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A NEW PROOF OF THE REALISATION OF CUBIC TABLEAUX

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Abstract

By means of the dynamics on trees introduced by Emerson, DeMarco and McMullen, we give a new proof of the realisation of cubic tableaux.

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1. Introduction

The dynamics of cubic polynomials has been studied extensively. A classic work is due to Branner and Hubbard [1, 2]. In [2], they introduced a powerful tool, the *tableau*, which was used to describe combinatorial structures of the Julia sets of cubic polynomials. They also considered the realisation problem of tableaux. Namely, when can a tableau with a single critical point be realised by a cubic polynomial?

Recall that for a cubic polynomial f with complex coefficients, the set

$$K(f) = \{z \in \mathbb{C} \mid \text{the sequence } \{f^{\circ n}(z)\}_{n\geq 1} \text{ is bounded}\}$$

is called the *filled-in Julia set* of f, where $f^{\circ n}$ is the nth iteration of f. The complement $\Omega(f) := \mathbb{C} \setminus K(f)$ is the *basin of infinity*, and the *Julia set* $J(f)$ is defined as the boundary of $K(f)$. It is well known that the *escape rate* function $G : \mathbb{C} \to [0, +\infty)$, defined by

$$G(z) = \lim_{n\to\infty} \frac{1}{3^n} \log^+ |f^{\circ n}(z)|,$$

is continuous and satisfies $G(f(z)) = 3G(z)$ and $K(f) = G^{-1}(0)$; see [1, 8].

Suppose that the Julia set of a cubic polynomial f is disconnected. Then there exists at least one critical point which lies in the basin $\Omega(f)$. Let c_0 and c_1 be two critical points of f and suppose that c_0 is the faster-escaping one, which means that $G(c_0) \geq G(c_1)$. Write $r_0 = G(c_0) > 0$. For every $k \geq 0$, the locus $G^{-1}([0, r_0/3^{k-1}))$ is the disjoint union of a finite number of open topological discs. Each such open disc is called a *puzzle piece* P_k of depth k. For each integer $N \geq 1$ such that $G(c_0)/3^N \leq G(c_1) < G(c_0)/3^{N-1}$ and $0 \leq k < N+1$, the *critical puzzle piece* $P_k(c_1)$ is defined as the component of $G^{-1}([0, r_0/3^{k-1}))$ containing c_1. Here N is allowed to be infinity, which means c_1 is not escaping, and therefore $G(c_1) = 0$ and there are infinitely many critical puzzle pieces.

The (cubic) *tableau* or *marked grid* of f of size N is an array $\{M(j,k) \in \{0,1\} \mid j,k \geq 0 \text{ and } j+k \leq N\}$, defined by the condition

$$M(j,k) = 1 \quad \text{if and only if } f^{\circ k}(c_1) \in P_j(c_1).$$

A position $M(j,k)$ is said to be *marked* if $M(j,k) = 1$. The marked grid can be depicted as a subset in the 4th quadrant of the $\mathbb{Z}^2$-lattice, where $j \geq 0$ denotes the distance to the positive x-axis and $k \geq 0$ denotes the distance to the negative y-axis. Branner and Hubbard [2] showed that the marked grids of cubic polynomials must satisfy some rules. An (abstract) *marked grid of size N* (allowed to be infinity) is an array $M = \{M(j,k) \in \{0,1\} \mid j,k \geq 0 \text{ and } j+k \leq N\}$ which satisfies the following rules [2, 4].

(R0) For each $n \leq N$, $M(n,0) = M(0,n) = 1$.
(R1) If $M(j,k) = 1$, then $M(i,k) = 1$ for all $i \leq j$.
(R2) If $M(j,k) = 1$, then $M(j-i, k+i) = M(j-i, i)$ for all $0 \leq i \leq j$.
(R3) If $j+k < N$, $M(j,k) = 1$, $M(j+1,k) = 0$, $M(j-i,i) = 0$ for $0 < i < m$, and $M(j-m+1, m) = 1$, then $M(j-m+1, k+m) = 0$.
(R4) If $j+k < N$, $M(j,k) = 1$, $M(1,j) = 0$, $M(j+1,k) = 0$, and $M(j-i,k+i) = 0$ for all $0 < i < j$, then $M(1, j+k) = 1$.

The rule (R4) was omitted in [2], but is necessary in their proof. The original rule (R4) first appeared in the literature in [6], but was just a special case of (R4) stated above. Later, the correct statement was found by Kiwi [7], and independently by DeMarco and McMullen [3].

A marked grid M of size N is said to be *realised* by a cubic polynomial f, if the tableau of f coincides with M. Branner and Hubbard proved the following theorem.

THEOREM 1.1 (Realisation of tableaux, [2]). *Every marked grid can be realised by a cubic polynomial.*

In general, the polynomial which realises a given marked grid is not unique. In this paper, we will use dynamics on trees to give a new proof of Theorem 1.1. The main idea is to show that a marked grid of size N can be realised by a 'nice' cubic children preserving map (see Section 3) of the same size, and this nice cubic children preserving map can be extended to a polynomial-like tree with degree three. Since every cubic polynomial-like tree can be realised by a cubic polynomial (Theorem 2.2), Theorem 1.1 then follows.

2. Dynamics on trees and preliminary results

To record the combinatorial information of the dynamics of f on $\Omega(f)$, dynamics on trees was introduced in [5, 3] successively.

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First, we recall some definitions related to dynamics on abstract trees defined in [3]. A *simplicial tree* T is a nonempty, connected, locally finite, one-dimensional simplicial complex without cycles. The set of vertices and edges (unoriented, closed) of T will be denoted by $V(T)$ and $E(T)$, respectively. The edges adjacent to a given vertex $v \in V(T)$ form a finite set $E_v(T)$, whose cardinality $\mathrm{val}(v)$ is defined as the *valence* of v. The space of *ends* of T, denoted by ∂T, is the totally disconnected space obtained as the inverse limit of the set of connected components of $T \setminus K$ as K ranges over all finite subtrees.

A map $F : T_1 \to T_2$ between simplicial trees is called a *branched covering* if: (1) F is proper, open and continuous; and (2) F is *simplicial* (every edge of T_1 maps linearly to another edge in T_2). A *local degree function* for a branched covering $F : T_1 \to T_2$ is a map $\deg : E(T_1) \cup V(T_1) \to \{1, 2, 3, \ldots\}$ satisfying for every $v \in V(T_1)$ the inequality

$$2\deg(v) - 2 \geq \sum_{e \in E_v(T_1)} (\deg(e) - 1), \tag{2.1}$$

as well as for every $e \in E_v(T_1)$ the equality

$$\deg(v) = \sum_{e' \in E_v(T_1) : F(e') = F(e)} \deg(e'). \tag{2.2}$$

The *global degree* $\deg(F)$ is defined by

$$\deg(F) = \sum_{F(e_1)=e_2} \deg(e_1) = \sum_{F(v_1)=v_2} \deg(v_1) \tag{2.3}$$

for any edge e_2 and vertex v_2 in T_2. It is easy to verify that (2.3) is well defined by using (2.2) and the connectedness of T_2. In this paper, we only consider the case in which $\deg(F) = 3$.

Definition 2.1. Let $F : T \to T$ be a branched covering map of a simplicial tree to itself. Two points $x, y \in T$ are in the same *grand orbit* if $F^{\circ m}(x) = F^{\circ n}(y)$ for some $m, n > 0$. The branched covering map $F : T \to T$ is called *polynomial-like* if:

(1) there exists a unique isolated end $\infty \in \partial T$;
(2) there is a local degree function satisfying (2.1) and (2.2);
(3) the tree T has no *leaves* (vertices of valence one); and
(4) the grand orbit of any vertex includes a vertex of valence three or more.

Moreover, the 2-tuple (T, F) is called a polynomial-like tree.

Theorem 2.9 in [3] shows that if a branched covering map $F : T \to T$ has a local degree function which satisfies (2.1) and (2.2), then the local degree function is unique. More detailed information about polynomial-like trees can be found in [3].

Let f be a cubic polynomial with disconnected Julia set, and T be the quotient space, which is a simplicial tree obtained by identifying points in each component of the level set of G into a single point. Then f induces a map $F : T \to T$ on the quotient

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tree. The 2-tuple $\tau(f) = (T, F)$ is called the *quotient* of f. By [3, Theorem 3.1], the quotient $\tau(f)$ of f is a polynomial-like tree. One of the main results in [3] is the following theorem.

THEOREM 2.2 (Realisation of trees, [3]). *Every polynomial-like tree* (T, F) *arises as the quotient of a polynomial* f.

Let T be a simplicial tree which has at least one isolated end. (Note that the leaves of T are not included.) Choose one of them, marked by ∞. Then every vertex close enough to ∞ has valence two. If T has one vertex of valence three or more, then there must exist a unique vertex v_0 of valence three or more which is closest to ∞, and we call it the *base* of T. If every vertex of T has valence less than three, choose one vertex with valence two, marked by v_0.

The *combinatorial height function* $h : V(T) \to \mathbb{Z}$ is defined by setting $|h(v)|$ to be the minimal number of edges needed to connect v to v_0. Moreover, the sign of h is determined by the condition that $h(v) \le 0$ if v lies on the unique shortest path from v_0 to ∞, while $h(v) > 0$ otherwise.

A vertex v is called a *child* of vertex w if $h(v) = h(w) + 1$ and there exists an edge connecting v with w.

DEFINITION 2.3. A simplicial map $F : T \to T$ is called *children preserving* if for every $v \in V(T)$, the image of every child of v is also a child of $F(v)$.

Note that a polynomial-like tree must be children preserving. It is easy to see that if $h(v) = k$, then $h(F(v)) = k + U(F)$ for a children preserving map $F : T \to T$, where $U(F)$ is a constant depending only on F. Moreover, unlike the polynomial-like tree, a children preserving map need not be surjective.

A *branch* of T is a sequence $(v_k)_{k=1}^{n}$, where $h(v_k) = k$ and v_{k+1} is a child of v_k for $1 \le k < n$, and the positive integer n is required to be largest in the sense that either $n = \infty$ or v_n has valence one.

DEFINITION 2.4. A children preserving map $F : T \to T$ is called *cubic* of *size* $N > 0$ if:

(a) $U(F) = -1$; and
(b) there exists a degree function $\deg : V(T) \cup E(T) \to \{1, 2, 3\}$ satisfying:
  (b1) $\deg(s_k) = 2$ for a branch $(s_k)_{k=1}^{N}$, where $h(s_k) = k$ for $1 \le k < N + 1$;
  (b2) $\deg(s_l) = 3$ for $l \le 0$, where $s_0 = v_0$ is the base point and $h(s_l) = l \le 0$;
  (b3) $\deg(v) = 1$ for every vertex $v \in T \setminus \bigcup_{k<N+1}\{s_k\}$; and
  (b4) when v_1 and v_2 are the two ends of an edge $e \in E(T)$, if $h(v_1) = h(v_2) + 1$, then $\deg(e) = \deg(v_1)$.

Since $\{s_k\}_{k<0}$ is not important in the dynamics on T, the unique sequence $S := (s_k)_{k<N+1}$ is called the *critical branch* of the cubic children preserving map $F : T \to T$.

The condition (a) in Definition 2.4 means that the images of whole trees go ahead one step towards the isolated end ∞ under the action of F. Note that we only consider the realisation problem up to level N. So a cubic polynomial with two escaping critical points c_0 and c_1 such that $G(c_0)/3^N \leq G(c_1) < G(c_0)/3^{N-1}$ may also realise a marked grid with size N. In this case, the polynomial-like tree induced by the cubic polynomial has two escaping critical points (which means $U(F) = -2$). But we can also obtain a well-defined polynomial-like tree by removing the grand orbit of $\tau(c_1)$ up to level N, where τ is the quotient map by identifying points in each component of the level set of G into a single point. In fact, for any marked grid of size $N < \infty$, there exists at least one cubic polynomial one of whose critical points is not escaping, and this polynomial realises the marked grid.

3. The extension of cubic children preserving maps

For a marked grid of size N, we need to construct a cubic children preserving map of the same size such that the dynamics is well defined. Also, we must show that the cubic children preserving map constructed from the marked grid can be extended to a polynomial-like tree with deg = 3; then Theorem 1.1 follows from Theorem 2.2.

Let $F' : T' \to T'$ and $F : T \to T$ be two cubic children preserving maps. The map $F' : T' \to T'$ is called an *extension* of $F : T \to T$ if T is a subtree of T', the restriction $F'|T$ coincides with F and they have the same degree function defined on T. Let $C(v)$ denote the set of all the children of v.

Lemma 3.1 (Extension lemma). *Let $S = (s_k)_{k<N+1}$ denote the critical branch of a cubic children preserving map $F : T \to T$ of size $N \geq 1$. Then $F : T \to T$ can be extended to a polynomial-like tree of degree three if and only if:*

(1) *for $v \in V(T) \setminus S$, if $v_1, v_2 \in C(v)$ and $v_1 \neq v_2$, then $F(v_1) \neq F(v_2)$;*
(2) *for $i \neq 0$, if $u \in C(s_i)$ and $u \neq s_{i+1}$, then $F(u) \neq F(s_{i+1})$;*
(3) *if s_i has three different children u, v, w and $F(u) = F(v)$, then $F(w) \neq F(u)$; and*
(4) *for any $v \in V(T)$, $\sum_{F(u)=v} \deg(u) \leq 3$.*

Proof. If a cubic children preserving map $F : T \to T$ can be extended to a polynomial-like tree $F' : T' \to T'$ of degree three, then (2.1)–(2.3) hold for $F' : T' \to T'$. It is straightforward to verify that (2.2) implies (1)–(3) and (2.3) implies (4). This proves the 'only if' part.

We give the proof of the 'if' part by induction. Consider the base point s_0 (namely, v_0) of T. We know that s_0 has at most two children by condition (4). If $C(s_0) = \{s_1\}$, we add a child to s_0 with degree one such that (2.3) holds. Denote the extended cubic children preserving map by $F_1 : T_1 \to T_1$. Equation (2.2) also holds for those vertices in T_1 with combinatorial height less than 1.

Assume that (2.1)–(2.3) hold for those vertices in T_k with combinatorial height less than k for some $k \geq 1$. For each vertex $v \in T_k$ such that $h(v) = k$, check the children of v. If (2.2) is not true, add sufficiently many children with degree one to v such that (2.2) holds. Now we obtain a cubic children preserving map $F_{k+1} : T_{k+1} \to T_{k+1}$, which is the extension of $F_k : T_k \to T_k$, such that (2.2) holds for the vertices in T_{k+1} with combinatorial height less than $k + 1$.

In this way, we can extend $F : T \to T$ to a cubic children preserving map $F_\infty : T_\infty \to T_\infty$ with size ∞. Moreover, the formulas (2.1)–(2.3) hold for every vertex and edge in T_∞. Considering the four conditions of polynomial-like trees which are required in Section 2, it follows that $F_\infty : T_\infty \to T_\infty$ is a polynomial-like tree of degree three since the global degree satisfies $\deg(F_\infty) = \sum_{F_\infty(v)=s_0} \deg(v) = 3$. The proof is complete. □

Remark 3.2. The four conditions in Lemma 3.1 are the key to computing the tree codes of cubic polynomials in [3]. The conditions (2)–(4) are no longer necessary for the realisation of tableaux with multi-critical points.

Definition 3.3. A cubic children preserving map $F : T \to T$ is called *nice* if it satisfies the conditions in Lemma 3.1.

4. From marked grids to nice cubic children preserving maps

In this section, we want to extract a nice cubic children preserving map from a given marked grid with fixed size. We say a marked grid of size N is *realised* by a nice cubic children preserving map of the same size, if it satisfies the condition

$$M(j, k) = 1 \quad \text{if and only if } \deg(F^{\circ k}(s_{j+k})) \geq 2.$$

Lemma 4.1. *Every marked grid of size N can be realised by a nice cubic children preserving map of the same size.*

Proof. We use induction to prove this lemma. For $N = 1$, let $T_1 = (s_k)_{k=1}^{-\infty}$ be such that s_k is the unique child of s_{k-1}. Define $F_1 : T_1 \to T_1$ by $F_1(s_k) = s_{k-1}$, where $k \leq 1$. The degree function is defined by $\deg(s_1) = 2$ and $\deg(s_i) = 3$ for $i \leq 0$. Obviously, $F_1 : T_1 \to T_1$ is a nice cubic children preserving map which realises the unique marked grid of size 1.

If $N = 2$, there are two different marked grids of size 2, which correspond to whether the position $M(1, 1)$ is marked or not, respectively. If $M(1, 1)$ is marked, by adding a child s_2 of degree two to s_1, we obtain a simplicial tree T_2. After extending $F_1 : T_1 \to T_1$ to $F_2 : T_2 \to T_2$ such that $F_2(s_k) = s_{k-1}$ for all $k \leq 2$, the nice cubic children preserving map $F_2 : T_2 \to T_2$ is a realisation of the marked grid of size $N = 2$ with $M(1, 1) = 1$. Similarly, if $M(1, 1)$ is not marked, by adding a child s_2 of degree two to s_1 and a child v_1 of degree one to s_0, we obtain a simplicial tree T_2'. Now extend $F_1 : T_1 \to T_1$ to $F_2' : T_2' \to T_2'$ such that $F_2'(s_2) = v_1$ and $F_2'(v_1) = s_0$. The nice cubic children preserving map $F_2' : T_2' \to T_2'$ is a realisation of the marked grid of size $N = 2$ with $M(1, 1) = 0$.

Suppose that any marked grid of size $k \leq N - 1$ for $N < +\infty$ can be realised by a nice cubic children preserving map $F_k : T_k \to T_k$ of the same size. According to the process of induction, the whole simplicial tree T_k is in the forward orbit of the critical branch $(s_i)_{i \leq k}$. Let M be a given marked grid of size N. The following arguments are based on walking along the path towards the northeast orientation with starting point $M(N, 0)$ and finishing point $M(0, N)$.

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The first thing is to add a child s_N of degree two to s_{N-1}. Then let us walk to the position $M(N-1,1)$. If $M(N-1,1)$ is marked, then the whole grid of size N is marked by rule (R2). Define $F_N(s_N) = s_{N-1}$, so that we obtain a nice cubic children preserving map $F_N : T_N \to T_N$ which is a realisation of M with $M(N-1,1)$ marked.

Assume that $M(N-1,1)$ is not marked. Add a child v_{N-1} of degree one to $F_{N-1}(s_{N-1})$, and define $F_N(s_N) = v_{N-1}$. Now walk to the position $M(N-2,2)$. If $M(N-2,2)$ is marked, define $F_N(v_{N-1}) = s_{N-2}$. It is straightforward to verify that $F_N : T_N \to T_N$ is a nice cubic children preserving map which realises the marked grid where $M(N-1,1)$ is not marked and $M(N-2,2)$ is marked.

Assume that $M(N-2,2)$ is not marked. For any position $M(m,n)$ in M, denote the triangle above $M(m,n)$ by $Q(m,n) = \{(M(j,k)) \mid j+k \le m+n, 0 \le j \le m \text{ and } n \le k \le m+n\}$. If $Q(N-2,2) \ne Q(N-2,1)$, add a child v_{N-2} of degree one to $F^{\circ 2}_{N-1}(s_{N-1})$, and define $F_N(v_{N-1}) = v_{N-2}$. Then move to the position $M(N-3,3)$. If $Q(N-2,2) = Q(N-2,1)$, it follows that $M(N-2,1)$ is not marked. Define $F_N(v_{N-1}) = F_{N-1}(s_{N-1})$. The four conditions in Lemma 3.1 are very easy to verify.

Suppose that we have defined the map F_N on the orbit $s_N \mapsto F_N(s_N) \mapsto \cdots \mapsto F^{\circ k}_N(s_N)$ for some $k \ge 1$ (notice that F_N has no definition on $F^{\circ k}_N(s_N)$), where $v_{N-i} = F^{\circ i}_N(s_N)$ is the new added child of $F^{\circ i}_{N-1}(s_{N-1})$ with degree one for $1 \le i \le k$. Moreover, we know that $M(N-i,i) = 0$ for $1 \le i \le k$ since the construction will be completed if we meet a marked position.

Now move to the position $M(N-k-1,k+1)$. We use similar arguments as before. If $M(N-k-1,k+1)$ is marked, define $F_N(v_{N-k}) = s_{N-k-1}$. In this case, we only need to verify condition (4) in Lemma 3.1. In fact, until now, except for v_{N-k}, s_{N-k-1} cannot have any other preimages of degree one according to the construction. It follows that $\sum_{F(u)=s_{N-k-1}} \deg(u) \le 3$.

We should remember that either we are adding new children of degree one or the construction is completed. If $M(N-k-1,k+1)$ is not marked, check $Q(N-k-1,k+1)$ and $Q(N-k-1,j)$ for $0 < j \le k$. If $Q(N-k-1,k+1) \ne Q(N-k-1,j)$ for any $0 < j \le k$, add a child v_{N-k-1} to $F^{\circ(k+1)}_{N-1}(s_{N-1})$ and define $F_N(v_{N-k}) = v_{N-k-1}$. Then we move to the position $M(N-k-2,k+2)$. If there exists some j such that $Q(N-k-1,k+1) = Q(N-k-1,j)$, we also need to check something. If $M(N-k-1,k)$, $M(N-k-1,j-1)$ and $M(N-k,j-1)$ are all marked, add a child v_{N-k-1} to $F^{\circ(k+1)}_{N-1}(s_{N-1})$ and define $F_N(v_{N-k}) = v_{N-k-1}$. Then we move to the position $M(N-k-2,k+2)$. Otherwise, we define $F_N(v_{N-k}) = F^{\circ j}_{N-k-j-1}(s_{N-k-j-1})$. Similarly, the four conditions in Lemma 3.1 are all guaranteed by the rules of tableaux and the construction.

In this way, we can always obtain a nice cubic children preserving map which realises the marked grid of size N. By the induction, the proof is complete. □

REMARK 4.2. Regarding the proof of Lemma 4.1, we would like to say some words about the case where $M(N-2,2) = 0$ and $Q(N-2,2) = Q(N-2,1)$. In fact, in order to satisfy condition (2) in Lemma 3.1, we need to check $M(N-2,1)$, $M(N-2,0)$ and $M(N-1,1)$. Since $M(N-2,1) = 0$, we define $F_N(v_{N-1}) = F_{N-1}(s_{N-1})$ directly.

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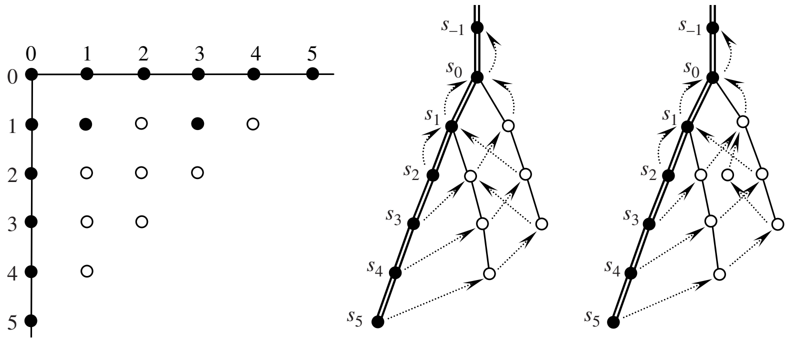

FIGURE 1. A marked grid of size $N = 5$, which can be realised by exactly two nice cubic children preserving maps of the same size. The tree in the middle is constructed by the method stated in Lemma 4.1. The dark spots denote the critical positions and the spots with empty interior denote the noncritical positions.

TABLE 1. The number of marked grids and nice cubic children preserving maps up to size $N = 21$. The notations MG_N and NC_N denote the number of marked grids and nice cubic children preserving maps respectively.

N	1	2	3	4	5	6	7
MG_N	1	2	4	8	16	33	69
NC_N	1	2	4	8	18	42	103
N	8	9	10	11	12	13	14
MG_N	144	303	641	1361	2895	6174	13188
NC_N	260	670	1753	4644	12433	33581	91399
N	15	16	17	18	19	20	21
MG_N	28229	60515	129940	279415	601742	1297671	2802318
NC_N	250452	690429	1913501	5328648	14902959	41841737	117887513

But if we are in the case where $M(N-k-1, k+1) = 0$ and $Q(N-k-1, k+1) = Q(N-k-1, j)$ for some $0 < j \leq k$, then we have to make an argument as in the proof of Lemma 4.1.

A marked grid may be realised by two or more nice cubic children preserving maps (see Figure 1 and Section 5). It is straightforward to verify that for $N \leq 4$, the marked grid of size N can only be realised by a unique nice cubic children preserving map of the same size.

PROOF OF THEOREM 1.1. Combine Theorem 2.2 and the results of Lemmas 3.1 and 4.1. □

5. The number of marked grids and nice cubic children preserving maps

By the rules (R0)–(R4) stated in the Introduction, we can calculate the number of different marked grids with fixed size. Similarly, by the four conditions stated in Lemma 3.1, we can calculate the number of different nice cubic children preserving maps with fixed size (see Table 1).

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Note that in [3], the number of different nice cubic children preserving maps (which is called the tree number) has been calculated up to size 17. Here we extend the result to size 21.

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