COMMUNTANTS OF TOEPLITZ OPERATORS ON THE BALL AND ANNULUS

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In this paper we study commutants of Toeplitz operators with polynomial symbols acting on Bergman spaces of various domains. For a positive integer n, let V denote the Lebesgue volume measure on $\mathbb{C}^n$. If Ω is a domain in $\mathbb{C}^n$, then the Bergman space $L_a^2(\Omega)$ is defined to be the set of all analytic functions from Ω into $\mathbb{C}$ such that

$$\int_\Omega |f|^2\, dV < \infty.$$

The Bergman space $L_a^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega, dV)$ with inner product given by

$$\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)}\, dV(z), \quad \text{for } f, g \in L^2(\Omega, dV).$$

Let $\mathbf{P}$ denote the orthogonal projection of $L^2(\Omega, dV)$ onto $L_a^2(\Omega)$. For a function $f \in L^\infty(\Omega)$, we define the Toeplitz operator $T_f: L_a^2(\Omega) \to L_a^2(\Omega)$ and the Hankel operator $H_f: L_a^2(\Omega) \to L_a^2(\Omega)^\perp$ by $T_f g = \mathbf{P}(fg)$ and $H_f g = (I - \mathbf{P})fg$. Clearly, for every $f \in L^\infty(\Omega)$, T_f and H_f belong to $L(L_a^2(\Omega))$, where $L(L_a^2(\Omega))$ is the set of all bounded linear operators on $L_a^2(\Omega)$. The *commutant* of T_f, denoted by $\{T_f\}'$, is the set of all $S \in L(L_a^2(\Omega))$ such that $ST_f = T_f S$. Much work has been done in studying commutants of Toeplitz operators defined on the Bergman and Hardy spaces of the unit disk $\mathbb{D}$. The reader can see **[1]** and **[2]**, **[3]**, **[4]**, **[5]** and **[6]** for further references.

In this paper we will investigate commutants of certain Toeplitz operators for the cases $\Omega = B_n = \{z \in \mathbb{C}^n : |z| < 1\}$ and $\Omega = A = \{z \in \mathbb{C} : r < |z| < 1\}$ for some $r > 0$. In what follows, we will use $T(\Omega)$ to denote the norm closed subalgebra of $L(L_a^2(\Omega))$ generated by all Toeplitz operators, and $H^\infty(\Omega)$ to denote the algebra of bounded analytic functions on Ω. Our first result is an extension of the following theorem: see **[3]**.

THEOREM A. *Let $n \in \mathbb{N}$ and let $S \in T(\mathbb{D})$ commute with T_{z^n}. Then $S = T_\Psi$, for some $\Psi \in H^\infty(\mathbb{D})$.*

This paper is organized as follows. In §1 we prove a theorem for $\Omega = B_n$ that is analogous to the above Theorem A. In §2 we study commutants of certain Toeplitz operators $T_p \in L(L_a^2(A))$, where p is a polynomial. We give a sufficient condition for $\{T_p\}' = \{T_\Psi : \Psi \in H^\infty(A)\}$ for the case where p is a polynomial with non-negative coefficients. To our knowledge, nobody has studied commutants of Toeplitz operators defined on the Bergman space of an annulus. We hope our paper will initiate more work in that direction.

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1. The unit ball in $\mathbb{C}^n$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, and $z \in \mathbb{C}^n$, we use the standard notation $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. The main result in this section is the following theorem.

THEOREM 1. *For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, if $S \in T(B_n)$ commutes with $T_{z_i^{\alpha_i}}$, for all $i = 1, \dots, n$, then $S = T_\psi$ for some $\psi \in H^\infty(B_n)$.*

Proof. Let $g_k = Sz^k$, for $k = (k_1, \dots, k_n)$, where $0 \le k_i \le \alpha_i - 1$ for all $i = 1, \dots, n$. Then for any such k

$$\begin{aligned} Sz^{l\alpha+k} &= S\prod_{i=1}^{n} z_i^{l_i\alpha_i+k_i} = S\prod_{i=1}^{n} (T_{z_i^{\alpha_i}})^{l_i} \prod_{i=1}^{n} z_i^{k_i} \\ &= \prod_{i=1}^{n} (T_{z_i^{\alpha_i}})^{l_i} Sz^k = \prod_{i=1}^{n} z_i^{l_i\alpha_i} g_k = z^{l\alpha} g_k, \end{aligned} \tag{1}$$

for $l = (l_1, \dots, l_n)$ and $l_i = 0, 1, 2, \dots$. Note that $\{C(m_1, \dots, m_n) \prod_{i=1}^{n} z_i^{m_i}\}$ is an orthonormal basis for $L_a^2(B_n)$, with the appropriate constants $C(m_1, \dots, m_n)$. We let $X_k = X_{(k_1,k_2,\dots,k_n)} =$ span$\{C(l_1\alpha_1 + k_1, \dots, l_n\alpha_n + k_n) \prod_{i=1}^{n} z_i^{l_i\alpha_i+k_i}, l = 0, 1, 2, \dots\}$. Then it is clear that

$$L_a^2(B_n) = \bigoplus_k X_k = \bigoplus_{k_1=0}^{\alpha_1-1} \bigoplus_{k_2=0}^{\alpha_2-1} \dots \bigoplus_{k_n=0}^{\alpha_n-1} X_{(k_1,k_2,\dots,k_n)}.$$

Thus each $f \in L_a^2(B_n)$ can be written as $f = \sum_k f_k$ with $f_k \in X_k$. Each $f_k \in X_k$ has the expansion $f_k = \lim_N \sigma_N$, where

$$\begin{aligned} \sigma_N = \sum_{l_1=0}^{N_1} \dots \sum_{l_n=0}^{N_n} &\left\langle f_k, C(l_1\alpha_1 + k_1, \dots, l_n\alpha_n + k_n) \prod_{i=1}^{n} z_i^{l_i\alpha_i+k_1} \right\rangle \\ &\cdot C(l_1\alpha_1 + k_1, \dots, l_n\alpha_n + k_n) \prod_{i=1}^{n} z_i^{l_i\alpha_i+k_i}, \end{aligned}$$

and $N = (N_1, \dots, N_n)$. Since the point evaluations are bounded on $L_a^2(B_n)$, we have $(f_k g_k)(z) = \lim_N (\sigma_N g_k)(z)$ for all $z \in B_n$. On the other hand, (1) implies $S\sigma_N = \sigma_N \dfrac{g_k}{z^k}$, so that $(Sf_k)(z) = \lim_N (S\sigma_N)(z) = \dfrac{f_k(z) g_k(z)}{z^k}$ for every $z \in B_n$. Thus we obtain that

$$Sf = \sum_k f_k \frac{g_k}{z^k} = \sum_{k_1=0}^{\alpha_1-1} \sum_{k_2=0}^{\alpha_2-1} \dots \sum_{k_n=0}^{\alpha_n-1} f_k \frac{g_k}{z^k}. \tag{2}$$

Next we need to prove the following two claims.

CLAIM 1. *$ST_{z_i} - T_{z_i}S$ is compact for each $i = 1, 2, \dots, n$.*

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Proof. If $S = T_\varphi$, for $\varphi \in L^\infty(B_n)$, then we easily see that $ST_{z_i} - T_{z_i}S = H^*_{\bar{z}_i}H_\varphi$. We also easily check that the function z_i belongs to the little Bloch space which is defined in the following way. For any $f \in \mathrm{Hol}(B_n)$, denote $f'(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\right)$. For any $\lambda \in B_n$ there exists a function $\phi_\lambda \in \mathrm{Aut}\,(B_n)$ such that $\phi_\lambda(0) = \lambda$. We define the invariant derivative

$$(\tilde{D}f)(\lambda) = |(f \circ \phi_\lambda)'(0)|.$$

The little Bloch space B_0 is the space of all holomorphic functions f on B_n that satisfy $(\tilde{D}f)(\lambda) \to 0$ as $|\lambda| \to 1$. By **[7]**, we know that $z_i \in B_0$ implies that $H^*_{\bar{z}_i}$ is compact. So the rest of the proof for Claim 1 follows easily in the same way as in **[3]**.

Claim 2. $\frac{g_k}{z^k} = g_0$, *where* $0 = (0, 0, \dots, 0)$.

Proof. We write $g_k = g_{(k_1, k_2, \dots, k_n)}$. Clearly we only need to prove that

$$\frac{g_{(k_1,k_2,\dots,k_n)}}{z_i} = g_{(k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_n)}$$

for any $1 \le k_i \le \alpha_i - 1$. Without loss of generality, we may assume $i = 1$ and prove that

$$\frac{g_{(k_1,k_2,\dots,k_n)}}{z_i} = g_{(k_1-1,k_2,\dots,k_n)}, \quad \text{for } 1 \le k_1 \le \alpha_1 - 1.$$

We calculate $ST_{z_1}f = \sum_k ST_{z_1}f_k$, where $f = \sum_k f_k$ with $f_k \in X_k$. By (1),

$$\begin{aligned} ST_{z_1}f_k &= \sum_{l_1=0}^{\infty} \dots \sum_{l_n=0}^{\infty} |C(l_1\alpha_1 + k_1, \dots, l_n\alpha_n + k_n)|^2 \left\langle f_k, \prod_{i=1}^n z_i^{l_i\alpha_i + k_i} \right\rangle ST_{z_1} \prod_{i=1}^n z_i^{l_i\alpha_i + k_i} \\ &= \sum_l |C(l_1\alpha_1 + k_1, \dots, l_n\alpha_n + k_n)|^2 \left\langle f_k, \prod_{i=1}^n z_i^{l_i\alpha_i + k_i} \right\rangle \prod_{i=1}^n z_i^{l_i\alpha_i + k_i} \frac{g_{(k_1+1,k_2,\dots,k_n)}}{z^k} \\ &= f_k \frac{g_{(k_1+1,k_2,\dots,k_n)}}{z^k}, \quad \text{if } k_1 < \alpha_1 - 1. \end{aligned}$$

We can also easily see that

$$T_{z_1}Sf = \sum_k \frac{g_k}{z_1^{k_1-1} z_2^{k_2} \dots z_n^{k_n}} f_k.$$

Therefore, we have

$$\begin{aligned} (ST_{z_1} - T_{z_1}S)f &= \sum_{k_1=0}^{\alpha_1-2} \sum_{k_2=0}^{\alpha_2-1} \dots \sum_{k_n=0}^{\alpha_n-1} \left[\frac{g_{(k_1+1,k_2,\dots,k_n)}}{\prod_{i=1}^n z_i^{k_i}} - \frac{g_{(k_1,k_2,\dots,k_n)}}{z_1^{k_1-1} z_2^{k_2} \dots z_n^{k_n}} \right] f_k \\ &\quad + \sum_{k_2=0}^{\alpha_2-1} \dots \sum_{k_n=0}^{\alpha_n-1} \left(\frac{z_1 g_{(0,k_2,\dots,k_n)}}{z_2^{k_2} z_3^{k_3} \dots z_n^{k_n}} - \frac{g_{(\alpha_1-1,k_2,\dots,k_n)}}{z_1^{\alpha_1-2} z_2^{k_2} \dots z_n^{k_n}} \right) \cdot f_{(\alpha_1-1,k_2,\dots,k_n)}. \end{aligned}$$

By Claim 1, $(ST_{z_1} - T_{z_1}S)|_{X_k} = M_\psi|_{X_k}$ is compact, where

$$\psi = \frac{g_{(k_1+1,k_2,\ldots,k_n)}}{z^k} - \frac{z_1 g_{(k_1,k_2,\ldots,k_n)}}{z_k}, \quad \text{for } 0 \le k_1 \le \alpha_1 - 2.$$

It follows that $M_{z^k}\psi : X_0 \to L_a^2(B_n)$ is compact too. Let $\varphi = z^k\psi$, for $0 \le k_1 \le \alpha_1 - 2$. The operator $M_\varphi|_{X_j} = M_{z^j}(M_\varphi|_{X_0})(M_{z^{-j}}|_{X_j})$ is compact for every j, so that M_φ is compact on $L_a^2(B_n)$. This implies that $\varphi \equiv 0$ and consequently

$$\frac{g_{(k_1+1,k_2,\ldots,k_n)}}{z_1} = g_{(k_1,k_2,\ldots,k_n)},$$

for all $0 \le k_1 \le \alpha_1 - 2$, which proves Claim 2.

Hence, by (2) and Claim 2 we have $Sf = \sum_k f_k \frac{g_k}{z^k} = g_0 \sum_k f_k = g_0 f$, and $g_0 \in H^\infty(B_n)$ is a multiplier of $L_a^2(B_n)$ (see [**5**]). The theorem is proved.

Theorem 1 has the following corollary.

COROLLARY 2. *Let* $u = (u_1, u_2, \ldots, u_n) \in \mathrm{Aut}(B_n)$ *and let* $S \in T(B_n)$ *commute with* $\{T_{u_i^{\alpha_i}} : i = 1, \ldots, n\}$, *where* $\alpha_i \in \mathbb{N}$. *Then* $S = T_\psi$ *for some* $\psi \in H^\infty(B_n)$.

Proof. For $u \in \mathrm{Aut}(B_n)$ define $V : L_a^2(B_n) \to L_a^2(B_n)$ by $Vf = f \circ u^{-1}$. Then

$$V^{-1}T_{z_i}Vf = V^{-1}(z_i . (f \circ u^{-1})) = (z_i \circ u)f = T_{u_i}f$$

and hence

$$T_{z_i^{\alpha_i}}V = VT_{u_i^{\alpha_i}}$$

for $\alpha_i \in \mathbb{N}$. Since S commutes with $T_{u_i^{\alpha_i}}$, for every $i = 1, \ldots, n$, it follows that $U = VSV^{-1} \in \{T_{z_i^{\alpha_i}} : i = 1, \ldots, n\}$. If we can show that $UT_{z_i} - T_{z_i}U$ is compact, then the corollary will follow easily from the proof of Theorem 1. Observe that

$$UT_{z_i} - T_{z_i}U = V(ST_{u_i} - T_{u_i}S)V^{-1}.$$

Since $S \in T(B_n)$, by a similar argument to that used in Claim 1, it suffices to show that $u_i \in B_0$. We denote $f = u_i = z_i \circ u$. If $u = \phi_a$, for some $a \in B_n$, then

$$(\tilde{D}f)(\lambda) = |(z_i \circ \phi_a \circ \phi_\lambda)'(0)| = |(z_i \circ \phi_{\phi_\lambda(a)})'(0)|.$$

For simplicity, we will denote $\xi = \phi_\lambda(a)$. Thus a simple computation shows that

$$(z_i \circ \phi_\xi)(z) = \frac{\xi_i - \dfrac{z . \bar{\xi}}{|\xi|^2}\xi_i - (1 - |\xi|^2)^{1/2}\left(z_i - \dfrac{z . \bar{\xi}}{|\xi|^2}\xi_i\right)}{1 - z . \bar{\xi}}.$$

If $i \neq j$, then

$$\frac{\partial(z_i \circ \phi_\xi)}{\partial z_j}(0) = -\frac{\bar{\xi}_j\xi_i}{|\xi|^2} + \xi_i\bar{\xi}_j + (1 - |\xi|^2)^{1/2}\frac{\bar{\xi}_j . \xi_i}{|\xi|^2}.$$

Now $|\xi| \to 1$ as $|\lambda| \to 1$, and therefore $\dfrac{\partial(z_i \circ \phi_\xi)}{\partial z_j}(0) \to 0$. Similarly, $\dfrac{\partial(z_i \circ \phi_\xi)}{\partial z_i}(0) \to 0$ as $|\lambda| \to 1$, which proves that $u_i \in B_0$. $\square$

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2. The annulus. We are also interested in studying Toeplitz operators defined on the Bergman spaces of nonsimply connected domains. The most basic and typical example is the annulus $A=\{z \in \mathbb{C}: r<|z|<1\}$. It is reasonable to ask when the commutant of T_p is equal to the set of analytic Toeplitz operators for the case where p is an arbitrary polynomial. The best result that we can obtain is under the assumption that all the polynomial coefficients are non-negative. In order to prove that result, we need the following well-known lemma.

LEMMA 3. *Let K_λ denote the reproducing kernel in $L_a^2(A)$ for $\lambda \in A$. If $h \in H^\infty(A)$ and $S \in L(L_a^2(A))$ commutes with T_h, then S^*K_λ is an eigenvector for T_h^*, for every $\lambda \in A$, with eigenvalue $\overline{h(\lambda)}$.*

Now, we are ready to state and prove the above mentioned theorem.

THEOREM 4. *Suppose that $p(z)=z+a_2z^2+\ldots+a_nz^n$, where $a_i \geq 0$ for $i=2,\ldots,n$. If $p(z)-p(1)$ has n distinct zeros, then*

$$\{T_p\}'=\{T_\psi : \psi \in H^\infty\}.$$

Proof. Consider the equation $p(z)-p(1)=0$; that is

$$z+a_2z^2+\ldots+a_nz^n=1+a_2+\ldots+a_n. \tag{3}$$

For $z \in \mathbb{D}$, we have

$$|z+a_2z^2+\ldots+a_nz^n|<1+a_2+\ldots+a_n$$

and consequently there is no solution of (3) in $\mathbb{D}$. If z is on the boundary of $\mathbb{D}$ and satisfies (3), then

$$\begin{aligned} 1+a_2+\ldots+a_n &= |z+a_2z^2+\ldots+a_nz^n| \\ &\leq |z|+a_2|z|^2+\ldots+a_n|z|^n \\ &= 1+a_2+\ldots+a_n. \end{aligned}$$

Therefore $z, z^2, \ldots, z^n$ are linearly dependent, so that

$$\arg(z+a_2z^2+\ldots+a_nz^n)=\arg z+2k\pi.$$

On the other hand, $z+a_2z^2+\ldots a_nz^n$ is positive so that $\arg z=2m\pi$, showing that $z=1$. Therefore $z=1$ is the only solution of (3) in $\bar{\mathbb{D}}$, and other zeros $z_1, z_2, \ldots, z_{n-1}$ are outside $\bar{\mathbb{D}}$. We now choose a positive number ε which is small enough such that $K(z_i,\varepsilon) \cap \bar{\mathbb{D}}=\varnothing$ for every i, and $K(z_i,\varepsilon) \cap K(z_j,\varepsilon)=\varnothing$, for $i \neq j$. Here $K(z_i,\varepsilon)=\{z \in \mathbb{C}: |z-z_i|<\varepsilon\}$. Define a function $F: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$F(\lambda,z)=z+a_2z^2+\ldots+a_nz^n-\lambda-a_2\lambda^2-\ldots-a_n\lambda^n.$$

Then $z \to F(1,z)$ has $n-1$ zeros outside $\bar{\mathbb{D}}$. From the assumption on the zeros of $p(z)-p(1)$, it follows that $\dfrac{\partial F}{\partial z}(1,z_i)=\{p(z)-p(1)\}'(z_i) \neq 0$. Thus by the Implicit Function Theorem, for each i, there exists an open neighborhood W_i, $1 \in W_i$, and a continuous

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map $\varphi_i : W_i \to \mathbb{C}$ such that $\varphi_i(1) = z_i$ and $F(\lambda, \varphi_i(\lambda)) = 0$ for all $\lambda \in W_i$. By continuity of φ_i, there exists an open subset $V_i \subset W_i$ such that $1 \in V_i$ and $\varphi(V_i) \subset K(z_i, \varepsilon)$. Let $V = \bigcap_{i=1}^{n-1} V_i$. Then V is an open neighborhood of 1 and $U = V \cap A$ is a nonempty open subset of A. If $\lambda \in U$ is fixed, then $\lambda \in V_i$ for each i and thus $(\lambda, \varphi_1(\lambda)), (\lambda, \varphi_2(\lambda)), \ldots, (\lambda, \varphi_{n-1}(\lambda))$ are zeros of F. Hence $\varphi_1(\lambda), \ldots, \varphi_{n-1}(\lambda)$ are roots of the equation $p(z) - p(\lambda) = 0$. Since $\varphi_i(\lambda) \in K(z_i, \varepsilon)$, we have $\varphi_i(\lambda) \neq \varphi_j(\lambda)$ for $i \neq j$. Thus for each $\lambda \in U$, the equation $p(z) - p(\lambda) = 0$ has exactly $n-1$ roots outside $\bar{\mathbb{D}}$. This implies that there is exactly one root in $\bar{A}$.

If $g \in \text{Range } T_{p-p(\lambda)}$, it is clear that $g(\lambda) = 0$. On the other hand, if $g \in L_a^2(A)$ and $g(\lambda) = 0$, then $g(z) = (z - \lambda)u(z)$, where $u \in L_a^2(A)$. Since $p(z) - p(\lambda) = (z - \lambda)q(z)$, with a polynomial q, we can write

$$g(z) = [p(z) - p(\lambda)]\frac{u(z)}{q(z)}.$$

By the previous discussion, the polynomial $q(z)$ has no roots in $\bar{\mathbb{D}}$ so that $\dfrac{u(z)}{q(z)} \in L_a^2(A)$. Thus $q(z) \in \text{Range } T_{p-p(\lambda)}$. In other words, $\text{Range } T_{p-p(\lambda)} = \{g \in L_a^2(A) : g(\lambda) = 0\}$, for all $\lambda \in U$. Let $S \in \{T_p\}'$. Because of Lemma 3,

$$S^*K_\lambda \in \text{Ker } T^*_{p-p(\lambda)} = [\text{Range } T_{p-p(\lambda)}]^\perp = \overline{\text{span}\{K_\lambda\}}.$$

Thus $S^*K_\lambda = \overline{f(\lambda)}K_\lambda$, where f is some function defined on U. For any $u \in L_a^2(A)$, we have

$$(Su)(\lambda) = \langle Su, K_\lambda\rangle = \langle u, S^*K_\lambda\rangle = f(\lambda)u(\lambda), \text{for each } \lambda \in U.$$

For such λ, we further have

$$[(ST_z - T_zS)u](\lambda) = [S(zu)](\lambda) - [zSu](\lambda) = 0.$$

Thus $(ST_z - T_zS)u \equiv 0$, so that $ST_z = T_zS$. Therefore we have $S = T_\psi$, where $\psi \in H^\infty(A)$. The theorem is proved.

Finally, we would like to point out that on the Hardy and Bergman spaces of the unit disk, Theorem 4 is true without the assumption on the zeros of $p(z) - p(1)$. Is the same true on $L_a^2(A)$?

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