotent e, with $e\mathscr{R}^*a$ and such that $e\omega bb^{-1}$ and $a=eb$. Put $e_1=b^{-1}eb$, then $e_1\mathscr{L}^*a$, $e_1\omega b^{-1}b=g$ and $be_1=(bb^{-1})eb=ey=a$, and so $a\leqq_1 b$.

A non-zero element of an abundant semigroup is said to be *primitive* if it is minimal amongst the non-zero elements of S with respect to $\leqq$. Since the restriction of $\leqq$ to $E(S)$ is ω, this definition coincides with the usual definition when applied to the idempotents.

Proposition 2.9. *An abundant semigroup is primitive with respect to* ω *if and only if it is primitive with respect to* $\leqq$.

Proof. Suppose that every non-zero idempotent is minimal in the set of non-zero idempotents, and let x and y be two non-zero elements of S with $x\leqq y$. Then for each idempotent f, with $f\mathscr{R}^*y$ there exists an idempotent e with $e\mathscr{R}^*x$, $e\omega f$ and $x=ey$. But by the primitivity of the idempotents $e\omega f$ implies $e=f$. Hence $x=ey=fy=y$.

The converse is clear.

The following result is now immediate, by Corollary 5.2 of [**15**].

Corollary 2.10. *An abundant semigroup without zero is primitive, if and only if the natural partial order is the identity relation. In particular, an abundant semigroup without a zero and satisfying the regularity condition is completely $\mathscr{J}^*$-simple if and only if its natural partial order is the identity relation.*

Note that it can be shown that the minimal elements in an arbitrary IC abundant semigroup without zero form a primitive subsemigroup and order ideal.

Proposition 2.11. *Let S be an abundant semigroup and let U be an abundant subsemigroup with $E(U)$ an order ideal of $E(S)$. Then:*

(i) *For $x, y \in U$ if $x \leqq y$ in U then $x \leqq y$ in S.*

(ii) *For $x, y \in U$ if $x \leqq y$ in S then $x \leqq y$ in U.*

Proof. Case (i) requires no proof.

Case (ii). By Lemma 1.6 of [**11**] U has the property that $\mathscr{L}^*(U)=\mathscr{L}^*(S)\cap(U\times U)$ and dually. Suppose that $x, y \in U$ with $x \leqq y$ in S. Then for each y^+ there is an x^+ such that $x^+\omega y^+$ and $x=x^+y$. Since U is abundant there is an idempotent f in U with $f\mathscr{R}^*(U)y$: by the property above this means that we may take $y^+=f$ in S and so by the comment above there is an idempotent x^+ in S with $x^+\omega f$ and $x=x^+y$. This means in particular that x^+ actually belongs to U, under our assumption. Appealing to the fact that $\mathscr{R}^*(U)=\mathscr{R}^*(S)\cap(U\times U)$ it is immediate that $x^+\mathscr{R}^*(U)x$. We have shown that if $x \leqq_r y$ in S then $x \leqq_r y$ in U: we may apply a similar argument to the order $\leqq_1$, and our result follows.

Proposition 2.12. *Let S be abundant and let U be an abundant subsemigroup. Then $E(U)$ is an order ideal of $E(S)$ if and only if U is an order ideal of S with respect to the natural partial order.*

Proof. Suppose that $E(U)$ is an order ideal of $E(S)$. Let y be an element of U and let x be an element of S with $x \leqq y$. Choose an idempotent f in U with $f\mathscr{R}^*(U)y$, by Lemma 1.6 [**11**] again it follows that $f\mathscr{R}^*(S)y$, consequently there exists an idempotent e with $e\mathscr{R}^*(S)x$, $e\omega f$ and $x=ey$. But $e\in E(U)$, since $E(U)$ is an order ideal of $E(S)$ and so $x=ey\in U\cdot U\subseteq U$.

Let S be a concordant semigroup. Following Armstrong [**4**] we may define the **-trace* of S to be the partial groupoid tr*(S) equipped with a partial binary operation "·", defined in the following way,

$$a\cdot b=\begin{cases} ab \text{ if } L_a^*\cap R_b^*\cap E(S)\neq\varnothing \\ \text{undefined otherwise.}\end{cases}$$

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If we extend the multiplication on $\operatorname{tr}^*(S)$ by setting all undefined products equal to 0, adjoined in the usual way, then the resulting structure is a primitive concordant semigroup. The connection with the natural partial order is demonstrated by the next result.

Theorem 2.13. *Let S be a concordant semigroup and let $x, y \in S$. Then there exist elements u and v such that,*

(i) *$u \leqq x$ and $v \leqq y$*

(ii) *$xy = u \cdot v$*

and if $x \cdot y$ exists in $\operatorname{tr}^*(S)$ and u and v satisfy (i) and (ii) then $x = u$ and $y = v$.

Proof. Let e and f be idempotents with $e\mathscr{L}^*x$ and $f\mathscr{R}^*y$ and let $h \in S(e,f)$. By Proposition 1.3(i), $xy = (xh)(hy)$, also $xh\mathscr{L}^*h\mathscr{R}^*hy$ so in fact $xy = (xh)\cdot(hy)$. Put $u = xh$ and $v = hy$. It is evident that $u \leqq x$ and $v \leqq y$.

Now suppose that $x \cdot y$ exists in $\operatorname{tr}^*(S)$. Then the product xy lies in $R_x^* \cap L_y^*$. Let u and v satisfy conditions (i) and (ii). Then $uv \in R_u^* \cap L_v^*$ and $xy = uv$. But then $R_x^* \cap R_u^* \neq \emptyset$ implies xR^*u, however, since $u \leqq x$ we may apply Proposition 2.7 and so $u = x$. Similarly we may show that $v = y$.

Let $(X, \leqq)$ and $(Y, \leqq)$ be quasi-ordered sets. Then a mapping $\phi: X \to Y$ is said to *reflect* quasi-orders if and only if for all pairs of elements $y, y' \in X\phi$ with $y' \leqq y$ and $x \in X$ with $x\phi = y$ then there exists an $x' \in X$ such that $x' \leqq x$ and $x'\phi = y'$.

Proposition 2.14. *Let $\phi: S \to T$ be a good homomorphism between abundant semigroups, each of which satisfies the regularity condition. Then the mapping ϕ preserves and reflects the natural partial orders of S and T.*

Proof. That ϕ preserves the natural partial orders presents no problems.

Now let $u, v \in S\phi$ with $u \leqq v$ and let $y \in S$ with $y\phi = v$. If f is an idempotent with $f\mathscr{R}^*y$ then if $f\phi = f'$ we have $f'\mathscr{R}^*v$, since the mapping is good. From the fact that $u \leqq v$ there exists an idempotent e' with $e'\mathscr{R}^*u$ such that $e'\omega f'$ and $u = e'v$. By Proposition 1.9 there exists an idempotent $e \in \omega(f)$ such that $e\phi = e'$. Now put $x = ey$, then $x\phi = e\phi \cdot y\phi = e'v = u$. From $f\mathscr{R}^*y$ and the fact that $\mathscr{R}^*$ is a left congruence $ef\mathscr{R}^*ey$, that is $e\mathscr{R}^*x$. Therefore, we have,

$$R_x^* = R_{ey}^* \leqq R_e^* \leqq R_f^* = R_y^*$$

and so $x \leqq y$.

We will conclude this section by applying some of the preceding ideas to semihereditary monoids. Recall by Theorem 1 of Dorofeeva [9] that semihereditary monoids are just the abundant monoids in which incomparable principal right (left) ideals are disjoint. By Theorem 2 of Fountain [12] semihereditary monoids with central idempotents have the property that the images of non-minimal elements under the connecting homomorphisms are regular: we may generalise this result to arbitrary semihereditary monoids.

Proposition 2.16. *Let S be a semihereditary monoid and let a be an element of S then either a is minimal or $[a]\backslash\{a\}$ consists entirely of regular elements.*

Proof. Let a be a non-minimal element of S: then there exists at least one element z with $z \leqq a$ and $a \neq z$. Choose an idempotent a^+, by Proposition 2.5 there exists an idempotent z^+ such that $z^+ \omega a^+$ and $z = z^+ a$. Note that $z^+ \neq x^+$, for otherwise $z = z^+ a = a^+ a = a$. Furthermore, from $z \leqq a$ we have that $z = az^*$ for some z^*. In particular, $z = z^+ a = az^* \in z^+ S \cap aS$. By the comment preceding this theorem we must have either, $z^+ S \subseteq aS$ or $aS \subseteq z^+ S$: that is, either $z^+ = as$ or $a = z^+ t$ for some elements s and t of S. Suppose that $a = z^+ t$, then $z^+ a = z^+(z^+ t) = z^+ t = a$, but this implies that $z = a$, contradicting our assumption. This means that we must have $z^+ = as$, but then, $z^+ a = z^+ a \cdot s \cdot z^+ a$ and so $z = zsz$, implying that z is regular.

Corollary 2.17. *In a semihereditary monoid without minimal elements, the regular elements are dense.*

3. Locally type A semigroups

A partial order $\leqq$ on a semigroup S is said to be *compatible* with the multiplication if whenever $a \leqq b$ and $c \leqq d$ then $ac \leqq bd$. The first result of this section establishes a necessary condition for the natural partial order $\leqq = \leqq_l \cap \leqq_r$ to be compatible with the multiplication on an abundant semigroup. Recall first that by Proposition 1.5(i) that each local submonoid is abundant.

Proposition 3.1. *If the natural partial order of an abundant semigroup is compatible with the multiplication then the semigroup is locally adequate.*

Proof. Let e be an idempotent of an abundant semigroup S and let $f, h \in \omega(e)$. Since both $f \leqq e$ and $h \leqq e$, and under the assumption that $\leqq$ is compatible with the multiplication, $fh \leqq e^2 = e$. By Proposition 2.7(i) the element fh is therefore an idempotent and so $fh \omega e$. This shows that the idempotents of each local submonoid form bands.

Let u and v be any two idempotents in the local submonoid eSe, and suppose that in addition $u \mathscr{R}^* v$ in eSe. From the fact that $u \omega e$ and $v \omega e$, both $u \leqq e$ and $v \leqq e$. Since idempotents are regular $u \mathscr{R} v$ and so $uv = v$ and $vu = u$. But then applying the compatibility of $\leqq$,

$$v = uv \leqq ue = u \quad \text{and} \quad u = vu \leqq ve = v.$$

Hence $u = v$. We have shown that each local submonoid is an abundant semigroup in which each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class contains a unique idempotent and which satisfies the regularity condition. By Proposition 1.3 of **[14]** the local submonoid is adequate.

Since idempotent-connectedness is inherited by local submonoids the following is immediate by Proposition 1.5.

Corollary 3.2. *Every IC abundant semigroup with a natural partial order compatible with the multiplication is locally type A.*

Example 3.3. Let S be the eight element semigroup whose multiplication table is given below: this semigroup is taken from Example 2.2 of [**14**].

	e	f	g	h	z	a	b	c
e	e	f	g	z	z	a	b	c
f	f	f	z	z	z	b	b	z
g	g	z	g	z	z	c	z	c
h	z	z	z	h	z	z	z	z.
z	z	z	z	z	z	z	z	z
a	z	z	z	a	z	z	z	z
b	z	z	z	b	z	z	z	z
c	z	z	z	c	z	z	z	z

It can be verified that S is an adequate semigroup with semilattice of idempotents $E=\{e,f,g,h,z\}$. In fact each local submonoid of S is a semilattice, so S is an adequate locally inverse semigroup. If S were type A then for each idempotent e in S and each element a in S we would have $eS^1 \cap aS^1 = eaS^1$—see Proposition 1.5 of [**14**]. Now, $fa=b$ and $fS^1=\{f,z,b\}$ and $aS^1=\{a,z\}$ and so $fS^1 \cap aS^1=\{z\}$. But $faS^1=bS^1=\{z,b\}$ which means that $fS^1 \cap aS^1 \neq faS^1$. So S is not type A, which implies that S is not IC.

We will show that the partial order on S is not compatible with the multiplication. It is clear that $f \leqq e$ and $a \leqq a$. Now $fa=b$ and $ea=a$ however b is not a restriction of a: for $aE=\{z,a\}$, and evidently b does not belong to the set on the right-hand side, but $b \leqq a$ would imply the existence of an idempotent e' such that $b=ae'$.

This example shows that in general the condition in Proposition 3.1 is not sufficient, we need to assume that the semigroup is IC. On the other hand the regularity condition is not necessary, for on every primitive abundant semigroup without zero the natural partial order is the identity and consequently it is trivially compatible with the multiplication. We have so far only been able to prove the converse of Corollary 3.2 under the assumption that the semigroup satisfies the regularity condition.

Theorem 3.4. *The natural partial order on a concordant semigroup is compatible with the multiplication if and only if the semigroup is locally type A.*

Proof. It remains to be shown that in a locally type A concordant semigroup the partial order is compatible with the multiplication. Let $x, y, u, v \in S$ with $x \leqq u$ and $y \leqq v$. Choosing idempotents u^* and v^+ there are idempotents e and f with $e \in \omega(u^*) \cap E(L_x^*)$ and $x=ue$, and $f \in \omega(v^+) \cap E(R_y^*)$ with $y=fy$. Let $k \in S(e,f)$, since $e\omega u^*$ and $f\omega v^+$ we have $k \in \omega^l(u^*) \cap \omega^r(v^+)$. From this we may deduce,

(i) $u^*k \in E$, $u^*k\omega u^*$ and $u^*k\mathscr{L}k$,

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(ii) $ek \in E$, $ek\omega e$ and $ek\mathscr{L}k$,

(iii) $kv^+ \in E$ and $kv^+\omega v^+$.

From (i) and (ii), u^*k, $ek \in \omega(u^*)$ and $u^*k\mathscr{L}ek$ from which it is easy to see that $u^*k = ek$. Now, $xk = (ue)k = u(ek) = u(u^*k) = uk$, similarly $ky = kv$. We have, $xy = (xk)(ky) = (uk)(kv) = ukv$.

Now, $kv = (kv^+)v$ and from (iii) $kv^+ \in \omega(v^+)$, applying Proposition 1.1 there exists an idempotent $e_1 \in \omega(v^*)$ such that $(kv^+)v = ve_1$. This gives us, $xy = ukv = u[(kv^+)v] = uve_1$. Similarly $uk = u(u^*k)$, by (i) $u^*k\omega u^*$ so applying Proposition 1.1 once again there exists an idempotent $f_1 \in \omega(u^+)$ such that $u(u^*k) = f_1 u$. Hence, $xy = ukv = [u(u^*k)]v = f_1 uv$. From Proposition 2.4 this yields the required inequality, $xy \leqq uv$.

The behaviour of subsemigroups, homomorphic images and direct products of locally type A concordant semigroups is summarised below.

Proposition 3.5. *Let S be a concordant locally type A semigroup. Then it has the following properties:*

(i) *If T is an abundant subsemigroup which is an order ideal of S and which satisfies the regularity condition then it is a concordant and locally type A subsemigroup.*

(ii) *If $\theta: S \to T$ is a good homomorphism onto a semigroup T then T is also concordant and locally type A.*

(iii) *The direct product of a family of concordant locally type A semigroups is locally type A.*

Proof. (i) is by Proposition 2.12, (ii) is by Theorem 1.10 and Proposition 1.9 and (iii) is straightforward.

If we dispense with the regularity condition on the semigroup T in case (i) above, then T is still an IC abundant and locally type A subsemigroup, furthermore since it is an order ideal of S the natural partial order on T is just the restriction of the order on S, this means that the partial order on T is compatible with the multiplication on T. However T need not be concordant as the following example shows.

Example 3.6. Let $S = M(\mathbb{Q}^*; \{1,2\}, \{1,2\}; P)$ where $P = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ then S is completely simple and so regular with all its idempotents primitive. Any subset of S is an order ideal. Consider

$$T = \{(x)_{ij} : x \geqq 1,\ i, j \in \{1,2\}\}.$$

Then T is abundant with primitive idempotents and

$$E(T) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

In fact $T = M(X; \{1,2\}, \{1,2\}; P)$, where $X = \{x \in \mathbb{Q}^* : x \geqq 1\}$. The idempotents of T do not generate a regular subsemigroup because the entries 2 in P are not units in X—see Theorem 5.1 of **[15]**.

The structure theory of locally type A semigroups is discussed in detail in **[19]** and **[20]**. Type A semigroups are the subject of **[21]**.

4. Primitive good congruences on abundant semigroups

In this section S will be assumed to be concordant. A semigroup S is said to be *categorical at zero* if and only if for all elements a, b and c of S, $ab \neq 0$ and $bc \neq 0$ implies that $abc \neq 0$. It is easy to see that the intersection of any family of good congruences on an abundant semigroup is again good: this enables us to define the smallest good congruence containing a relation R which we denote by R^{gc}. A congruence ρ is said to be 0-*restricted* if and only if $\{0\}$ is a ρ-class. Define a relation $\beta(S)$ on S as follows:

$$\beta(S) = \{(x, y)\colon \text{there exists } z \in S\backslash\{0\} \text{ s.t. } z \leqq x \text{ and } z \leqq y\} \cup \{(0,0)\}.$$

Proposition 4.1. *Let S be categorical at zero and suppose that $\beta(S) = \beta$ is a good congruence, then β is the finest* 0*-restricted, primitive good congruence on S.*

Proof. It is clear that under the conditions of the theorem, β will be 0-restricted. We will show first that S/β is primitive. Let $x', y' \in S/\beta$, where we write $x' = x\beta$, with $x' \leqq y'$ and $x' \neq 0$. By Proposition 2.15 we may assume that $x \leqq y$. If $x = 0$ then $x' = 0$. On the other hand if $x \neq 0$ then $(x, y) \in \beta$ and so $x' = y'$.

Now let σ be any 0-restricted, primitive good congruence on S and let $(x, y) \in \beta$. Then $x = 0$ if and only if $y = 0$, so if $x = 0$ then $(x, y) \in \beta$. If on the other hand $x \neq 0$ then $y \neq 0$ and so there exists a non-zero element z with $z \leqq x$ and $z \leqq y$. Since σ is 0-restricted, $z \neq 0$ implies that $z\sigma$ is a non-zero element in S/σ. Since $z \leqq x$ we have $z\sigma \leqq x\sigma$, likewise $z\sigma \leqq y\sigma$. But by assumption S/σ is primitive and so $z\sigma = x\sigma = y\sigma$. Hence $(x, y) \in \sigma$, that is $\beta \subseteq \sigma$.

Proposition 4.2. *For a semigroup S which is categorical at zero, the following conditions are equivalent:*

(i) *For each $e \in E(S)\backslash\{0\}$ $\omega(e)\backslash\{0\}$ is a directed set.*

(ii) *$\beta(S)$ is an equivalence relation.*

(iii) *$\beta(S) = \beta(S)^{gc}$.*

Proof. (i) implies (ii). Suppose that for each $e \in E(S)\backslash\{0\}$ the set $\omega(e)\backslash\{0\}$ is directed under the natural partial order. We need only show that $\beta(S)$ is transitive. To this end let $(x, y), (y, z) \in \beta$. Then either $x = y = z = 0$ or none of them is zero. In the first case $(x, z) \in \beta$ and the result follows. On the other hand, suppose that none of them is zero. Then there exist non-zero elements u_1 and u_2 such that $u_1 \leqq x$, $u_1 \leqq y$ and $u_2 \leqq y$, $u_2 \leqq z$. Choose an idempotent f with $f\mathscr{R}^*y$. Then there exist idempotents e_i for $i = 1, 2$ satisfying: $e_i\mathscr{R}^*u_i$, $e_i\omega f$ and $u_i = e_i y$. Now $e_i\mathscr{R}^*u_i$ and so $e_i u_i = u_i$, furthermore, $u_i \neq 0$ so it follows that $e_i \neq 0$. This means that $e_i \in \omega(f)\backslash\{0\}$ for $i = 1, 2$. By assumption this set is directed, so there is an element $g \in \omega(f)\backslash\{0\}$ with $g\omega e_i$ for $i=1,2$. Now, $f\mathscr{R}^*y$ and so $gf\mathscr{R}^*fy$ since $\mathscr{R}^*$ is a left congruence. But then $g\mathscr{R}^*gy$ also $gy=gfy$, $gf=g\neq 0$ and $fy=y\neq 0$ so by the fact that S is categorical at zero $gy\neq 0$. Furthermore,

$$gy=ge_1y=gu_1 \text{ and } gy=ge_2y=gu_2.$$

From $gy\mathscr{R}^*g\omega^r e_i\mathscr{R}^*u_i$ and the fact that $g\in E(R^*_{gy})$ it follows that $gu_i \leqq u_i$ for $i=1,2$. With the result above this means that we have, $gy=gu_1\leqq u_1\leqq x$ and $gy=gu_2\leqq u_2 \leqq z$, which gives us $(x,z)\in\beta$ as required.

(iii) implies (i). Assume that β is a good congruence. Let $e\in E(S)\backslash\{0\}$ and let $f,g\in\omega(e)\backslash\{0\}$. Then $(f,e),(e,g)\in\beta$ and by transitivity we obtain $(f,g)\in\beta$. But by definition there exists a non-zero element z such that $z\leqq f$ and $z\leqq g$. By Proposition 2.7 this means that z must be idempotent, giving the desired result.

(ii) implies (iii). We need to show that β is compatible with the multiplication and good, we will proceed in a number of steps.

Let $x,y,c\in S$ with $x\leqq y$ and $x\neq 0$. Then $cx=0$ if and only if $cy=0$: for choosing an idempotent y^+ there exists an idempotent e belonging to $\omega(y^+)\cap R^*_x$ such that $x=ey$, and so $y^+x=y^+ey=ey=x$. Now suppose that $cy=0$ then $cy^+=0$, from the above $cx=cy^+x=0$. Conversely, if $cx=0$ then $cy^+x=0$. We have shown that $y^+x=x\neq 0$, if cy^+ were non-zero too we would have $cy^+x\neq 0$ by the fact that S is categorical at zero. This forces $cy^+=0$ and hence $cy=0$ as required.

Following on from the previous step we will now show that $(cx,cy)\in\beta$. If one of cx or cy is zero, then so is the other by the result above, immediately giving $(cx,cy)\in\beta$.

Therefore we may assume that $cx\neq 0\neq cy$. Choose idempotents y^+ and y^*. From the fact that $x\leqq y$ we may pick an idempotent x^+ with $x^+\omega y^+$ such that $x=x^+y$. Now $x^+\in\langle y^+\rangle$ and so $x^+y=y(x^+\alpha)$ where $\alpha:\langle y^+\rangle\to\langle y^*\rangle$ is the connecting isomorphism. Note that by Lemma 3.3 of [11] $e\alpha\mathscr{L}^*x$, so we may put $x^*=e\alpha$ and we have, $x=ey=yx^*$ with $x^*\in\omega(y^*)$.

Now pick an element c^* and let $h\in S(c^*,y^+)$ and $k\in S(c^*,x^+)$. By applying Lemma 1.4 there exist idempotents with,

$$h_1\in E(L^*_{cy})\cap\omega(y^*) \quad \text{and} \quad k_1\in E(L^*_{cx})\cap\omega(x^*).$$

But from the fact that $x^*\omega y^*$ we also have,

$$k_1\in E(L^*_{cx})\cap\omega(y^*).$$

This means that $x^*,h_1,k_1\in\omega(y^*)\backslash\{0\}$—furthermore every element of this set is β-related to y^*, so that $x^*\beta h_1\beta k_1$. We now use the assumption that β is an equivalence to obtain,

$$\omega(x^*)\cap\omega(h_1)\cap\omega(k_1)\backslash\{0\}\neq\varnothing.$$

Let l belong to this set and define the element z by,

$$z=cy\cdot l=cy(x^*l)=c(yx^*)l=cx\cdot l.$$

Note that $z \neq 0$: by assumption the product $cy \neq 0$, also yl cannot be zero, for if it were we would have $y^*l=0$ which means that $x^*y^*l=0$ but from $x^*\omega y^*$ and $l\omega x^*$ this would give $l=0$, which contradicts our choice of l. We now use the fact that S is categorical at zero.

We will now show that $z \leqq cy$ and $z \leqq cx$. First of all $l \in E(L_z^*)$: since $l \in \omega(h_1)$ and $h_1 \in E(L_{cy}^*) \cap \omega(y^*)$, it follows that $h_1 \mathscr{L}^* cy$ and because $\mathscr{L}^*$ is a right congruence, $h_1 l \mathscr{L}^* cyl$ that is $l \mathscr{L}^* z$. From the fact that,

$$z = cyl \mathscr{L}^* l\omega h_1 \mathscr{L}^* cy$$

we obtain $L_z^* \leqq L_{cy}^*$. This means that $z \leqq cy$, similarly we may show that $z \leqq cx$. We have demonstrated that $(cx, cy) \in \beta$.

We may now complete the proof that β is a congruence. Suppose that $(u, v) \in \beta$ where c is an arbitrary element of S. Then if $u=v=0$ it is immediate that $(cu, cv) \in \beta$. Suppose now that $u, v \neq 0$. Then there exists a non-zero element z such that $z \leqq u$ and $z \leqq v$. Hence by the above arguments both $(cz, cu) \in \beta$ and $(cz, cv) \in \beta$. But β is an equivalence relation therefore $(cu, cv) \in \beta$.

Lastly, we will show that β is a good congruence. Let $(ax, ay) \in \beta$, if $ax=0$ and $ay=0$ then $a^*x=0$ and $a^*y=0$ so that $(a^*x, a^*y) \in \beta$. We may assume therefore that $ax, ay \neq 0$. Then there exists a non-zero element z such that $z \leqq ax$ and $z \leqq ay$. From the definition of the natural partial order there are idempotents f and g with $f \mathscr{R}^* z$ and $g \mathscr{L}^* z$ and such that,

$$z = f \cdot ax, R_z^* \leqq R_{ax}^* \quad \text{and} \quad z = ax \cdot g, L_z^* \leqq L_{ax}^*.$$

In particular $z = f \cdot ax = ax \cdot g$: we may likewise find idempotents f_1 and g_1 with $z = f_1 \cdot ay = ay \cdot g_1$ from the fact that $z \leqq ay$. Since $axg = ayg_1$ we have $a^*xg = a^*yg_1 = z_1$. Clearly $z_1 \mathscr{L}^* z$ so that $g \mathscr{L}^* z_1$ and it follows that $z_1 \leqq a^*x$. A similar argument shows that $z_1 \leqq a^*y$—which means that $(a^*x, a^*y) \in \beta$. Together with a dual argument this gives the result.

Corollary 4.3. *In a concordant locally type A semigroup which is categorical at zero β is the finest* 0*-restricted primitive good congruence.*

Proof. Let e be a non-zero idempotent and let $f, g \in \omega(e)$. If $fg=0$ then $feg=0$, but since the semigroup is categorical at zero either $f=fe=0$ or $g=eg=0$. Consequently from the fact that $\omega(e)$ is a semilattice, it follows that $\omega(e) \setminus \{0\}$ is directed.

Now let S be any concordant semigroup not containing a zero and in which each subset of the form $\omega(e)$ is directed under the natural partial order. The semigroup $S^0 = S \cup \{0\}$ is clearly categorical at zero. The set of non-zero elements of $S^0/\beta(S^0)$ forms a completely $\mathscr{J}^*$-simple semigroup, and the restriction of $\beta(S^0)$ to S is the finest completely $\mathscr{J}^*$-simple congruence on S.

Theorem 4.4. *Let S be a concordant semigroup without zero in which each subset of the form $\omega(e)$ is directed and define*

$$\beta(S)=\{(x, y)\colon \text{there exists } z\in S \text{ such that } z\leqq x \text{ and } z\leqq y\}.$$

*Then β is the finest good congruence on S such that S/β is completely $\mathcal{I}^*$-*simple.

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