Canad. Math. Bull. Vol. **67** (1), 2024, pp. 49–59
http://dx.doi.org/10.4153/S0008439523000486

A note about charts built by Eriksson-Bique and Soultanis on metric measure spaces

Luca Gennaioli and Nicola Gigli

Abstract. This note is motivated by recent studies by Eriksson-Bique and Soultanis about the construction of charts in general metric measure spaces. We analyze their construction and provide an alternative and simpler proof of the fact that these charts exist on sets of finite Hausdorff dimension. The observation made here offers also some simplification about the study of the relation between the reference measure and the charts in the setting of RCD spaces.

1 Introduction

In the recent, very interesting, paper [ES21], the authors provided a general construction of charts on metric measure spaces, key features of their notion being: the compatibility with Sobolev calculus (and thus in particular with the differential calculus as developed by Cheeger in [Che99] and by the second author in [Gig15]), a very general existence result, and notable consequences in terms of the structure of the Sobolev spaces (see also [ERS22a, ERS22b]). An example in this latter direction is the proof that the space $W^{1,p}(\mathrm{X})$, $p \in (1,\infty)$, is reflexive as soon as the space X can be covered by a countable number of sets with finite Hausdorff measure (the "previous best" result appeared in [ACD14] and required the metric to be locally doubling).

A crucial step in [ES21] is the proof that if $\varphi : E \subset \mathrm{X} \to \mathbb{R}^n$ is a "p-independent weak chart," then n is bounded from above by the Hausdorff dimension of E: more precisely, the authors prove the following.

Proposition 1.1 *Suppose* $\varphi \in \mathrm{Lip}(\mathrm{X}, \mathbb{R}^n)$ *is* p*-independent on* U. *Then* $n \leq \mathrm{d}im_{\mathcal{H}}(U)$.

For the precise meaning of "p-independent weak chart," we refer to Definition 2.23; for the purpose of this introduction, we shall limit ourselves to point out that in the smooth setting, this would be equivalent to requiring the image of the differential of φ at every point to span the whole tangent space of $\mathbb{R}^d$. Starting from this result, the existence of actual charts is obtained via a suitable maximality argument.

Interestingly, this upper bound is proved via means that have, in principle, little to do with analysis in nonsmooth setting: key ingredients are indeed the elliptic regularity result in [DR] and the study of the structure of the set of nondifferentiability points of Lipschitz functions in [AM16].

Received by the editors December 14, 2022; revised April 27, 2023; accepted April 27, 2023.
Published online on Cambridge Core June 9, 2023.
AMS subject classification: **30Lxx, 53A35, 53C23.**
Keywords: Charts, metric measure spaces.

This sort of procedure has a recent analog in the theory of RCD spaces. Let us recall indeed that, in [MN14], it has been proved that finite-dimensional RCD spaces admit bi-Lipschitz charts covering almost all the space. In [DMR, KM18, MN14], no information about the behavior of the reference measure with respect to these charts has been provided: this topic has been later studied in [KM18] where, relying in a way or another on [DR] and [AM16], it has been proved that $\varphi_*(\mathfrak{m}_{|E}) \ll \mathscr{L}^n$ for a Mondino–Naber chart $\varphi : E \to \mathbb{R}^n$.

Of particular interest for the discussion here is the fact that in [GP21] only the results in [DR] have been used, whereas in [KM18] also those in [AM16] were necessary. Comparing this with the results in [ES21], it is natural to wonder whether the use of [AM16] is really crucial or can be avoided: this is the question motivating the present note. Of course, there is nothing wrong in using a well-established result in doing research; our study is simply motivated by the desire of better understanding the interesting construction done in [ES21]. The result of our investigation is that [AM16] is not really needed and the line of thought presented here simplifies not only some of the steps done in [ES21], but also some of those in [GP21] (see Section 3).

Another remark that we make, consequence of the studies in [ES21], is that the dimension of the (co)tangent module (in the sense of [Gig15]) on a subset $E \subset \mathrm{X}$ is bounded from above from the Hausdorff dimension of E (see Remark 3.6).

2 Preliminaries

2.1 Test plans and Sobolev functions

In this section, we shall recall the definition of Sobolev space following the approach in [AGS14]. We say that a triple $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ is a metric measure space if (X, d) is a complete and separable metric measure space and $\mathfrak{m}$ is a Radon measure which is finite on balls. For the rest of the paper, p, q will be conjugate exponents, namely, $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.1 We say that a probability measure π on $C([0,1];\mathrm{X})$ is a *q-test plan* if it is concentrated on $AC([0,1];\mathrm{X})$ and the following two conditions are met:

(1) $\exists\ C = C(\pi) > 0$ such that $e_{t\,\sharp}\pi \le C\mathfrak{m}$, where $\mathfrak{m}$ is the reference measure on X and $e_t : C([0,1];\mathrm{X}) \to \mathrm{X}$ is the evaluation map $e_t(\gamma) = \gamma_t$.

(2) The following quantity, called *kinetic energy*, is finite:

$$\mathrm{K.E.}(\pi) = \int \int_0^1 |\dot\gamma_t|^q \,\mathrm{d}t\,\mathrm{d}\pi(\gamma),$$

where $|\dot\gamma_t| = \lim_{h\to 0} \frac{\mathrm{d}(\gamma_{t+h},\gamma_t)}{h}$ is the metric derivative of the curve γ.

With this notion at hand, we can introduce the Sobolev space $\mathrm{W}^{1,p}(\mathrm{X}, \mathrm{d}, \mathfrak{m})$.

Definition 2.2 We say that a function $f : \mathrm{X} \to \mathbb{R}$ belongs to the Sobolev space $W^{1,p}(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ if $f \in \mathrm{L}^p(\mathfrak{m})$ and if

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi(\gamma) \le \int \int_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma) \quad \forall \pi \text{ q-test plan}, \tag{2.1}$$

with $G : \mathrm{X} \to \mathbb{R}_+$ being a Borel function belonging to $\mathrm{L}^p(\mathfrak{m})$.

https://doi.org/10.4153/S0008439523000486 Published online by Cambridge University Press

Remark 2.3 It is easy to see that the set of functions G satisfying (2.1) is a closed convex set; hence, it admits an element of minimal norm: we will call such an element *p-weak upper gradient* and we will denote it by $|Df|_p$. With a little bit of work, it is possible to prove that the function $|Df|_p$ is such that $|Df|_p \le G$ $\mathfrak{m}$-a.e. for every other G satisfying (2.1).

2.2 The language of L^p-normed L^∞-modules

We now switch our attention to the theory of $\mathrm{L}^p(\mathfrak{m})$-normed $\mathrm{L}^\infty(\mathfrak{m})$-modules developed by the second author in [Gig18]: the following material can be found there, unless otherwise stated.

Definition 2.4 ($\mathrm{L}^p(\mathfrak{m})$-normed module) We say that a Banach space $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is an $\mathrm{L}^p(\mathfrak{m})$-normed $\mathrm{L}^\infty(\mathfrak{m})$-module if there exists a bilinear continuous map $\cdot :$ $\mathrm{L}^\infty(\mathfrak{m}) \times \mathcal{M} \to \mathcal{M}$ which makes $\mathcal{M}$ a module with unity over the ring of $\mathrm{L}^\infty(\mathfrak{m})$ functions and another map $|\cdot| : \mathcal{M} \longrightarrow \mathrm{L}^p(\mathfrak{m})$ with nonnegative values such that

$$\| |v| \|_{\mathrm{L}^p(\mathfrak{m})} = \|v\|_{\mathcal{M}}, \tag{2.2}$$

$$|f \cdot v| = |f||v| \qquad \mathfrak{m} - \text{a.e.} \tag{2.3}$$

for all $v \in \mathcal{M}$, $f \in \mathrm{L}^\infty(\mathfrak{m})$. We call $\cdot$ the multiplication and $|\cdot|$ the *pointwise norm*.

Remark 2.5 Note that the pointwise norm is continuous thanks to the triangular inequality; in fact,

$$\| |v| - |w| \|_{\mathrm{L}^p(\mathfrak{m})} \le \| |v - w| \|_{\mathrm{L}^p(\mathfrak{m})} = \|v - w\|_{\mathcal{M}}.$$

Moreover, with a little bit of abuse of notation, we will write fv instead of $f \cdot v$ and write $\mathrm{L}^p(\mathfrak{m})$-normed module instead of $\mathrm{L}^p(\mathfrak{m})$-normed $\mathrm{L}^\infty(\mathfrak{m})$-module.

A related interesting concept is the one of *localization* of a module; indeed, it is easy to see that the following object

$$\mathcal{M}_{|E} := \{\chi_E v : \ v \in \mathcal{M}\}$$

is a submodule of $\mathcal{M}$ and it clearly inherits the normed structure from $\mathcal{M}$.

Definition 2.6 (**Local independence**) Let $\mathcal{M}$ be an $\mathrm{L}^p(\mathfrak{m})$-normed $\mathrm{L}^\infty(\mathfrak{m})$-module and $A \in \mathscr{B}(X)$ with $\mathfrak{m}(A) > 0$, and we say that a family $v_1, \dots, v_n \in \mathcal{M}$ is independent on A if, for every $f_1, \dots, f_n \in \mathrm{L}^\infty(\mathfrak{m})$,

$$\sum_{i=1}^n f_i v_i = 0 \quad \mathfrak{m} - \text{a.e. on } A \implies f_i = 0 \quad \mathfrak{m} - \text{a.e. on } A \quad \forall i = 1, \dots, n. \tag{2.4}$$

In the spirit of linear algebra, we shall also define what is the *span* of a set of vectors.

Definition 2.7 (**Span**) Let $\mathcal{M}$ be an $\mathrm{L}^p(\mathfrak{m})$-normed $\mathrm{L}^\infty(\mathfrak{m})$-module, $V \subset \mathcal{M}$ a subset, and $A \in \mathscr{B}(X)$. We denote with $\mathrm{Span}_A(V)$ the closure in $\mathcal{M}$ of the

https://doi.org/10.4153/S0008439523000486 Published online by Cambridge University Press

$L^\infty(\mathfrak{m})$-linear combinations of elements of V. Moreover, we say that $\mathrm{Span}_A(V)$ is the space generated by V on A.

After this definition, the one of basis and of dimension for an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$ arise naturally.

Definition 2.8 We say that a finite family $v_1, \dots, v_n \in \mathcal{M}$ is a basis on $A \in \mathscr{B}(X)$ if it is independent on A and $\mathrm{Span}_A\{v_1, \dots, v_n\} = \mathcal{M}_{|A}$. If the above happens, we say that the *local dimension* of $\mathcal{M}$ on A is n and in case $\mathcal{M}$ has not dimension k for any $k \in \mathbb{N}$, we say that it has infinite dimension.

It can be proved that the notion of dimension is well posed, namely, if we have $v_1, \dots, v_n$ generating $\mathcal{M}$ on a set A and $w_1, \dots, w_m$ are independent on A, then $n \geq m$. Ultimately, this means that two different bases must have the same cardinality.

Building over these tools, we have the following proposition.

Proposition 2.9 *Let $\mathcal{M}$ be an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then there is a unique partition $\{E_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ of X, up to $\mathfrak{m}$-a.e. equality, such that:*

(1) *for every $i \in \mathbb{N}$ such that $\mathfrak{m}(E_i) > 0$, $\mathcal{M}$ has dimension* i *on E_i,*
(2) *for every $E \subset E_\infty$ with $\mathfrak{m}(E) > 0$, $\mathcal{M}$ has infinite dimension on* E.

2.3 Pullback of a normed module

We now introduce the notion of *pullback module* which, roughly speaking, is nothing but a module over a space X obtained by pulling back a module on another space Y via a certain map.

Definition 2.10 (Pullback) Let $(\mathrm{X}, \mathrm{d_X}, \mathfrak{m}_\mathrm{X})$ and $(\mathrm{Y}, \mathrm{d_Y}, \mathfrak{m}_\mathrm{Y})$ be metric measure spaces, $\varphi : \mathrm{X} \longrightarrow \mathrm{Y}$ a map of bounded compression, and $\mathcal{M}$ an $L^p(\mathfrak{m}_\mathrm{Y})$-normed module. Then there exists a unique, up to unique isomorphism, couple $(\varphi^*\mathcal{M}, \varphi^*)$ with $\varphi^*\mathcal{M}$ being an $L^p(\mathfrak{m}_\mathrm{X})$-normed module and $\varphi^* : \mathcal{M} \longrightarrow \varphi^*\mathcal{M}$ being a linear and continuous operator such that:

(1) $|\varphi^* v| = |v| \circ \varphi$ holds $\mathfrak{m}_\mathrm{X}$-a.e., for every $v \in \mathcal{M}$,
(2) the set $\{\varphi^* v : v \in \mathcal{M}\}$ generates $\varphi^*\mathcal{M}$ as a module.

At this point, one can try to understand what is the relation between the dimension of a module and the one of its pullback via the map φ and in order to do so we need to introduce a sort of *left inverse* of the pullback operator φ^*. To do so, let us assume that $\varphi_\sharp \mathfrak{m}_\mathrm{X} = \mathfrak{m}_Y$ to simplify the exposition.

For $f \in L^p(\mathfrak{m}_\mathrm{X})$ nonnegative, we put

$$\mathrm{Pr}_\varphi(f) := \frac{\mathrm{d}\varphi_\sharp(f\mathfrak{m}_\mathrm{X})}{\mathrm{d}\mathfrak{m}_\mathrm{Y}}, \tag{2.5}$$

and in a natural way, we set $\mathrm{Pr}_\varphi(f) := \mathrm{Pr}_\varphi(f^+) - \mathrm{Pr}_\varphi(f^-)$ for general $f \in L^p(\mathfrak{m}_\mathrm{X})$.

For the next proposition, we need to recall the classical Disintegration theorem. The statement below is taken from [AGS08, Theorem 5.3.1] (see also [Fre06, Chapter 452] and [Bog07, Chapter 10.6]).

Theorem 2.11 (Disintegration) *Let* X, Y *be complete and separable metric spaces, let* $\mu \in \mathcal{P}(\mathsf{X})$*, let* $\pi : \mathsf{X} \to \mathsf{Y}$ *be a Borel map, and let* $\nu = \pi_\sharp \mu \in \mathcal{P}(\mathsf{Y})$*. Then there exists a* ν*-a.e. uniquely determined Borel family of probability measures* $\{\mu_y\}_{y\in\mathsf{Y}} \subseteq \mathcal{P}(\mathsf{X})$ *such that* $\mu_x(\mathsf{X}\backslash\pi^{-1}(\{y\})) = 0$ *for* ν*-a.e.* $y \in \mathsf{X}$ *and*

$$\int_{\mathsf{X}} f \,\mathrm{d}\mu = \int_{\mathsf{Y}} \left(\int_{\pi^{-1}(\{y\})} f \,\mathrm{d}\mu_y \right) \mathrm{d}\nu(y) \tag{2.6}$$

for every Borel map $f : \mathsf{X} \to [0, +\infty]$.

Remark 2.12 Two remarks are in order here: the first one is that the above theorem in [AGS08] is stated for Radon separable metric space, but in our setting, it suffices to state it for complete and separable ones (which in particular are Radon), and the second is that the result easily extends to any $f : \mathsf{X} \to \mathbb{R}$ Borel provided, for example, that $f \in L^1(\mu)$.

We now recall some properties of the map Pr_φ.

Proposition 2.13 *The operator* $\mathrm{Pr}_\varphi : \mathrm{L}^p(\mathfrak{m}_{\mathsf{X}}) \longrightarrow \mathrm{L}^p(\mathfrak{m}_Y)$ *is linear, continuous, and*

$$\mathrm{Pr}_\varphi(f)(y) = \int_{\mathsf{X}} f(x) \,\mathrm{d}\mathfrak{m}_y(x) \quad \mathfrak{m}_{\mathsf{Y}} - a.e., \quad \forall f \in \mathrm{L}^p(\mathfrak{m}_{\mathsf{X}}), \tag{2.7}$$

where $y \mapsto m_y$ *denotes the disintegration of* $\mathfrak{m}_{\mathsf{X}}$ *with respect to the map* φ*. Finally, it holds*

$$|\mathrm{Pr}_\varphi(f)| \le \mathrm{Pr}_\varphi(|f|) \quad \mathfrak{m}_{\mathsf{Y}} - a.e. \tag{2.8}$$

Proof Linearity is a consequence of the linearity of the integral. Formula (2.8) is also trivial, while for (2.7), we have, for any $A \in \mathscr{B}(\mathsf{Y})$,

$$\int_A \mathrm{Pr}_\varphi(f)(y) \,\mathrm{d}\mathfrak{m}_{\mathsf{Y}} = \int_A \mathrm{d}\varphi_\sharp(f \,\mathrm{d}\mathfrak{m}_{\mathsf{X}}) = \int_{\varphi^{-1}(A)} f(x) \,\mathrm{d}\mathfrak{m}_{\mathsf{X}},$$

and by the properties of the disintegration, we have

$$\int_{\varphi^{-1}(A)} f(x) \,\mathrm{d}\mathfrak{m}_{\mathsf{X}} = \int_{\mathsf{Y}} \int_{\varphi^{-1}(A)} f(x) \,\mathrm{d}\mathfrak{m}_y(x) \,\mathrm{d}\mathfrak{m}_{\mathsf{Y}}(y) = \int_A \int_{\mathsf{X}} f(x) \,\mathrm{d}\mathfrak{m}_y(x) \,\mathrm{d}\mathfrak{m}_{\mathsf{Y}}(y),$$

therefore proving (2.7).

To prove continuity, note that the case $p = \infty$ is due to formula (2.8), while continuity in $\mathrm{L}^p(\mathfrak{m})$ for every $p \in [1, +\infty)$ follows from the following:

$$\int_{\mathsf{Y}} |\mathrm{Pr}_\varphi|^p \,\mathrm{d}\mathfrak{m}_{\mathsf{Y}} = \int_{\mathsf{Y}} \left| \int_{\mathsf{X}} f(x) \,\mathrm{d}\mathfrak{m}_y(x) \right|^p \mathrm{d}\mathfrak{m}_{\mathsf{Y}}(y) \le \int_{\mathsf{Y}} \int_{\mathsf{X}} |f(x)|^p \,\mathrm{d}\mathfrak{m}_y(x) \,\mathrm{d}\mathfrak{m}_{\mathsf{Y}}(y) = \|f\|^p_{\mathrm{L}^p(\mathfrak{m})},$$

where we used Jensen's inequality and the properties of the disintegration. ∎

https://doi.org/10.4153/S0008439523000486 Published online by Cambridge University Press

In the case of a general $\mathrm{L}^p(\mathfrak{m}_X)$-normed module, the continuous operator $\mathrm{Pr}_\varphi : \varphi^*\mathcal{M} :\longrightarrow \mathcal{M}$ can be characterized by the following properties:

$$g\mathrm{Pr}_\varphi(v) = \mathrm{Pr}_\varphi(g \circ \varphi v), \quad \forall v \in \mathcal{M} \quad \forall g \in \mathrm{L}^\infty(\mathfrak{m}_X), \tag{2.9}$$

$$\mathrm{Pr}_\varphi(g\varphi^* v) = \mathrm{Pr}_\varphi(g)v, \quad \forall v \in \mathcal{M} \quad \forall g \in \mathrm{L}^\infty(\mathfrak{m}_X), \tag{2.10}$$

with the bound $|\mathrm{Pr}_\varphi(V)| \le \mathrm{Pr}_\varphi(|V|)$ still holding $\mathfrak{m}_Y$-a.e. for every $V \in \varphi^*\mathcal{M}$.

With these objects, we are now able to describe the structure of the pullback module; in particular (as one can expect by reasoning via pre-composition), the pullback of an n-dimensional module $\mathcal{M}$ over E is an n-dimensional module over $\varphi^{-1}(E)$ (see also [Pas18]).

Proposition 2.14 *Let $\mathcal{M}$ be an $\mathrm{L}^p(\mathfrak{m}_Y)$-normed module over the m.m.s. $(\mathrm{Y}, d_\mathrm{Y}, \mu)$, and let $E \in \mathcal{B}(Y)$ be a Borel set where $\mathcal{M}$ has dimension* n, *with $\{v_1, \dots, v_n\}$ being a basis. Let $(\mathrm{X}, d_\mathrm{X}, \mathfrak{m})$ be another m.m.s., and let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a Borel map such that $\varphi_\sharp \mathfrak{m}_X = \mathfrak{m}_Y$, then $\{\varphi^* v_1, \dots, \varphi^* v_n\}$ is a basis of $\varphi^*\mathcal{M}$ over $\varphi^{-1}(E)$.*

Proof We first prove that $\{\varphi^* v_1, \dots, \varphi^* v_n\}$ generate $\varphi^*\mathcal{M}$ over $\varphi^{-1}(E)$.

First recall that $\varphi^*\mathcal{M}$ is generated (as module) by $\{\varphi^* v : v \in \mathcal{M}\} =: V$. Let us show that $V \subseteq \mathrm{Span}_{\varphi^{-1}(E)}\{\varphi^* v_1, \dots, \varphi^* v_n\}$: pick $w \in V$, then there exists $v \in \mathcal{M}$ such that $w = \varphi^* v$ so that there exists $(A_j)_j \subseteq \mathscr{B}(\mathrm{X})$ partition of E and $(g_i^j)_{j \in \mathbb{N}} \subset \mathrm{L}^\infty(\mathfrak{m}_Y)$ $\forall i = 1, \dots, n$ such that

$$\chi_{A_j} v = \sum_{i=1}^n g_i^j v_i \qquad \forall j \in \mathbb{N}.$$

Using the linearity of the pullback map and the fact that $\varphi^*(gv) = g \circ \varphi \varphi^* v$ for all $v \in \mathcal{M}$, $g \in \mathrm{L}^\infty(\mathfrak{m}_Y)$, we get

$$\chi_{\varphi^{-1}(A_j)} w = \sum_{i=1}^n g_i^j \circ \varphi \varphi^* v_i.$$

Finally, since the pullback module has a natural structure of $\mathrm{L}^p(\mathfrak{m})$-normed $\mathrm{L}^\infty(\mathfrak{m})$-module, we get that $\mathrm{Span}_{\varphi^{-1}(E)}\{\varphi^* v_1, \dots, \varphi^* v_n\}$ is closed, proving the first result.

We now turn to local independence: assume by contradiction $\{\varphi^* v_1, \dots, \varphi^* v_n\}$ are not independent on $\varphi^{-1}(E)$, then there exist $f_1, \dots, f_n \in \mathrm{L}^\infty(\mathfrak{m}_X)$ such that $\sum_{i=1}^n f_i \varphi^* v_i = 0$ $\mathfrak{m}$-a.e. with (upon relabeling indexes) $|f_1| > 0$ $\mathfrak{m}$-a.e. on some subset $\tilde{E}$ of positive measure. Without loss of generality, possibly considering a smaller set, we shall assume $f_1 > 0$ $\mathfrak{m}$-a.e. so that

$$\sum_{i=1}^n f_i \varphi^* v_i = 0 \quad \mathfrak{m} - \text{a.e. on } \tilde{E} \implies \sum_{i=1}^n \mathrm{Pr}_\varphi(f_i) v_i = 0 \quad \mathfrak{m} - \text{a.e. on } \tilde{E}.$$

However, note that $\mathrm{Pr}_\varphi(f_1) > 0$ on some set of positive $\mathfrak{m}_Y$ measure, contradicting the independence of the v_is. ∎

Definition 2.15 We say that the space of $\mathrm{L}^\infty(\mathfrak{m})$-linear and continuous maps $L : \mathcal{M} \to L^1(\mathfrak{m})$ is the dual module of the module $\mathcal{M}$, and we shall denote this space by $\mathcal{M}^*$.

https://doi.org/10.4153/S0008439523000486 Published online by Cambridge University Press

Remark 2.16 Being $\mathcal{M}$ $\mathrm{L}^p(\mathfrak{m})$-normed, we can endow $\mathcal{M}^*$ with a natural structure of $\mathrm{L}^q(\mathfrak{m})$-normed module.

2.4 The cotangent and tangent modules

We are now in position to speak about the differential of a Sobolev function as the following proposition shows.

Proposition 2.17 *Let* $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ *be a metric measure space, then there exists a unique (up to unique isomorphism) couple* $(\mathrm{L}^p(\mathrm{T}^*\mathrm{X}), \mathrm{d}_p)$ *where* $\mathrm{L}^p(\mathrm{T}^*\mathrm{X})$ *is an* $\mathrm{L}^p(\mathfrak{m})$-*normed* $\mathrm{L}^\infty(\mathfrak{m})$-*module and* $\mathrm{d}_p : W^{1,p}(\mathrm{X}) \to \mathrm{L}^p(\mathrm{T}^*\mathrm{X})$ *is a linear and continuous operator such that:*

(1) $|\mathrm{d}_p f| = |Df|_p$ $\mathfrak{m}$-*a.e. for every* $f \in W^{1,p}(\mathrm{X})$,
(2) *the set* $\{\mathrm{d}f : f \in W^{1,p}(\mathrm{X})\}$ *generates* $\mathrm{L}^p(\mathrm{T}^*\mathrm{X})$.

Remark 2.18 We will call *1-forms* the elements of $\mathrm{L}^p(\mathrm{T}^*\mathrm{X})$, in analogy with the section of the cotangent bundle on a Riemannian manifold.

Definition 2.19 We denote with $\mathrm{L}^q(\mathrm{TX})$ the dual module of $\mathrm{L}^p(\mathrm{T}^*\mathrm{X})$, and we call its elements vector fields or vectors.

Besides the differential of a Sobolev function introduced in Proposition 2.17, one can give another definition which exploits the fact that the map is Lipschitz and such that $\varphi_\sharp \mathfrak{m}_\mathrm{X} \le C\mathfrak{m}_\mathrm{Y}$ for some C¿0 (namely a map of *bounded compression*): this class of maps is that of *bounded deformation*. In this direction, we need to recall the notion of *pullback of forms*: in order to distinguish it from the pullback of a module, we shall proceed denoting with $\omega \mapsto [\varphi^*\omega]$ the pullback map and with φ^* the pullback of 1-forms which is the following.

Definition 2.20 Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation, then we define $\varphi^* : \mathrm{L}^p(\mathrm{T}^*\mathrm{Y}) \to \mathrm{L}^p(\mathrm{T}^*\mathrm{X})$ to be the linear map such that $\varphi^*(\mathrm{d}f) = \mathrm{d}(f \circ \varphi)$ for all $f \in W^{1,p}(\mathrm{Y})$ and $\varphi^*(g\omega) = g \circ \varphi\varphi^*\omega$ for all $g \in \mathrm{L}^\infty(\mathrm{Y})$ and $\omega \in \mathrm{L}^p(\mathrm{T}^*\mathrm{Y})$.

Remark 2.21 It is easy to see that, thanks to the regularity properties of φ, the pullback of 1-forms φ^* is well defined.

Definition 2.22 Given $\varphi : \mathrm{X} \longrightarrow \mathrm{Y}$ of bounded deformation, we define for all $p \ge 1$ its p-differential as an operator $\underline{\mathrm{d}_p\varphi} : \mathrm{L}^q(\mathrm{TX}) \longrightarrow \varphi^*\big(\mathrm{L}^p(\mathrm{T}^*\mathrm{Y})\big)^*$ such that

$$[\varphi^*\omega](\underline{\mathrm{d}_p\varphi}(v)) = \varphi^*\omega(v) \quad \forall v \in \mathrm{L}^q(\mathrm{TX}), \ \ \forall \omega \in \mathrm{L}^p(\mathrm{T}^*\mathrm{Y}). \tag{2.11}$$

In the recent work [ES21], the authors provide some "charts" over Borel sets $(E_i)_{i\in\mathbb{N}}$ partitioning the metric measure space $\mathfrak{m}$-a.e.: we will briefly recall here the definition.

https://doi.org/10.4153/S0008439523000486 Published online by Cambridge University Press

Definition 2.23 We say that $\varphi : \mathrm{X} \to \mathbb{R}^N$ is an EBS chart over the Borel set E if it is a Lipschitz map with the following properties:

(1) (p-independence) $\operatorname{ess\,inf}_{\nu\in\mathbb{S}^{N-1}}|D(\nu\cdot\varphi)|_p > 0$ $\mathfrak{m}$-a.e on E.
(2) (maximality) There is no other Lipschitz map $\varphi : \mathrm{X} \to \mathbb{R}^M$ with $M > N$ which is p-independent on a subset of E of positive measure.

The authors proved that the condition of p-independence over a set E is equivalent to the fact that the $\mathrm{L}^p(\mathrm{T}^*\mathrm{X})$ module over E is generated by the differentials of the components of the chart: in other words, $\{\mathrm{d}_p\varphi^1, \ldots, \mathrm{d}_p\varphi^N\}$ is a basis for $\mathrm{L}^p(\mathrm{T}^*\mathrm{X})_{|E}$ (see Lemma 6.3 in [ES21]), and as a consequence of Theorem 1.4.7 in [Gig18], we are able to deduce that $\mathrm{L}^q(\mathrm{TX})_{|E}$ is also an N-dimensional normed module.

3 Main result

In this section, we give an alternative proof to Proposition 4.13 in [ES21]. First, we remark that with $\underline{\mathrm{d}_p\varphi}$, we will denote the differential of a map of bounded deformation in the sense of Definition 2.22, whereas with $\mathrm{d}_p f$, we denote the differential in the sense of Proposition 2.17. Lastly, let us assume that $\mathfrak{m}$ is a finite measure: we can do so because of the inner regularity of the measure $\mathfrak{m}$. Indeed, if for a Borel map $\psi : \mathrm{X} \to \mathbb{R}^n$ we have $\psi_\sharp(\mathfrak{m}_{|E_k}) << \mathscr{L}^n$ for every $k \in \mathbb{N}$ with $(E_k)_k$ compact, such that $E_k \subseteq E_{k+1}$ and $\mathfrak{m}(E\setminus \cup_k E_k) = 0$, then $\psi_\sharp(\mathfrak{m}_{|E}) << \mathscr{L}^n$.

We begin with the following simple lemma, which follows standard arguments in linear algebra.

Lemma 3.1 *Let $\mathcal{M}$ be an $\mathrm{L}^p(\mathfrak{m})$-normed module, and let $\mathcal{M}^*$ be its dual module. Assume that $\mathcal{M}$ has dimension* n *over* E*: then $\{v_1, \ldots, v_n\}$ and $\{\omega_1, \ldots, \omega_n\}$ are basis of $\mathcal{M}^*$ and $\mathcal{M}$ (respectively) over* E *if and only if $\det[\omega_i(v_j)]_{ij} > 0$ $\mathfrak{m}$-a.e. on* E.

Proof Define $A_{ij} := [\omega_i(v_j)]_{ij}$, and let us assume first that $\det A > 0$ $\mathfrak{m}$-a.e. It is clearly sufficient to prove the independence: assume by contradiction that $\sum_{i=1}^n g_i v_i = 0$ $\mathfrak{m}$-a.e. on some subset B of positive measure, for some $g_1, \ldots, g_n$ which are not all zero on B (in the measure theoretic sense). Then consider $\mathrm{g} := (g_1, \ldots, g_n)$ and note that $A\mathrm{g} \neq 0$ $\mathfrak{m}$-a.e. on B because of the condition on the determinant. However, $(A\mathrm{g})_i = \sum_{j=1}^n g_j v_j(\omega_i)$=0 $\mathfrak{m}$-a.e. on B for every $i = 1, \ldots, n$, which is clearly a contradiction. This argument trivially applies for $\{\omega_1, \ldots, \omega_n\}$ as well by considering the transpose of A.

Assume now that $\{\omega_1, \ldots, \omega_n\}$ and $\{v_1, \ldots, v_n\}$ are basis over E of $\mathcal{M}$ and $\mathcal{M}^*$, respectively, and by contradiction, let $\det A = 0$ $\mathfrak{m}$-a.e. on a Borel subset C of positive measure. Then there exists a further measurable subset (which we will not relabel) C of positive measure and $\mathrm{g} \in \mathrm{L}^\infty(\mathfrak{m})^n$ for which $A\mathrm{g} = 0$ and $\mathrm{g} \neq 0$ $\mathfrak{m}$-a.e. on C. The latter system of equations means that we have

$$v_i\left(\sum_{j=1}^n g_j\omega_j\right) = 0 \quad \mathfrak{m}-\text{a.e. on } C, \;\; \forall i = 1, \ldots, n. \tag{3.1}$$

Set $\tilde{\omega} = \sum_{j=1}^n g_j\omega_j$ and suppose that $|\tilde{\omega}| \neq 0$ $\mathfrak{m}$-a.e. on C, then there exists a nonzero continuous functional $\ell \in \mathcal{M}'$ (which is the Banach dual) such that $\ell(\chi_C\tilde{\omega}) = \|\chi_C\tilde{\omega}\|_{\mathcal{M}}$

https://doi.org/10.4153/S0008439523000486 Published online by Cambridge University Press

and there exists $L \in \mathcal{M}^*$ (see Proposition 1.2.13 in [Gig18]) such that

$$\ell(\omega) = \int_X L(\omega)\,\mathrm{d}\mathfrak{m} \quad \forall \omega \in \mathcal{M}.$$

In our case, this means that $\|\chi_C \tilde{\omega}\|_{\mathcal{M}} = \int_C L(\tilde{\omega})\,\mathrm{d}\mathfrak{m} > 0$, so that there must be a Borel set of positive measure where $\chi_C L(\tilde{\omega}) > 0$, which contradicts (3.1) since there exists $D \subset C$ with $\mathfrak{m}(D) > 0$ such that $\chi_D L = \sum_{i=1}^n f_i v_i$ for some $f_1, \dots, f_n \in \mathrm{L}^\infty(\mathfrak{m})$. ■

Lemma 3.2 *Let φ be an EBS chart over the Borel set* E, *and let $\{v_1, \dots, v_n\} \in \mathrm{L}^p(\mathrm{TX})$ be independent over* E, *then $\{\underline{\mathrm{d}_p\varphi}(v_1), \dots, \underline{\mathrm{d}_p\varphi}(v_n)\} \in \varphi^*\mathrm{L}^p_\mu(T\mathbb{R}^n)$ are independent over the same set, where $\mu = \varphi_\sharp(\mathfrak{m}_{|E})$ and $L^p_\mu(T\mathbb{R}^n)$ is the tangent module built over $(\mathbb{R}^n, d_{eucl}, \mu)$.*

Proof Consider $f_1, \dots, f_n \in \mathrm{L}^\infty(\mathfrak{m})$ such that

$$\sum_{i=1}^n f_i\,\underline{\mathrm{d}_p\varphi}(v_i) = 0 \quad \mathfrak{m}-\text{a.e. on } E,$$

then set $v := \sum_{i=1}^n f_i v_i$. Note that the maps $\Pi^j : \mathbb{R}^n \longrightarrow \mathbb{R}$ being the projection on the jth component are all 1-Lipschitz with respect to the Euclidean distance, and for this reason, they belong to $W^{1,p}(\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}, \mu)$: following equation (2.11), we have that, for every $j = 1, \dots, n$ and choosing $\omega = \mathrm{d}_p\Pi_j$,

$$0 = \mathrm{d}_p\varphi^j(v) = \sum_{i=1}^n f_i\,\mathrm{d}_p\varphi^j(v_i) \quad \mathfrak{m}-\text{a.e. on } E,$$

where φ^j is the jth component of the map φ.

Being the matrix $A = (A_{ij})_{ij} = \langle \mathrm{d}_p\varphi^j, v_i\rangle$ such that $\det A > 0$ $\mathfrak{m}$-a.e., the equations above can be rewritten as $A\underline{\mathrm{f}} = 0$ $\mathfrak{m}$−a.e. on E with $\underline{\mathrm{f}} = (f_1, \dots, f_n)$, meaning $\underline{\mathrm{f}} = 0$ thanks to Lemma 3.1. ■

The following result is borrowed from [LPR21, Proposition 4.5] where only the metric measure space $(\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}, \mu)$ is considered.

Proposition 3.3 *Assume that there exists a Borel set* E *such that* $\dim \mathrm{L}^p_\mu(\mathrm{T}^*\mathbb{R}^n)_{|E} = n$ *for some $p \in (1, +\infty)$, then $\mu_{|E} << \mathscr{L}^n$.*

Remark 3.4 It is in the proof of the latter proposition that the results contained in [DR] are used.

Now we are in place to apply Proposition 3.3 to prove the following.

Theorem 3.5 *Let $\varphi : \mathrm{X} \to \mathbb{R}^N$ be a* p-*independent weak chart over a Borel set* E *of positive measure and with $p \geq 1$, then $\mu = \varphi_\sharp(\mathfrak{m}_{|E}) << \mathscr{L}^N$ and $N \leq dim_H(E)$.*

Proof For the moment, assume that $p \in (1, +\infty)$, and without loss of generality, assume that E to be compact. Thanks to Lemma 3.2, we deduce that $\varphi^*\mathrm{L}^p_\mu(\mathrm{T}^*\mathbb{R}^N)$ has dimension N over the set E, meaning that $\mathrm{L}^p_\mu(\mathrm{T}^*\mathbb{R}^N)$ has dimension N over the set $\varphi(E)$. Being the latter module top dimensional, by Proposition 3.3, we have that $\mu << \mathscr{L}^N$, which is the first part of the statement. The second part is immediate since if we had $N > \dim_H(E)$ we would get $\mathcal{H}^N(E) = 0$ and since the map φ is Lipschitz this implies $\mathcal{H}^N(\varphi(E)) = \mathscr{L}^N(\varphi(E)) \leq C \cdot 0 = 0$, so that by absolute continuity $\mu(\varphi(E)) = \mathfrak{m}(E) = 0$, which is clearly a contradiction.

For the case $p = 1$, note that, since the measure $\mathfrak{m}$ is finite, we have $|D(v \cdot \varphi)|_1 \leq |D(v \cdot \varphi)|_p$ $\mathfrak{m}$-a.e. and for every $v \in \mathbb{S}^{N-1}$, meaning that φ is also p-independent and the same argument applies. ∎

Remark 3.6 By virtue of the latter theorem, one can see that a control on the Hausdorff dimension l of a subset E of a metric measure space grants that the dimension of $L^p(T^*X)_{|E}$ is bounded by l; hence, the cotangent module is finite-dimensional there. Moreover, the proof presented here simplifies the one in [GP21] since there the authors needed to build independent vector fields in $L^2(TX)$ with $L^2(\mathfrak{m})$-integrable divergence and push them to $\mathbb{R}^n$ keeping them independent and regular: to do so, they had to use additional properties of the map $\Pr_\varphi$ and the bi-Lipschitz regularity of their chart φ was essential. Here, instead, we mainly exploit the properties of $\mathbb{R}^n$.

Acknowledgment We wish to thank Elefterios Soultanis for the numerous conversations we had with him while working on this manuscript.

References

[AM16] G. Alberti and A. Marchese, *On the differentiability of Lipschitz functions with respect to measures in the Euclidean space*. Geom. Funct. Anal. **26**(2016), no. 1, 1–66. https://doi.org/10.1007/s00039-016-0354-y

[ACD14] L. Ambrosio, M. Colombo, and S. Di Marino, *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope*. Accepted at Adv. Stud. Pure Math., 2014. arXiv:1212.3779

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2008, pp. x + 334.

[AGS14] L. Ambrosio, N. Gigli, and G. Savaré, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*. Invent. Math. **195**(2014), no. 2, 289–391. https://doi.org/10.1007/s00222-013-0456-1

[Bog07] V. I. Bogachev, *Measure theory , Vols. I and II*, Springer, Berlin, 2007, Vol. I: xviii + 500 pp., Vol. II: xiv + 575 pp. https://doi.org/10.1007/978-3-540-34514-5

[Che99] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*. Geom. Funct. Anal. **9**(1999), no. 3, 428–517.

[DMR] G. De Philippis, A. Marchese, and F. Rindler, *On a conjecture of Cheeger*. Preprint, 2016, arXiv:1607.02554

[DR] G. De Philippis and F. Rindler, *On the structure of $\mathscr{A}$-free measures and applications*. Accepted at Ann. Math., 2016. arXiv:1601.06543

[ERS22a] S. Erikkson-Bique, T. Rajala, and E. Soultanis, *Tensorization of p-weak differentiable structures*. Preprint, 2022. arXiv:2206.05046

[ERS22b] S. Erikkson-Bique, T. Rajala, and E. Soultanis, *Tensorization of quasi-Hilbertian Sobolev spaces*. Preprint, 2022. arXiv:2209.03040

[ES21] S. Erikkson-Bique and E. Soultanis, *Curvewise characterizations of minimal upper gradients and the construction of a Sobolev differential*. Preprint, 2021. arXiv:2102.08097

[Fre06] D. H. Fremlin, *Measure theory: topological measure spaces. Parts I and II,* Vol. 4, Torres Fremlin, Colchester, 2006, Part I: 528 pp., Part II: 439 + 19 pp. (errata), corrected second printing of the 2003 original.

[Gig15] N. Gigli, *On the differential structure of metric measure spaces and applications*. Mem. Amer. Math. Soc. **236**(2015), no. 1113, vi + 91. https://doi.org/10.1090/memo/1113

[Gig18] N. Gigli, *Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below*. Mem. Amer. Math. Soc. **251**(2018), no. 1196, v + 161. https://doi.org/10.1090/memo/1196

[GP21] N. Gigli and E. Pasqualetto, *Behaviour of the reference measure on RCD spaces under charts*. Comm. Anal. Geom. **29**(2021), no. 6, 1391–1414. https://doi.org/10.4310/CAG.2021.v29.n6.a3

[KM18] M. Kell and A. Mondino, *On the volume measure of non-smooth spaces with Ricci curvature bounded below*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18**(2018), no. 2, 593–610.

[LPR21] D. Lučić, E. Pasqualetto, and T. Rajala, *Characterisation of upper gradients on the weighted Euclidean space and applications*. *Ann. Mat. Pura Appl. (4)* **200**(2021), no. 6, 2473–2513. https://doi.org/10.1007/s10231-021-01088-4

[MN14] A. Mondino and A. Naber, *Structure theory of metric-measure spaces with lower Ricci curvature bounds*. Accepted at J. Eur. Math. Soc., 2017. arXiv:1405.2222

[Pas18] E. Pasqualetto, *Structural and geometric properties of RCD spaces*. Ph.D. thesis, SISSA, 2018.

SISSA, Via Bonomea 256, Trieste, Italy
e-mail: luca.gennaioli@sissa.it nicola.gigli@sissa.it