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A Banach space of functions of generalized variation

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In this note we show that $BV_k[a, b]$, the linear space of functions of bounded kth variation on $[a, b]$, is a Banach space under the norm $\|\cdot\|_k$, where

$$\|f\|_k = \sum_{s=0}^{k-1} |f^{(s)}(a)| + V_k(f; a, b) .$$

Introduction

It is well known that $BV[a, b]$, the class of functions of bounded variation on $[a, b]$ is a Banach space under the norm $\|\cdot\|_1$, where

$$\|f\|_1 = |f(a)| + V_1(f; a, b) .$$

We generalize this result by showing that when k is an integer greater than one, $BV_k[a, b]$ is a Banach space under the norm $\|\cdot\|_k$, where

$$\|f\|_k = \sum_{s=0}^{k-1} |f^{(s)}(a)| + V_k(f; a, b) ,$$

and where, for convenience of notation, $f^{(s)}(a)$ means $f_+^{(s)}(a)$, and $f_+^{(s)} = \left(f_+^{(s-1)}\right)_+'$. The definitions of $BV_k[a, b]$ and $V_k(f) \equiv V_k(f; a, b)$

Received 21 July 1976. In a recent private communication the author has learnt that Dr Frank Huggins, University of Texas at Arlington, has shown, in particular, that $BV_2[a, b]$ is a Banach space under $\|\cdot\|_2$. He thanks him for this communication.

can be found in Russell [1].

We also take the opportunity to improve some results of Russell [1]. In particular we present a sharper version of Theorem 4, and take this opportunity to point out that "a set of measure zero" can be replaced by "a countable set" in Theorem 12.

Preliminaries

We readily observe that $\|\cdot\|_k$ satisfies all properties of a norm except possibly that $\|f\|_k = 0$ implies $f = 0$. Accordingly, if $\|f\|_k = 0$, then $V_k(f;\, a,\, b) = 0$, and this implies that

$$Q_k(f;\, x_i,\, \ldots,\, x_{i+k}) = 0$$

for any $(k+1)$ points $x_i, \ldots, x_{i+k}$ in $[a, b]$. Using a well known property of divided differences, we conclude that f must be a polynomial of degree $(k-1)$ at most. That $f = 0$ now follows readily.

We now improve our characterization of $BV_k[a, b]$. In Russell [1] it was shown that

> $BV_k[a, b] = \{f : f = f_1 - f_2$, where f_1 and f_2 are
> 0-, 1-, ..., k-convex functions having right and left $(k-1)$th Riemann * derivatives at a and b respectively$\}$.

If $k = 2$ it follows immediately that the $(k-1)$th Riemann * derivatives at a and b can be replaced by the usual right and left hand derivatives respectively. Assume now that $k \geq 3$, and that $f \in BV_k[a, b]$. Then according to Theorem 12 of Russell [1], $f^{(k-2)}$ is continuous on $[a, b]$ and belongs to $BV_2[a, b]$. Thus $f_+^{(k-1)}(a)$ and $f_-^{(k-1)}(b)$ must exist. We summarize the previous discussion in the following

THEOREM 1. *If k is an integer greater than or equal to one, then*

> $BV_k[a, b] = \{f : f = f_1 - f_2$, *where* f_1 *and* f_2 *are*
> *0-, 1-, ..., k-convex functions having right and left $(k-1)$th derivatives at a and b respectively*$\}$.

Our next result is an improved version of Theorem 4 of Russell [1].

THEOREM 2. *If* $f \in BV_{k+1}[a, b]$ *, and* $k \geq 0$ *, then* $Q_k(f; y_0, y_1, \ldots, y_k)$ *is bounded when* $a \leq y_i \leq b$ *,* $i = 0, 1, \ldots, k$ *. More precisely,*

(1) $|Q_k(f; y_0, y_1, \ldots, y_k) - Q_k(f; a_0, a_1, \ldots, a_k)| \leq V_{k+1}(f; a, b)$ *,*

where $a_0, a_1, \ldots, a_k$ *is a fixed* π *subdivision of* $[a, b]$ *.*

Proof. Let $a_0, a_1, \ldots, a_k$ be a fixed π subdivision of $[a, b]$, and let $y_0, y_1, \ldots, y_k$ be another π subdivision of $[a, b]$ such that $a_0 < \ldots < a_{k-1} < y_0 < a_k < y_1 < \ldots < y_k$. Re-label the points $a_0, \ldots, a_k$, $y_0, \ldots, y_k$ as $x_0, x_1, \ldots, x_{2k+1}$, where

$$x_i = a_i \ , \quad i = 0, 1, \ldots, k-1 \ ,$$

$$x_k = y_0 \ ,$$

$$x_{k+1} = a_k \ ,$$

$$x_i = y_{i-k-1} \ , \quad i = k+2, \ldots, 2k+1 \ .$$

Using Theorem 1 of Russell [1], we obtain

$$\begin{aligned}
&Q_k(f; y_0, y_1, \ldots, y_k) - Q_k(f; a_0, a_1, \ldots, a_k) \\
&\quad = \alpha_1 Q_k(f; y_0, a_k, y_1, \ldots, y_{k-1}) + \beta_1 Q_k(f; a_k, y_1, \ldots, y_k) \\
&\qquad - \alpha_2 Q_k(f; a_0, a_1, \ldots, a_{k-1}, y_0) - \beta_2 Q_k(f; a_1, \ldots, a_{k-1}, y_0, a_k) \\
&\qquad\qquad \text{where } \alpha_1 + \beta_1 = 1 = \alpha_2 + \beta_2 \\
&\quad = \alpha_1 Q_k(f; x_k, x_{k+1}, \ldots, x_{2k}) + \beta_1 Q_k(f; x_{k+1}, \ldots, x_{2k+1}) \\
&\qquad - \alpha_2 Q_k(f; x_0, x_1, \ldots, x_k) - \beta_2 Q_k(f; x_1, \ldots, x_{k+1}) \\
&\quad = [Q_k(f; x_k, \ldots, x_{2k}) - Q_k(f; x_0, \ldots, x_k)] \\
&\qquad + \beta_1 [Q_k(f; x_{k+1}, \ldots, x_{2k+1}) - Q_k(f; x_k, \ldots, x_{2k})] \\
&\qquad + \beta_2 [Q_k(f; x_0, \ldots, x_k) - Q_k(f; x_1, \ldots, x_{k+1})]
\end{aligned}$$

$$= \sum_{i=1}^{k} [Q_k(f; x_i, \ldots, x_{i+k})-Q_k(f; x_{i-1}, \ldots, x_{i+k-1})]$$
$$+ \beta_1[Q_k(f; x_{k+1}, \ldots, x_{2k+1})-Q_k(f; x_k, \ldots, x_{2k})]$$
$$+ \beta_2[Q_k(f; x_0, \ldots, x_k)-Q_k(f; x_1, \ldots, x_{k+1})]$$
$$= \alpha_2[Q_k(f; x_1, \ldots, x_{k+1})-Q_k(f; x_0, \ldots, x_k)]$$
$$+ \sum_{i=2}^{k} [Q_k(f; x_i, \ldots, x_{i+k})-Q_k(f; x_{i-1}, \ldots, x_{i+k-1})]$$
$$+ \beta_1[Q_k(f; x_{k+1}, \ldots, x_{2k+1})-Q_k(f; x_k, \ldots, x_{2k})] .$$

Taking absolute values now, and noting that $0 \le \alpha_2 \le 1$, $0 \le \beta_1 \le 1$, gives the required inequality.

An argument similar to that above establishes (1) in cases corresponding to other relative distributions of the sets of points $y_0, \ldots, y_k$ and $a_0, \ldots, a_k$.

COROLLARY. *If* $f \in BV_{k+1}[a, b]$ *, and* $k \ge 0$ *, then*

(2) $$\sup_{\pi} |Q_k(f; x_0, \ldots, x_k)| - \inf_{\pi} |Q_k(f; x_0, \ldots, x_k)| \le V_{k+1}(f; a, b) .$$

REMARK. The inequality (2) is best possible as illustrated by the case $k = 1$, $a = 0$, $b = 1$, and

$$f(x) = \begin{cases} 0 , & 0 \le x \le \tfrac{1}{2} , \\ x - \tfrac{1}{2} , & \tfrac{1}{2} \le x \le 1 . \end{cases}$$

THEOREM 3. *If* $f \in BV_{k+1}[a, b]$ *, and* $k \ge 0$ *, then* $f \in BV_k[a, b]$ *, and*

(3) $$V_k(f; a, b) \le k(b-a)\left[V_{k+1}(f; a, b) + \inf_{\pi} |Q_k(f; x_0, \ldots, x_k)|\right] .$$

Proof. The first part of the theorem follows from Theorem 10 of Russell [1].

For the second part, it follows from Theorem 2, Corollary, that for any π subdivision $x_i, \ldots, x_{i+k}$ of $[a, b]$,

$$(x_{i+k}-x_i)\,|Q_k(f;\,x_i,\,\ldots,\,x_{i+k})|$$
$$\le (x_{i+k}-x_i)\left[V_{k+1}(f;\,a,\,b) + \inf_{\pi}|Q_k(f;\,x_0,\,\ldots,\,x_k)|\right] .$$

Summing from $i = 0$ to $i = n - k$, and taking the supremum gives (3).

REMARK. The constant in (3) is best possible as illustrated by the case $k = 1$, $a = 0$, $b = 1$, $f(x) \equiv x$.

Main results

THEOREM 4. *If* $\{g_n\}$ *is a sequence of functions in* $BV_{k+1}[a,\,b]$, $k \ge 0$, *such that* $\|g_n\|_{k+1} \to 0$ *as* $n \to \infty$, *then* $\|g_n\|_k \to 0$ *as* $n \to \infty$.

Proof. It follows immediately from Theorem 10 of Russell [1] that $g_n \in BV_k[a,\,b]$ for all n. Since $\|g_n\|_{k+1} \to 0$ as $n \to \infty$, given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\|g_n\|_{k+1} < \varepsilon$$

whenever $n > N(\varepsilon)$. Hence, whenever $n > N(\varepsilon)$,

$$\sum_{s=0}^{k}\left|g_n^{(s)}(a)\right| < \varepsilon , \tag{4}$$

and

$$V_{k+1}(g_n;\,a,\,b) < \varepsilon . \tag{5}$$

Since $g_n \in BV_{k+1}[a,\,b]$, $\left|g_n^{(k)}(a)\right|$ exists and is less than ε whenever $n > N(\varepsilon)$, by (4). Hence, whenever $n > N(\varepsilon)$,

$$\inf_{\pi}|Q_k(g_n;\,x_0,\,\ldots,\,x_k)| < 2\varepsilon k! .$$

It now follows from Theorem 3 that $V_k(g_n;\,a,\,b) \to 0$, and hence that $\|g_n\|_k \to 0$ as $n \to \infty$.

We now consider a Cauchy sequence $\{f_n\}$ in $BV_k[a,\,b]$. Consequently, for each $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$(6) \qquad \sum_{s=0}^{k-1} \left|f_m^{(s)}(a)-f_n^{(s)}(a)\right| + V_k(f_m-f_n;\ a,\ b) < \varepsilon\ ,$$

whenever m, n exceed $N(\varepsilon)$.

If $\{f_n\}$ is a Cauchy sequence in $BV_k[a, b]$, it follows from Theorem 4 that $\{f_n(x)\}$ is a Cauchy sequence for each $x \in [a, b]$. Accordingly, we define

$$f(x) = \lim_{n\to\infty} f_n(x)\ , \quad x \in [a, b]\ .$$

THEOREM 5. *If* $f_n \in BV_k[a, b]$ *for all* n *, and* $f(x) = \lim_{n\to\infty} f_n(x)$ *,* $x \in [a, b]$ *, then* $f \in BV_k[a, b]$.

Proof. Let $S_\pi(f)$ denote an approximating sum for $V_k(f;\ a,\ b)$. Let $\{f_n\}$ be a Cauchy sequence in $BV_k[a, b]$, so that for each $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $\|f_m-f_n\|_k < \varepsilon$ whenever m and n exceed $N(\varepsilon)$. Therefore, whenever m, n exceed $N(\varepsilon)$,

$$S_\pi(f_m-f_n) = \sum_{i=0}^{n-k} |Q_{k-1}(f_m-f_n;\ x_i,\ \ldots,\ x_{i+k-1})-Q_{k-1}(f_m-f_n;\ x_{i+1},\ \ldots,\ x_{i+k})| < \varepsilon$$

for all π subdivisions of $[a, b]$. Letting $m \to \infty$ in the last inequality gives

$$S_\pi(f-f_n) \le \varepsilon$$

for all π subdivisions of $[a, b]$, and whenever $n > N(\varepsilon)$. Let n_0 be a fixed integer exceeding $N(\varepsilon)$, and let $\sup_\pi S_\pi(f_{n_0}) = K_{n_0}$. Then

$$S_\pi(f) \le S_\pi(f-f_{n_0}) + S_\pi(f_{n_0}) \le \varepsilon + K_{n_0}$$

for all π subdivisions of $[a, b]$. Hence $f \in BV_k[a, b]$, as required.

It now remains to show that $\|f_n-f\|_k \to 0$ as $n \to \infty$; that is, that

(7) $$\sum_{s=0}^{k-1}\left|f_n^{(s)}(a)-f^{(s)}(a)\right| + V_k\left(f_n-f\right) \to 0 \quad \text{as} \quad n \to \infty .$$

It is clear that $V_k\left(f_n-f\right)$ and $f_n(a)$ both converge to 0 as $n \to \infty$, so we now show that $f_n^{(s)}(a) - f^{(s)}(a) \to 0$ as $n \to \infty$ when $s = 1, 2, \ldots, k-1$.

We first observe that $f^{(s)}(a)$ exists when $s = 1, 2, \ldots, k-1$, because $f \in BV_k[a, b]$. Let $s = k - 1$. Then it follows from Theorem 12 of Russell [1] that $f^{(k-1)}(x)$ exists on $[a, b]$, except possibly on a countable set. For each n, let $A_n = \left\{x : f_n^{(k-1)}(x) \text{ exists}\right\}$, so that $[a, b]\backslash A_n$ is countable. Let $x > a$, and $x \in A = \bigcap_1^\infty A_n$. Since $V_k\left(f_m-f_n\right) < \varepsilon$ whenever m, n exceed $N(\varepsilon)$,

$$|Q_{k-1}\left(f_m-f_n; x, x+h, \ldots, x+(k-1)h\right) - Q_{k-1}\left(f_m-f_n; a, a+h, \ldots, a+(k-1)h\right)| < \varepsilon ,$$

for all π subdivisions of $[a, b]$ such that $a + (k-1)h < x$. Letting $h \to 0$ gives

$$\left|\left[f_m^{(k-1)}(x)-f_n^{(k-1)}(x)\right] - \left[f_m^{(k-1)}(a)-f_n^{(k-1)}(a)\right]\right| \le (k-1)!\varepsilon ,$$

whenever $x \in A$ and m, n exceed $N(\varepsilon)$. Therefore, using (6), it follows that

$$\begin{aligned}\left|f_m^{(k-1)}(x)-f_n^{(k-1)}(x)\right| &\le (k-1)!\varepsilon + \left|f_m^{(k-1)}(a)-f_n^{(k-1)}(a)\right| \\ &< (k-1)!\varepsilon + \varepsilon ,\end{aligned}$$

whenever $x \in A$ and m, n exceed $N(\varepsilon)$. Thus $\left\{f_n^{(k-1)}(x)\right\}$ converges uniformly to $\phi(x)$, say, on A.

Since $f_n^{(k-2)}$ is absolutely continuous on $[a, b]$, it follows that

$$f_n^{(k-2)}(x) - f^{(k-2)}(a) = \int_a^x f_n^{(k-1)}(t)dt ,$$

and hence that

$$f^{(k-2)}(x) - f^{(k-2)}(a) = \int_a^x \phi(t)dt .$$

Consequently $f^{(k-1)}(x)$ exists, almost everywhere, and equals $\phi(x)$. In particular,

$$f^{(k-1)}(a) = \lim_{n\to\infty} f_n^{(k-1)}(a) .$$

Similarly,

$$f^{(s)}(a) = \lim_{n\to\infty} f_n^{(s)}(a) \quad \text{when} \quad s = 1, 2, \ldots, k-2 .$$

Returning to (7) we see that $\|f_n-f\|_k \to 0$ as $n \to \infty$, and so we conclude that $BV_k[a, b]$ is a Banach space under $\|\cdot\|_k$.

Reference

[1] A.M. Russell, "Functions of bounded kth variation", *Proc. London Math. Soc.* (3) **26** (1973), 547-563.

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