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"Vortex Rings in a Compressible Fluid."

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In a paper recently printed in the Society's *Proceedings*, I considered the effect of compressibility in the fluid on the motion of straight vortices; the present paper treats of circular vortex rings in a compressible fluid. The circle passing through the centres of the circular cross sections of the vortex filament will be called the "circular axis," and the perpendicular to the plane of the circular axis through its centre, the "axis" of the vortex. In the notation employed, a denotes the radius of the circular axis, and e that of the cross section of the filament, while ω represents vorticity, and ρ density. It is also convenient to denote the area of the cross section—*i.e.*, πe^2, by σ. Following Helmholtz, it will be supposed that e/a is always very small, and that the cross section is truly circular. Certain small inconsistencies in the ordinary theory following from this last assumption will be pointed out, though they do not seem seriously to affect the general applicability of the results. The axis of the vortex ring is taken as axis of z, and z, r, θ are the ordinary cylindrical co-ordinates. It is also convenient to denote by r' the distance of a point from the circular axis of a ring, and by ψ the inclination of this distance to the plane of the circular axis. The effects of the vorticity and variation in density may be considered separately.

The components of vorticity at any point of the ring are $\xi = -\omega\sin\theta$, $y = \omega\cos\theta$. There is obviously symmetry about oz, and the velocity at any point can be resolved into w parallel to oz and u along the perpendicular on oz. Since e/a is very small, we shall, following the common practice in calculating the velocity, regard the vortex filament as concentrated in the circular axis, and so in Lamb's formulæ* replace the vortex element $dxdydz$ by $\frac{ma}{\omega}d\theta$; where $m, = \pi e^2\omega$, is the strength of the vortex. Supposing the origin taken in the instantaneous position of the centre of the circular axis, we find from these formulæ for the velocity in the fluid at the point r, θ, z outside the vortex

$$w = \frac{ma}{2\pi}\int_0^{2\pi}\frac{(a - r\cos\theta)d\theta}{(z^2 + a^2 + r^2 - 2ar\cos\theta)^{\frac{3}{2}}} \quad \cdots \quad \cdots \quad (1),$$

$$u = \frac{ma}{2\pi}\int_0^{2\pi}\frac{z\cos\theta d\theta}{(z^2 + a^2 + r^2 - 2ar\cos\theta)^{\frac{3}{2}}} \quad \cdots \quad \cdots \quad (2).$$

We may at once transform (1) into

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\frac{1}{\sqrt{z^2 + (r+a)^2}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}\right] \quad \cdots \quad \cdots \quad (3),$$

$$\text{where } k^2 = \frac{4ra}{z^2 + (r+a)^2},\ 2\phi = \pi - \theta \quad \cdots \quad \cdots \quad (4).$$

Thus if, as usual, F_1 denote the complete elliptic integral of the first order,

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \cdots \quad \cdots \quad (5).$$

When k is nearly unity, an approximate value is $F_1(k) = \log(4/k_1)$,

$$\text{where } k_1{}^2 = 1 - k^2 = \frac{z^2 + (r-a)^2}{z^2 + (r+a)^2} \quad \cdots \quad \cdots \quad (6).$$

Thus, for points near the surface of the filament, where z and $r - a$ are both very small, an approximate value is

$$w = -\frac{2ma}{\pi}\frac{d}{da}\left[\{z^2 + (r+a)^2\}^{-\frac{1}{2}}\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\}\right].$$

This gives

$$w = \frac{2ma}{\pi}\{z^2 + (r+a)^2\}^{-\frac{3}{2}}\left[(r+a)\log\left\{4\left(\frac{z^2 + (r+a)^2}{z^2 + (r-a)^2}\right)^{\frac{1}{2}}\right\} - \frac{2r(z^2 + r^2 - a^2)}{z^2 + (r-a)^2}\right] \quad \cdots \quad \cdots \quad (7).$$

For points outside, but in contact with the surface of the filament, we have $r = a + e\cos\psi, z = e\sin\psi$. Substituting these values in (7) and

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* "Motion of Fluids"—Equations (15), p. 152.

retaining the principal terms, we get for the velocity in the fluid just outside the filament

$$w=\frac{m}{2\pi a}\left\{\log\frac{8a}{e}-1+\cos^2\psi-\frac{2a\cos\psi}{e}\right\}\quad\ldots\quad\ldots\qquad(8).$$

The first and last are much the most important terms.

The most important term omitted is $-\frac{me\cos\psi}{2\pi a^2}\log\frac{8a}{e}$.

From (2), using k and ϕ in the same sense as before, we easily find

$$u=\frac{m}{\pi r}\frac{d}{dz}\left[\{z^2+(r+a)^2\}^{\frac{1}{2}}E_1(k)-\frac{z^2+r^2+a^2}{\{z^2+(r+a)^2\}^{\frac{1}{2}}}F_1(k)\right]\quad\ldots\qquad(9);$$

where E_1 is the complete elliptic integral of the second order.

For points near the surface of the filament, we saw that approximately

$$F_1(k)=\log(4/k_1);$$

also

$$E_1(k)=(1-k^2)\left\{F_1(k)+\frac{kd}{dk}F_1(k)\right\}\quad\ldots\quad\ldots\qquad(10).*$$

From its approximate value we get $k\frac{dF_1(k)}{dk}=\frac{k^2}{k_1^2}=\frac{4ar}{z^2+(r-a)^2}$;

thus an approximate value is

$$E_1(k)=\frac{z^2+(r-a)^2}{z^2+(r+a)^2}\left[\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\}+\frac{4ar}{z^2+(r-a)^2}\right]\quad\ldots\qquad(11).$$

Substituting these values in (9), carrying out the differentiations and reducing, we obtain the approximate value

$$u=\frac{2maz}{\pi}\left\{z^2+(r+a)^2\right\}^{-\frac{3}{2}}\left[\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\}-2+\frac{4ar}{z^2+(r-a)^2}\right]\quad\ldots\quad\ldots\qquad(12)$$

Using the same notation as before, and retaining the principal terms, we find for the velocity in the fluid just outside the filament

$$u=\frac{m}{\pi e}\sin\psi-\frac{m}{2\pi a}\sin\psi\cos\psi\quad\ldots\quad\ldots\qquad(13).$$

The most important term omitted is $\frac{me\sin\psi}{4\pi a^2}\log\frac{8a}{e}$.

From (8) and (13) we see that the velocity in the fluid just outside the vortex is composed of the two components

$$w=\frac{m}{2\pi a}\left\{\log\frac{8a}{e}-1\right\}\quad\ldots\quad\ldots\qquad(14),$$

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* Cayley's "Elliptic Functions."

parallel to the axis oz, and

$$\tau = \frac{m}{\pi e}\left(1 - \frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \qquad (15),$$

tangential to the surface of the filament in the plane containing oz.

The former component (14) is a motion *en masse*, the same at every point of the surface of the filament, and is shared by the vortex and the fluid bordering on it; the latter (15) represents the velocity of circulation of the fluid round the circular axis. This velocity is slightly greater on the concave, or inner, side of the filament, and less on the convex side than in the case of a straight vortex of the same strength and cross section.

This is one of the small inconsistencies already referred to; for if the vorticity ω be constant throughout the filament and the section truly circular, the velocity of circulation in the fluid just inside the surface of the filament must be ωe, while from (15) the fluid just outside has its velocity of circulation $= \omega e\left(1 - \frac{e}{2a}\cos\psi\right)$. There is thus a very slight absence of continuity in the motion on crossing the surface. In a perfectly frictionless fluid this may seem of absolutely no importance, but the following reasoning shows that an inconsistency of a precisely similar character exists in the hypothesis that ω can be constant throughout a truly circular section.

Using r' in the sense already indicated, let us not assume ω to be constant, but still suppose the velocity perpendicular to r'. Consider the elementary ring formed by the revolution about oz of the element $r'd\psi dr'$ of the cross section of the filament. The volume of this ring is $2\pi(a + r'\cos\psi)r'd\psi dr'$, and the areas of the two surfaces through which alone flow takes place are each $2\pi(a + r'\cos\psi)dr'$. The velocity is normal to these surfaces and equal to $\omega r'$. Thus, by the equation of continuity, if the fluid be incompressible, we get

$$\frac{d}{d\psi}\left[\left(1 + \frac{r'}{a}\cos\psi\right)\omega r'\right] + \frac{d}{dt}\left[\left(1 + \frac{r'}{a}\cos\psi\right)r'\right] = o.$$

Since $\frac{d\psi}{dt} = \omega$, this gives at once

$$\omega\left(1 + \frac{r'}{a}\cos\psi\right)^2 = \text{constant}.$$

Neglecting terms in $(r'/a)^2$, and denoting the mean value of the vorticity by Ω, this gives

$$\omega = \Omega\left(1 - \frac{2r'}{a}\cos\psi\right).$$

Thus, our fundamental hypothesis that ω and e are constant when logically carried out requires us to neglect the second term in (15), *i.e.*, velocities of order m/a. This consequently would lead us to neglect the second term in (14) also. This may explain the slight divergence in the results obtained by Prof. J. J. Thomson,* Mr T. C. Lewis,† and Mr W. M. Hicks.‡ The two former obtain the result (14), but the latter differs in the value of the small term.

We have next to consider the velocity due to variation in the density ρ of the fluid. From the equations (6) and (10) of Chap. VI. of Lamb's Treatise, it follows that the velocity due to change of density is expressed by the same formula as the force due to a gravitating mass of density $-\frac{1}{4\pi\rho}\frac{\delta\rho}{\delta t}$. If the ring be of small cross section σ we may in calculating the velocity, to the same degree of accuracy as when treating the vorticity, regard it as equal to the force due to a mass of line density $-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}$ concentrated in the circular axis $r = a$ of the ring. If ρ varied with the distance from the circular axis, but was independent of ψ, the accuracy would not be seriously affected.

If w_1 and u_1 denote the component velocities parallel and perpendicular to the axis of the ring, due to the variation in density alone, then from the above remarks it follows that

$$w_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}az\int_0^\pi \frac{d\theta}{(z^2 + r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} \quad \dots \quad \dots \quad (16),$$

$$u_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}a\int_0^\pi \frac{(r - a\cos\theta)d\theta}{(z^2 + r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} \quad \dots \quad \dots \quad (17).$$

Thence, k having its previous meaning, we find

$$w_1 = \frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\frac{d}{dz}\left[\{z^2 + (r + a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \dots \quad \dots \quad (18).$$

To the same degree of approximation as in the case of vorticity, we

* "Motion of Vortex Rings"—Equation (41), p. 33.

† "Quarterly Journal of Mathematics," XVI., 1879, pp. 338-347.

‡ "Philosophical Transactions," 1884, Part I., and 1885, Part II.

have for the velocity in the fluid comparatively near the surface of the filament

$$w_1 = -\frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}az\{z^2+(r+a)^2\}^{-\frac{3}{2}}\left[\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\} + \frac{4ra}{z^2+(r-a)^2}\right] \quad \ldots \quad \ldots \quad (19).$$

For points just outside the ring, using the previous notation and retaining the principal terms, we find

$$w_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\sin\psi\left(1-\frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \quad (20).$$

The most important term neglected is $-\frac{\sigma}{8\pi\rho}\frac{\delta\rho}{\delta t}\frac{e}{a^2}\sin\psi\log\frac{8a}{e}$.

From (17) we find

$$u_1 = \frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\frac{d}{dr}\left[\{z^2+(r+a)^2\}^{-\frac{1}{2}}F_1(k)\right] \quad \ldots \quad \ldots \quad (21);$$

and thence for points comparatively near the ring

$$u_1 = -\frac{\sigma}{\pi\rho}\frac{\delta\rho}{\delta t}a\{z^2+(r+a)^2\}^{-\frac{3}{2}}\left[(r+a)\log\left\{4\left(\frac{z^2+(r+a)^2}{z^2+(r-a)^2}\right)^{\frac{1}{2}}\right\} + 2a\frac{(r^2-a^2-z^2)}{z^2+(r-a)^2}\right] \quad \ldots \quad \ldots \quad (22).$$

For points just outside the ring we thence obtain the approximate value

$$u_1 = -\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\left[\frac{1}{a}\left(\log\frac{8a}{e}-1\right)+\frac{2}{e}\cos\psi\left(1-\frac{e}{2a}\cos\psi\right)\right] \quad \ldots \quad \ldots \quad (23).$$

The most important term neglected is $\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{e}{a^2}\cos\psi\log\frac{8a}{e}$.

From (20) and (23) we see that the velocity in the fluid just outside the filament is composed of the two components

$$u_1 = -\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\left\{\log\frac{8a}{e}-1\right\} \quad \ldots \quad \ldots \quad (24),$$

perpendicular to the axis oz, *i.e.*, tending to increase the radius a of the circular axis, and

$$v_1 = -\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\left(1-\frac{e}{2a}\cos\psi\right) \quad \ldots \quad \ldots \quad (25),$$

normal to the surface of the filament, *i.e.*, tending to increase e.

The first component u_1 represents a motion of the ring and surrounding fluid *en masse;* the second v_1 gives the rate of increase in the radius of the cross section consequent on the change in density. The slight variation in the rate of increase of e in different directions is a phenomenon exactly similar to that illustrated by the existence of the small term in (15). A consideration of the equation of continuity shows us, precisely as in the parallel case in vorticity, that if the density be uniform over the cross section it cannot vary so that *the cross section remain truly circular, unless velocities of the order* $\frac{\sigma}{\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}$ be negligible. In this case the second terms in both (24) and (25) must be neglected. It would even seem at first sight that the equation of continuity was inconsistent with the existence of the principal term of (24). For, since the mass of the ring is constant, $e^2 a\rho$ must be constant, and so

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t}+\frac{2}{e}\frac{\delta e}{\delta t}=-\frac{1}{a}\frac{\delta a}{\delta t} \quad \dots \quad \dots \qquad (26).$$

But $v_1=\frac{\delta e}{\delta t}$, and so, retaining only the principal term of (25) and putting $\sigma=\pi e^2$, we get $\frac{1}{\rho}\frac{\delta\rho}{\delta t}+\frac{2}{e}\frac{\delta e}{\delta t}=0 \quad \dots \quad \dots \qquad (27).$

Thus it might be thought from (26) that $\frac{\delta a}{\delta t}$ must vanish.

The true explanation is that $\frac{\delta a}{\delta t}$ does not vanish, but $\frac{1}{a}\frac{\delta a}{\delta t}$ is of an order of small quantities we agreed to neglect when we came to the conclusion that the second terms in (24) and (25) were negligible. To this degree of approximation, then, we see from (27) that $\sigma\rho$ is constant, and we may replace $-\frac{\sigma}{\rho}\frac{\delta\rho}{\delta t}$ by $\frac{\delta\sigma}{\delta t}$.

Combining the effects of vorticity and change of density, and retaining only the terms consistent with an exactly circular cross section, we finally obtain for the velocities of a thin filament

$$\left.\begin{aligned} w&=\frac{m}{2\pi a}\log\frac{8a}{e} \\ u&=\frac{1}{4\pi a}\frac{\delta\sigma}{\delta t}\log\frac{8a}{e} \end{aligned}\right\} \quad \dots \quad \dots \qquad (28).$$

The circular axis of the ring moves on the surface formed by the revolution about oz of the curve whose differential equation is

$$\frac{dz}{dx}=\frac{w}{u}=2m/\frac{\delta\sigma}{\delta t} \quad \dots \quad \dots \qquad (29).$$

If the rate of increase of the cross section be uniform this forms part of a right circular cone whose vertical angle is $2\tan^{-1}\left(\frac{1}{2m}\frac{\delta\sigma}{\delta t}\right)$.

It is easily seen that the case of a ring vortex in presence of an infinite plane, whether inclined or not to the plane of the circular axis, can be treated by the introduction of an "image" ring on the other side of the plane. The position and direction of rotation in the image were indicated in my previous paper; the cross section and density must be the same as at corresponding points in the real ring. However, the ring will remain circular with its circular axis in one plane only when it is parallel to the infinite plane, and the formulæ obtained above will be most usefully employed when the distance c of the ring from the plane is small compared to the radius a, though large compared to e. This case we proceed to treat.

Let us take the origin where the infinite plane is intersected by the common axis of the rings.

The components of the velocity at any point due to the real ring may be got from the preceding formulæ by writing $z-c$ for z, while the components of the velocity due to the image ring require the substitution of $z+c$ for z, and $-m$ for m.

For the velocity in the fluid immediately surrounding the ring the effect of the ring itself is given by (8), (13), (20), and (23), while the effect of the image may be got from (7), (12), (19), and (22) by writing $-m$ for m, $a+e\cos\psi$ for r and $2c+e\sin\psi$ for z. Combining the effects and retaining only the principal terms in accordance with the remarks already made as to the probable degree of accuracy of the method, I find

$$w=-\frac{m\cos\psi}{\pi e}-\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\sin\psi+\frac{m}{2\pi a}\log\frac{2c}{e}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{c}\quad\ldots\quad\ldots\qquad(30),$$

$$u=\frac{m\sin\psi}{\pi e}-\frac{\sigma}{2\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{e}\cos\psi-\frac{m}{2\pi c}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\log\left(\frac{32a^2}{ce}\right)\quad\ldots\qquad(31).$$

The two first terms in both (30) and (31) represent the velocity of circulation and the rate of increase of the radius of the cross section, while the two last terms in each represent a motion *en masse* shared by the ring and the surrounding fluid. Thus the velocity of the ring in the direction of the normal drawn *from* the infinite plane is

$$w=\frac{m}{2\pi a}\log\frac{2c}{e}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{c}\quad\ldots\quad\ldots\qquad(32),$$

while the rate of increase in the radius of the ring is

$$u=\frac{\delta a}{\delta t}=-\frac{m}{2\pi c}-\frac{\sigma}{4\pi\rho}\frac{\delta\rho}{\delta t}\frac{1}{a}\log\left(\frac{32a^2}{ce}\right)\quad\ldots\quad\ldots\qquad(33).$$

Neglecting at first any change in ρ we have from (32) and (33)

$$\frac{\delta c}{\delta t}=\frac{m}{2\pi a}\log\frac{2c}{e}\quad\ldots\quad\ldots\qquad(34),$$

$$\frac{\delta a}{\delta t}=-\frac{m}{2\pi c}\quad\ldots\quad\ldots\qquad(35);$$

while $ae^2=\text{constant}$.

It follows that $\frac{\delta^2 c}{\delta t^2}=\frac{m^2}{4\pi^2a^2c}\left\{\log\left(\frac{2c}{e}\right)^2-\frac{1}{2}\right\}$.

Thus $\frac{\delta^2 c}{\delta t^2}$ is always positive, for c must be greater than e and so $\log\left(\frac{2c}{e}\right)^2$ greater than $\frac{1}{2}$. Hence if m be negative, or the vortex be approaching the plane, its rate of approach continually diminishes; while if m be positive, or the vortex be receding, its rate of retreat continually increases so long as (34) and (35) apply.

From (35) we see that the aperture, $2a$, of the vortex increases or diminishes continually according as it is approaching to or receding from the plane.

Considering next the effect of the variation in density alone, we see from (32) and (33) that if its density be increasing the ring approaches the plane with a continually diminishing aperture, while if the density be diminishing the ring recedes from the plane with a continually increasing aperture. The exact relation between the variations in a, e and ρ is given by (26), but to the degree of accuracy obtained here this may, as in the case of a solitary ring, be replaced by (27). Using (27) in (32), and replacing σ by πe^2, we find

$$\frac{\delta c}{\delta t}=\frac{e}{2c}\frac{\delta e}{\delta t}=\frac{1}{4\pi c}\frac{\delta\sigma}{\delta t};$$

whence it follows that $c^2-\frac{1}{2}e^2=c^2-\frac{\sigma}{2\pi}=\text{constant}\quad\ldots\quad\ldots\qquad(36).$

In the cases to which our formulæ can be satisfactorily applied e/c is small, and so the total increase or diminution in the distance of the ring from the plane due to change in density alone is also small. Thus, in general, the effects of the vorticity will be much more important than the effects of variation in density. We conclude that if a vortex ring approach an infinite plane, its rate of approach is slightly greater if its density be increasing, and slightly less if its density be diminishing, than it would be if the density remained constant.

The most important terms due to the action of the image ring which we have neglected in (30) and (31) respectively may without much difficulty be found to be

$$\left.\begin{aligned} w_2 &= \mathrm{A}\cos(\psi - \alpha) \\ u_2 &= \mathrm{A}\sin(\psi - \alpha) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (37);$$

where, for shortness,

$$\left.\begin{aligned} \mathrm{A} &= \frac{e}{4\pi c^2}\left\{m^2 + \left(\frac{\sigma}{2\rho}\frac{\delta\rho}{\delta t}\right)^2\right\}^{\frac{1}{2}} \\ \alpha &= \tan^{-1}\left(\frac{\sigma}{2m\rho}\frac{\delta\rho}{\delta t}\right) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (38).$$

Since, as already explained, we are not warranted in retaining velocities of order $\frac{m}{a}$ it follows that the above terms represent an appreciable effect only when the vortex approaches so close to the plane that ae/c^2 becomes large.

From (37) it follows that the cross section of the filament tends to become slightly elliptical, the axes of the ellipse making with the infinite plane the angles $\frac{\alpha}{2} \pm \frac{\pi}{4}$. These axes are thus equally inclined to the plane when the fluid is incompressible. When the deviation of the cross section from the circular form becomes appreciable the accuracy of the preceding formulæ will be lessened, and they can certainly not be applied to the case of a vortex whose distance from an infinite plane is of the same order of quantities as the diameter of its cross section.

If in (32) – (36) we write $c/2$ for c, we get the case of two precisely equal ring vortices, with vorticities, however, in opposite directions, at a distance c. If we suppose a to become infinite, we deduce formulæ applicable to the case of straight vortex filaments. In particular, it will be noticed that (36) leads at once to a special case of the formula obtained in my previous paper for the distance of two straight vortices.