ON PLANE SECTIONS AND PROJECTIONS OF CONVEX SETS

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Let K be a three-dimensional convex body. It has been conjectured (cf. **3**) that one can always find a plane H such that the intersection $K \cap H$ is, in a certain sense, fairly circular. Instead of the plane section $K \cap H$ one can also consider the orthogonal projection of K onto H. Our aim in this paper is to prove some results concerning this type of problems. It appears that John has found similar theorems (cf. the remarks of Behrend, **1**, p. 717). His proof of the first inequality of our Theorem 1 has been published (**6**). It is based on a property of the ellipse of inertia which will not be used in the present paper.

A non-empty compact convex set S which is contained in some plane of euclidean three-dimensional space E^3 will be called a convex domain. Since S can be considered as a subset of a two-dimensional space we can define the radius $R(S)$ of the (smallest) circumscribed circle, the radius $r(S)$ of a (largest) inscribed circle, the diameter $D(S)$ (maximal width) and the thickness $d(S)$ (minimal width) of S. As a measure for the circular deviation of S we use the quotient

$$\rho(S) = \frac{r(S)}{R(S)} \tag{1}$$

which attains its maximum 1 if and only if S is a circle. The quotient

$$\delta(S) = \frac{d(S)}{D(S)} \tag{2}$$

will also be used, but it is actually a measure for the deviation of S from a domain of constant width, since $\delta(S) \leqq 1$ for any S and $\delta(S) = 1$ characterizes domains of constant width. A single point is considered to be a circle; in this case we define $\rho(S) = \delta(S) = 1$. Of course, many expressions similar to (1) and (2) which may also involve the area and the perimeter of S could be used. For some of these expressions, our method, together with known estimates of Behrend (**1**; **2**), could be used to prove results similar to the inequalities for $\rho(S)$ and $\delta(S)$ which are stated in the following theorem.

We write $M * H$ for the orthogonal projection of a subset M of E^3 onto a plane H.

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THEOREM 1. *Let K be a non-empty compact convex subset of E^3. There are planes G_1, G_2 with projections $K * G_1$, $K * G_2$ such that*

$$\rho(K * G_1) \geqq 1/2, \tag{3}$$

$$\delta(K * G_2) \geqq 1/\sqrt{2}. \tag{4}$$

If p is an interior point of K, there are planes H_1, H_2 which contain p and have the property that

$$\rho(K \cap H_1) \geqq 1/2, \tag{5}$$

$$\delta(K \cap H_2) \geqq 1/\sqrt{2}. \tag{6}$$

For each one of the inequalities (3), (4), (5), (6) *there are three-dimensional compact convex sets K such that the equality sign is necessary.*

The assumption that p is an interior point of K, and hence, that K has non-empty interior is easily seen to be essential for the validity of (5) and (6).

If K has the point p as centre, the projections and sections of Theorem 1 are also centrally symmetric. Since for any centrally symmetric domain T one has obviously $D(T) = 2R(T)$, $d(T) = 2r(T)$, and therefore $\rho(T) = \delta(T)$, it follows from (4) that (3) can be replaced by

$$\rho(K * G_2) \geqq 1/\sqrt{2} \tag{7}$$

and from (6), if p is an interior point of K, that (5) can be replaced by

$$\rho(K \cap H_2) \geqq 1/\sqrt{2}. \tag{8}$$

The examples which will be given to prove that (4) and (6) are best possible show also that (7) and (8) are best possible for the class of compact convex sets with an interior point as centre.

The second theorem which will be proved shows that the only properties of the sections and projections of K which are essential for the existence of the planes of Theorem 1 are the convexity and the continuous dependence of these domains on the planes which contain them.

We denote by $\mathbf{H}_p$ the class of all planes which contain a given point p. Let us assume that each H of $\mathbf{H}_p$ contains a convex domain $C(H)$. We say that $C(H)$ depends continuously on H if the convergence of a sequence $\{H_i\}$ of planes $H_i \in \mathbf{H}_p$ to some H in $\mathbf{H}_p$ implies the convergence of $\{C(H_i)\}$ to $C(H)$. Convergence of a sequence $\{X_i\}$ of compact convex sets to X, denoted by $X_i \to X$, means convergence of $\mu(X_i, X)$ to 0, where μ is the Hausdorff-Blaschke metric for compact convex sets of E^3. ($\mu(X, Y)$ is defined as the greatest lower bound of the set of all numbers x with the property that $X \subset Y + xS$, $Y \subset X + xS$. S is the unit ball in E^3.) Convergence of a sequence $\{L_i\}$ of straight lines which pass through a given point p means convergence of the corresponding pair of points on the unit sphere with p as centre. Convergence of a sequence of planes $H_i \in \mathbf{H}_p$ is defined as convergence of the sequence of lines which are orthogonal to H_i and contain p.

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THEOREM 2. *Let* $\mathbf{H}_p$ *be the class of all planes which contain a given point* p *and assume that each plane* H *of* $\mathbf{H}_p$ *contains a convex domain* $C(H)$ *which depends continuously on* H. *Then, there are planes* H_1, H_2 *in* $\mathbf{H}_p$ *such that*

$$\rho(C(H_1)) \geqq 1/2, \tag{9}$$

$$\delta(C(H_2)) \geqq 1/\sqrt{2}. \tag{10}$$

The constants $1/2$ *and* $1/\sqrt{2}$ *in* (9) *and* (10) *are best possible.*

As in Theorem 1, the functionals ρ and δ are defined by (1) and (2). Again, if all the domains $C(H)$ are centrally symmetric, (9) can be replaced by $\rho(C(H_2)) \geqq 1/\sqrt{2}$.

Since Theorem 1 is a simple consequence of Theorem 2, we present first the proof of Theorem 2. For this purpose we need the following three lemmas. The essential part of our first lemma states that the major axis of an ellipse L, i.e. the line segment in L of length $D(L)$, depends continuously on L, provided that L is not a circle. By an ellipse we mean an affine image of a closed circular disc in E^3. This includes the case where the ellipse degenerates to a line segment or a point. Note that here, as well as in Lemma 3, the ellipses may be contained in different planes of E^3; this causes a considerable complication of the corresponding proofs.

LEMMA 1. *Let* $\{L_i\}$ *be a sequence of ellipses which converges to a convex domain* L. *Then* L *is an ellipse and if none of the ellipses* L_i, L *is a circle, the sequence of the major axis of* L_i *converges to the major axis of* L.

Proof. If $\{\zeta_i\}$ is a sequence of affine transformations of E^3, we say that $\{\zeta_i\}$ is bounded or that $\{\zeta_i\}$ converges if for any system of basis vectors b_1, b_2, b_3 of E^3 the sequences $\{\zeta_i b_1\}$, $\{\zeta_i b_2\}$, $\{\zeta_i b_3\}$ are bounded or converge, respectively. Now, let U be a fixed unit circle (closed circular unit disc) in E^3. To every L_i one can find an affine transformation λ_i such that $L_i = \lambda_i(U)$. Since L is bounded, $\{L_i\}$ is bounded. It follows that $\{\lambda_i\}$ can be assumed to be bounded. Then $\{\lambda_i\}$ has a subsequence $\{\lambda_{ik}\}$ which converges to some affine transformation λ and $L_{ik} = \lambda_{ik}(U)$ converges to $\lambda(U)$. Because of $L_{ik} \to L$ we get $L = \lambda(U)$ which proves that L is an ellipse. Let us denote by A_i and A the major axes of L_i and L, respectively. If $\{A_i\}$ does not converge to A, one can find (by Blaschke's selection theorem) a subsequence $\{A_{im}\}$ of A_i which converges to some line segment A' in L, but not to A. Because of the continuity of the diameter $D(X)$ as a function of a compact set X, one has

$$D(A') = \lim_{m\to\infty} D(A_{im}) = \lim_{m\to\infty} D(L_{im}) = D(L),$$

and therefore $D(A') = D(A)$. This shows that $A' = A$ because the ellipse L contains only one line segment of maximal length since it is not a circle. Hence, we have $A_i \to A$, which completes the proof of Lemma 1.

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For proofs of the next lemma we refer to Behrend (**1**; **2**); Danzer, Laugwitz, and Lenz (**4**); Leichtweiss (**8**); and Zaguskin (**9**).

LEMMA 2. *Let F be a convex domain contained in some plane H of E^3. There is a unique ellipse $L(F)$ in H which contains F and has least possible area (and minimal diameter if F has dimension 0 or 1). Let us denote by $L'(F)$ and $L^*(F)$ the ellipses which are obtained from $L(F)$ by shrinking $L(F)$ with respect to its centre in the ratio $1/2$ and $1/\sqrt{2}$, respectively. Then $L'(F) \subset F$ and, if F is centrally symmetric, we have $L^*(F) \subset F$.*

$L(F)$ will be called the minimal ellipse of F. If L is an ellipse, but not a circle, we denote by $I(L)$ the straight line which contains the major axis of L. Recall that a single point is considered to be a circle. Instead of $I(L(F))$ we write simply $I(F)$. Our third lemma states that $I(F)$ depends continuously on F. It is worth mentioning that the minimal ellipse $L(F)$ does not, in general, depend continuously on F.

LEMMA 3. *Let $\{F_n\}$ be a sequence of convex domains in E^3 with $F_n \to F$. Assume that all the ellipses $L(F_n)$, $L(F)$ are different from a circle, but have a common centre. Then $I(F_n) \to I(F)$.*

Proof. F is either a line segment or two-dimensional. Let us first assume that F is a line segment. If $\{I(F_n)\}$ does not converge to $I(F)$ one can find (by Blaschke's selection theorem and Lemma 1) a subsequence $\{F_{n_i}\}$ of $\{F_n\}$ such that there are a line segment I_0 and an ellipse L_0 with $I(F_{n_i}) \to I_0$, $L(F_{n_i}) \to L_0$ and

$$I_0 \neq I(F). \tag{11}$$

Since F is a line segment, one has for the area $a(F)$ of F that $a(F) = 0$ and therefore $\lim_{i\to\infty} a(F_{n_i}) = 0$. Using the notation of Lemma 2 this implies that $\lim_{i\to\infty} a(L'(F_{n_i})) = 0$. Because of Lemma 2 we can deduce that $a(L_0) = \lim_{i\to\infty} a(L(F_{n_i})) = 0$. Hence, L_0 is also a line segment and L_0 contains F since $L(F_{n_i}) \to L_0$ and $F_{n_i} \subset L(F_{n_i})$. It follows that $I(L_0) = I(F)$ and together with Lemma 1 we obtain $I_0 = I(L_0) = I(F)$. This shows that (11) is not possible and $I(F_n) \to I(F)$ must hold.

Let us now assume that F is a two-dimensional convex domain. Because of Lemma 1 it is certainly sufficient to prove that in this case $L(F)$ depends continuously on F. We prove first the following statement:

If $\{A_i\}$ is a sequence of convex domains which converges to some A and if $\{\sigma_i\}$ is a sequence of affine transformations which converges to the identity transformation, we have $\sigma_i A_i \to A$.

Since $\mu(\sigma_i A_i, A) \leqq \mu(\sigma_i A_i, \sigma_i A) + \mu(\sigma_i A, A)$ and obviously

$$\lim_{i\to\infty} \mu(\sigma_i A, A) = 0,$$

it is sufficient to show that $\lim_{i\to\infty} \mu(\sigma_i A_i, \sigma_i A) = 0$. This is easy to see since for any $\epsilon > 0$ and sufficiently large indices i, $A_i \subset A + \epsilon S$ implies

$$\sigma_i A_i \subset \sigma_i A + \sigma_i(\epsilon S) \subset \sigma_i A + 2\epsilon S,$$

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and $A \subset A_i + \epsilon S$ implies $\sigma_i A \subset \sigma_i A_i + \sigma_i(\epsilon S) \subset \sigma_i A_i + 2\epsilon S$, so that $\mu(\sigma_i A_i, \sigma_i A) < 2\epsilon$ if $\mu(A, A_i) < \epsilon$. Recall that μ denotes the Hausdorff-Blaschke metric and S the closed unit ball in E^3.

Now, if $F_n \to F$ and $\{L(F_n)\}$ does not converge to $L(F)$, one can find (by Blaschke's selection theorem) a subsequence $\{F_{ni}\}$ of $\{F_n\}$ such that $L(F_{ni}) \to L_0$, where (by Lemma 1) L_0 is an ellipse and

$$L_0 \neq L(F). \tag{12}$$

L_0 is two-dimensional since $a(F) > 0$ implies, certainly, that $a(L_0) > 0$. Since the application of a non-singular fixed affine transformation to F and every F_n does not change the problem of proving $L(F_n) \to L(F)$, we may assume that $L(F)$ is a unit circle. To any F_{ni} one can obviously find an affine transformation τ_i such that $L(F_{ni}) = \tau_i L_0$, where $\{\tau_i\}$, and therefore also $\{\tau_i^{-1}\}$, converges to the identity transformation. If we write $A_i = \tau_i^{-1} F_{ni}$, we have

$$L(A_i) = L_0 \tag{13}$$

and, by the above remarks about A_i,

$$A_i \to F. \tag{14}$$

From (13) one obtains $A_i \subset L_0$ and therefore $F \subset L_0$ which shows that

$$a(L(F)) \leqq a(L_0). \tag{15}$$

Consider the plane which contains L_0; because of (13) it contains also every A_i. Denote by U the unit circle of this plane. (14) shows that for any $\epsilon > 0$ and all sufficiently large indices i, $A_i \subset F + \epsilon U$, and therefore

$$L_0 = L(A_i) \subset L(F + \epsilon U) = L(F) + \epsilon U$$

(recall that $L(F)$ is a circle). This is only possible if

$$L_0 \subset L(F). \tag{16}$$

(15) and (16) clearly imply $L_0 = L(F)$, in contradiction to (12). Hence, from $F_n \to F$ follows $L(F_n) \to L(F)$ and this, as we already remarked, completes the proof of the lemma.

Proof of Theorem 2. If X is a subset of E^3, we denote by X^+ the set obtained by central symmetrization of X, i.e. the set of all points of the form $\frac{1}{2}(x - y)$ with $x \in X$, $y \in X$. X^+ has the origin O of E^3 as a centre. If X has already a centre q, the effect of central symmetrization is a translation of X which moves q into O. Since for the class of compact convex sets Minkowski addition and reflection in O are obviously continuous set transformations, X^+ depends continuously on X.

Without any loss of generality we can always assume that $\mathbf{H}_p = \mathbf{H}_O$.

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Let us now make the assumption that none of the ellipses $L(C(H)) = L(H)$ is a circle. Then we can associate with each $L(H)$ the line $I(H) = I((L(H))^+)$ of Lemma 3. Because of the continuity assumptions of Theorem 2 we get from Lemma 3 that $I(H)$ depends continuously on H. To each line J through O corresponds the plane H through O which is orthogonal to J. With H we can associate $I(H)$ which is a line in H that contains O. This way we obtain a continuous mapping of the set of lines through O into itself, or, expressed differently, of the projective plane into itself. Such a mapping is known to have a fixed point (see, e.g., Fenchel (**5**) for an elementary proof). However, our mapping cannot have a fixed point since the line J and its image I are orthogonal to each other. This shows that the assumption that no $L(H)$ is a circle leads to a contradiction. Hence, there is an $H_1 \in \mathbf{H}_O$ such that the minimal ellipse $L(H_1)$ of $C(H_1)$ is a circle. Because of Lemma 2 we have

$$L'(C(H_1)) \subset C(H_1) \subset L(H_1)$$

and this yields immediately (9).

If we use the same method of proof for the centrally symmetric domains $(C(H))^+ = C^+(H)$ instead of $C(H)$ we obtain the existence of a plane $H_2 \in \mathbf{H}_O$ with the property that $L(H_2)$ is a circle and

$$L^*(C(H_2)) \subset C^+(H_2) \subset L(H_2).$$

This, together with the definition of L^*, shows that

$$r(C^+(H_2)) \geqq (1/\sqrt{2})R(C^+(H_2)). \tag{17}$$

From the symmetry of $C^+(H_2)$ follows

$$r(C^+(H_2)) = \tfrac{1}{2}d(C^+(H_2)) = \tfrac{1}{2}d(C(H_2)), \tag{18}$$

$$R(C^+(H_2)) = \tfrac{1}{2}D(C^+(H_2)) = \tfrac{1}{2}D(C(H_2)). \tag{19}$$

(17), (18), and (19) show that (10) holds.

That (9) and (10) are best possible follows from the examples which are presented in connection with the proof of Theorem 1.

Proof of Theorem 1. It is well known and easy to prove that under the assumptions of Theorem 1, $K * H$ and $K \cap H$ depend continuously on H. This remark shows that the inequalities (3), (4), (5), and (6) are an immediate consequence of Theorem 2 if we put $C(H) = K * H$ and $C(H) = K \cap H$, respectively.

It remains to be shown that the constants $1/2$ and $1/\sqrt{2}$ in Theorem 1 are best possible. Let T be a pyramid of height ϵ with a regular triangle of side-length 1 as base and the other three sides of equal length. If ϵ is sufficiently small, the projections of T are either triangles or quadrangles Q with $\rho(Q) = r(Q)/R(Q) < 1/2$. This proves that (3) is best possible. Using instead of T a right pyramid with a square base and of height ϵ, one sees easily that (4) is best possible if ϵ is small enough. T^+ has the same property and is

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centrally symmetric. To construct an example which shows that (5) cannot be improved, denote by Z a right cylinder of height 1 and as base a regular triangle of side-length ϵ. Let p be the centroid of Z. The intersection I of Z with a plane through p is either a triangle or a polygon with $\rho(I) < 1/2$ if ϵ is small enough. Taking instead of Z a cylinder with a small square base we find that (6) is best possible.

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