Canad. Math. Bull. Vol. **24** (3), 1981

THE STIELTJES MOMENT PROBLEM

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We shall apply the spectral theorem for self adjoint operators in Hilbert space to study an operator version of the Stieltjes moment problem [1]. In the course of the work we shall make use of the Friedrichs extension theorem which states that any non-negative symmetric operator in Hilbert space has a non-negative self adjoint extension.

Let X be some Hilbert space in which the inner product of two vectors u and v is denoted by $\langle u, v\rangle$. Let with each point t of some closed interval $[a, b]$ (finite or infinite) be associated a certain bounded self adjoint operator F_t, acting in X. We call F_t a *non-decreasing operator family* if

$$F_{t'} \leq F_{t''} \quad \text{for} \quad t' < t'' \qquad (t', t'' \in [a, b]).$$

For a non-decreasing operator family F_t the limit operators F_{t-0} and F_{t+0} exist in the sense that for any u we have

$$F_{t-0}u = \lim_{s\uparrow t} F_s u, \qquad F_{t+0}u = \lim_{s\downarrow t} F_s u.$$

We shall consider integrals of the form

$$J = \int_a^b g(t)\, dF_t,$$

where g is some continuous function on $[a, b]$, and assume that $F_a = 0$ and $F_t = F_{t-0}$ for $a < t < b$. We shall understand the above integral in the sense

$$\langle Ju, v\rangle = \int_a^b g(t)\, d\langle F_t u, v\rangle \qquad (u, v \in X).$$

The equation

$$J = \int_0^\infty g(t)\, dF_t$$

will be understood in the sense

$$\langle Ju, v\rangle = \lim_{N\to\infty} \int_0^N g(t)\, d\langle F_t u, v\rangle \qquad (u, v \in X).$$

Received by the editors June 18, 1979 and, in revised form, November 12, 1979.

https://doi.org/10.4153/CMB-1981-044-6 Published online by Cambridge University Press

PROPOSITION. *In order that the sequence $S_0, S_1, S_2, \ldots$ of bounded self adjoint operators in a Hilbert space X have a representation of the form*

$$S_k = \int_0^\infty t^k \, dF_t \qquad (k = 0, 1, 2, \ldots), \tag{1}$$

where F_t $(0 \le t < \infty)$ is some non-decreasing operator family, it is necessary and sufficient that for any $u_j \in X$ $(j = 0, 1, 2, \ldots)$ the following two conditions hold:

$$\sum_{j,k=0}^{n} \langle S_{j+k} u_j, u_k \rangle \ge 0 \qquad (n = 0, 1, \ldots), \tag{2}$$

$$\sum_{j,k=0}^{n} \langle S_{j+k+1} u_j, u_k \rangle \ge 0 \qquad (n = 0, 1, \ldots). \tag{3}$$

Proof. From the integral representation (1) we see that conditions (2) and (3) are necessary:

$$\sum_{j,k=0}^{n} \langle S_{j+k} u_j, u_k \rangle = \lim_{N\to\infty} \lim_{p\to\infty} \sum_{m=1}^{p} \langle \Delta_m F_t v_m, v_m \rangle,$$

$$\sum_{j,k=0}^{n} \langle S_{j+k+1} u_j, u_k \rangle = \lim_{N\to\infty} \lim_{p\to\infty} \sum_{m=1}^{p} t_m \langle \Delta_m F_t v_m, v_m \rangle,$$

where

$$t_m = (mN^2)/p, \qquad v_m = \sum_{j=0}^{n} t^j u_j, \qquad \Delta_m F_t = F_{t_{m+1}} - F_{t_m}$$

for $m = 1, 2, \ldots, p$.

Next we show that the conditions (2) and (3) are sufficient. We assume first that formula (2) is strictly positive, i.e., $\sum_{j,k=0}^{n} \langle S_{j+k} u_j, u_k \rangle = 0$ implies $u_j = 0$ for $j = 0, 1, \ldots, n$. We denote by V the set of all polynomials of the form

$$P(t) = \sum_{j=0}^{n} t^j u_j \qquad (u_j \in X; n = 0, 1, \ldots).$$

Let

$$Q(t) = \sum_{k=0}^{m} t^k v_k \qquad (v_k \in X; m = 0, 1, \ldots).$$

We define

$$(P, Q) = \sum_{j=0}^{n} \sum_{k=0}^{m} \langle S_{j+k} u_j, v_k \rangle. \tag{4}$$

It is straightforward to verify that (P, Q) has all the properties of an inner product (here we use the facts that the operators S_j are Hermitian and that formula (2) is strictly positive). Let W be the completion of V relative to the norm induced by this inner product. We define on V the operator H by

$$HP(t) = tP(t) = \sum_{j=0}^{n} t^{j+1} u_j.$$

Since the operators S_j are Hermitian and since V is dense in W we have that H is a symmetric operator in W. From condition (3) we get

$$(HP, P) = (tP, P) = \sum_{j,k=0}^{n} \langle S_{j+k+1} u_j, u_k \rangle \geq 0;$$

this means that H is a non-negative operator.

By virtue of the Friedrichs extension theorem, the operator H has a non-negative self adjoint extension $\tilde{H}$. Let E_t be the spectral family of the operator $\tilde{H}$:

$$\tilde{H} = \int_0^\infty t \, dE_t.$$

Then for any $u, v \in X$ we have

$$(5) \qquad \langle S_k u, v \rangle = (\tilde{H}^k u, v) = \int_0^\infty t^k \, d(E_t u, v) \qquad (k = 0, 1, \ldots).$$

We note that the operators E_t act in W but not in X; however $(E_t u, v)$ is a Hermitian bilinear functional in X and it is continuous:

$$0 \leq (E_t u, u) \leq (u, u) = \langle S_0 u, u \rangle \leq \|S_0\| \langle u, u \rangle.$$

By a theorem of Riesz there corresponds to it a bounded self adjoint operator F_t, acting in X and such that

$$(6) \qquad (E_t u, v) = \langle F_t u, v \rangle \qquad (u, v \in X).$$

F_t will be a non-decreasing operator family. Hence (5) and (6) imply (1).

In case the formula (2) is not strictly positive, we shall say that the polynomials $P, Q \in V$ are *equivalent* if

$$(P - Q, P - Q) = 0,$$

where (P, Q) is the bilinear functional defined in (4). The operator H (multiplying by t) maps equivalent polynomials into equivalent polynomials. To see this it sufficies to verify that (P, P) = 0 implies that $(tP, tP) = 0$. But this is clear from

$$(tP, tP)^2 = (P, t^2P)^2 \leq (P, P)(t^2P, t^2P).$$

By identification of equivalent polynomials we have reduced the case in which the formula (2) is not strictly positive but merely non-negative to the case in which the formula (2) was assumed to be strictly positive and the proof of the Proposition is finished.

Reference

1. N. Dunford and J. T. Schwartz, *Linear Operators, Part II: Spectral Theory*, Interscience Publishers, New York 1968; pp. 1253–1254.

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