A Geometrical Proof of a Theorem connected with the Tetrahedron.

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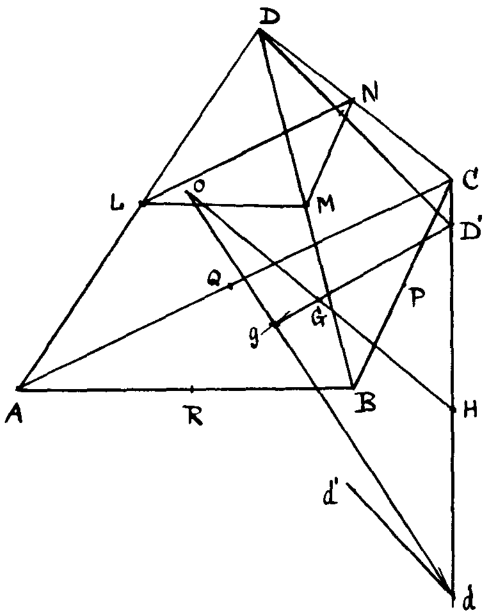
1. The six planes through the middle points of the edges of a tetrahedron perpendicular to the opposite edges are concurrent.

Let ABCD be the tetrahedron, P, Q, R, L, M, N the middle points of the edges as in the figure. Then, adopting the notation L.BC for the plane through L perpendicular to BC, we have L.BC ⊥ MN, M.CA ⊥ NL, N.AB ⊥ LM.



Therefore the three planes

I. L.BC, M.CA, N.AB meet in a straight line W (say) ⊥ plane LMN

Similarly

II.	L.BC, Q.BD, R.CD	"	" "	"	"	X	"	"	LQR
III.	M.CA, R.CD, P.AD	"	" "	"	"	Y	"	"	MRP
IV.	N.AB, P.AD, Q.BD	"	" "	"	"	Z	"	"	NPQ

Now the plane of W and X is L.BC; ∴ by I. and II. M.CA and R.CD must intersect in a straight line through the intersection of W and X; ∴ by III. Y passes through X. Similarly Z passes through X.

2. The point of intersection is the centre of the Hyperboloid whose generators are the perpendiculars from the vertices of the tetrahedron on the opposite faces.

Suppose DD′ is one of these perpendiculars and that d is the orthocentre of the △ABC. Draw dd' normal to the plane ABC. Now W passes through the orthocentre of the △LMN and is perpendicular to the plane of LMN; hence W lies in the plane of DD′ and dd'. Again the latter plane is tangent to the Asymptotic cone of the Hyperboloid, because DD′ and dd' are parallel generators of opposite species. (It is easy to show that dd' meets all the perpendiculars from the vertices on the opposite faces). Hence W, and similarly the other lines, lie in tangent planes of the Asymptotic cone. But the only point that these planes can have in common is the centre of the Hyperboloid.

3. The centroid of the tetrahedron is the middle point of the line joining the circumcentre and the centre of the Hyperboloid.

Suppose O is the circumcentre of the triangle ABC and g its centroid, then $Og : gd = 1 : 2$. Now if we join gD′ and divide it at G so that $g\mathrm{G} : \mathrm{GD}' = 1 : 3$, G will be the orthogonal projection of the centroid of the tetrahedron. Again the line W, as it passes through the orthocentre of the triangle LMN, must pass through H the the middle point of D′d, which is consequently the orthogonal projection of the centre of the Hyperboloid. We shall prove that O, G, H are collinear and that OG = GH.

For, in the triangle $gD'd$, we have

$$\frac{dH}{HD'}=1, \quad \frac{D'g}{Gg}=3, \quad \frac{Og}{Od}=\tfrac{1}{3},$$

$$\therefore \quad \frac{dH}{HD'}\cdot\frac{D'g}{Gg}\cdot\frac{Og}{Od}=1 \quad \therefore \quad \text{O, G, H are collinear.}$$

Again considering gGD' as a transversal to triangle OdH, we have

$$\frac{OG}{GH}\cdot\frac{HD'}{D'd}\cdot\frac{dg}{gO}=1\ ; \text{ but } \frac{HD'}{D'd}=\tfrac{1}{2}, \text{ and } \frac{dg}{gO}=2,$$

$$\therefore \quad OG=GH.$$

Hence the orthogonal projections (G, O, H) of the points in question are so related that G, O, H are collinear and OG = GH. Now the same is true of the orthogonal projections on any face of the tetrahedron; therefore the centroid, circumcentre, and centre of the Hyperboloid are related in the same manner.