A SLOW MOTION OF VISCOUS LIQUID CAUSED BY A SLOWLY MOVING SOLID SPHERE

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1. A slow steady motion of incompressible viscous liquid, bounded by an infinite rigid plane, which is generated when a rigid sphere of radius a moves steadily without rotation in a direction parallel to, and at a distance d from, the plane is considered. Use is made of bispherical coordinates, which were employed some years ago by G. B. Jeffery [**1**] and Stimson and Jeffery [**2**] in solving the axi-symmetrical problems in which the sphere is fixed and rotates about a diameter perpendicular to the plane, or when two spheres move without rotation along their line of centres in infinite liquid. The coordinate system has been used recently by Dean and O'Neill [**3**] in solving the problem in which the sphere is fixed and rotates about a diameter parallel to the plane.

Since the equations governing the motion of the liquid are linear, the solution of the problem in which the sphere has a uniform velocity (U, V, W) of translation and $(\Omega_1, \Omega_2, \Omega_3)$ of rotation, referred to a system of Cartesian coordinates in which the plane is given by $z = 0$ and the coordinates of the centre of the sphere by $(0, 0, d)$, may be obtained by combining the solutions of the problems in which only one of U, V, W, Ω_1, Ω_2, Ω_3 is non-zero. For those problems not discussed here, the solutions at any instant may be obtained by the method of [**1**] for the case when $\Omega_3 \neq 0$, by a method similar to that of [**2**] for the case when $W \neq 0$, and by the method of [**3**] for the cases when Ω_1 or Ω_2 is non-zero.

2. Let us suppose that the sphere has a velocity $(U, 0, 0)$; from the solution for this case may be deduced the solution for the case when the sphere has a velocity $(0, V, 0)$. The fluid velocity $\mathbf{V}$ must vanish at all points of the plane $z = 0$ and at any point of the sphere with cylindrical coordinates (r, θ, z), the cylindrical components (u, v, w) of $\mathbf{V}$ must satisfy the conditions

$$u = U\cos\theta, \quad v = -U\sin\theta, \quad w = 0. \tag{1}$$

3. If the liquid has a constant density ρ and coefficient of viscosity μ_1, the equations for the motion of the liquid are satisfied if the pressure p and u, v, w are given by

$$cp = \mu_1 UQ_1\cos\theta, \quad cu = \tfrac{1}{2}U[rQ_1 + c(U_2 + U_0)]\cos\theta, \tag{2}$$

$$cv = \tfrac{1}{2}Uc(U_2 - U_0)\sin\theta, \quad cw = \tfrac{1}{2}U[zQ_1 + 2cw_1]\cos\theta, \tag{3}$$

where c is a constant length which is defined later and Q_1, U_0, U_2 and w_1 are functions of r, z only, satisfying

$$L_0^2 U_0 = L_2^2 U_2 = L_1^2 Q_1 = L_1^2 w_1 = 0; \tag{4}$$

[Mathematika 11 (1964), 67–74]

the operators being defined by

$$L_m^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2} \quad (m = 0, 1, 2).$$

The equation of continuity is satisfied if

$$\left[3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right] Q_1 + c\left[\frac{\partial U_0}{\partial r} + \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) U_2 + 2\frac{\partial w_1}{\partial z}\right] = 0. \tag{5}$$

It is now convenient to introduce coordinates (ξ, η) defined by

$$r = \frac{c \sin\eta}{\cosh\xi - \cos\eta}, \quad z = \frac{c \sinh\xi}{\cosh\xi - \cos\eta} \quad (0 \leqslant \eta \leqslant \pi).$$

This definition determines the constant c of (2) and (3). The plane is defined by $\xi = 0$, and the sphere is defined by $\xi = \alpha > 0$, if $a = c \operatorname{cosech} \alpha$ and $d = c \coth \alpha$. The equations (4) for U_0, U_2, Q_1 and w_1 are satisfied [4] if

$$w_1 = (\cosh\xi - \mu)^{\frac{1}{2}} \sin\eta \sum_{n=1}^{\infty} [A_n \sinh(n+\tfrac{1}{2})\xi] P_n'(\mu), \tag{6}$$

$$Q_1 = (\cosh\xi - \mu)^{\frac{1}{2}} \sin\eta \sum_{n=1}^{\infty} [B_n \cosh(n+\tfrac{1}{2})\xi + C_n \sinh(n+\tfrac{1}{2})\xi] P_n'(\mu), \tag{7}$$

$$U_0 = (\cosh\xi - \mu)^{\frac{1}{2}} \sum_{n=0}^{\infty} [D_n \cosh(n+\tfrac{1}{2})\xi + E_n \sinh(n+\tfrac{1}{2})\xi] P_n(\mu), \tag{8}$$

$$U_2 = (\cosh\xi - \mu)^{\frac{1}{2}} \sin^2\eta \sum_{n=2}^{\infty} [F_n \cosh(n+\tfrac{1}{2})\xi + G_n \sinh(n+\tfrac{1}{2})\xi] P_n''(\mu), \tag{9}$$

where μ denotes $\cos\eta$, $P_n(\mu)$ the Legendre polynomial of order n, and the accents differentiations with respect to μ.

4. The boundary conditions on the plane may now be expressed as

$$U_0 = U_2 = -\frac{Q_1 \sin\eta}{2(1-\cos\eta)} \quad (\xi = 0), \tag{10}$$

and

$$w_1 = 0 \quad (\xi = 0), \tag{11}$$

and since $u = v = 0$ when $\xi = 0$, it is clear from the equation of continuity that $\partial w/\partial z = 0$ when $\xi = 0$. Hence

$$Q_1 = -2c\left.\frac{\partial w_1}{\partial z}\right|_{(\xi=0)} = -2c \lim_{\xi \to 0} \frac{w_1}{z}. \tag{12}$$

Equation (11) is clearly satisfied by w_1 given by (6), and equations (10) and (12) will be satisfied [3] if

$$B_n = (n-1)A_{n-1} - (2n+1)A_n + (n+2)A_{n+1} \quad (n \geqslant 1), \tag{13}$$

$$D_n = -\tfrac{1}{2}(n-1)n A_{n-1} + \tfrac{1}{2}(n+1)(n+2)A_{n+1} \quad (n \geqslant 0), \tag{14}$$

$$E_n = \tfrac{1}{2}(A_{n-1} - A_{n+1}) \quad (n \geqslant 2). \tag{15}$$

5. The boundary conditions (1) on the sphere may be expressed as

$$U_0 - 2U = U_2 = -\frac{Q_1 \sin\eta}{2(\cosh\alpha - \cos\eta)} \quad (\xi = \alpha), \tag{16}$$

and

$$w_1 = -\frac{Q_1 \sinh\alpha}{2(\cosh\alpha - \cos\eta)} \quad (\xi = \alpha). \tag{17}$$

Equations (16) and (17) can be shown to be satisfied by w_1, Q_1, U_0, U_2 given by (6), (7), (8) and (9), if

$$C_n = -2k_n\left[\frac{(n-1)A_{n-1}}{2n-1} - A_n + \frac{(n+2)A_{n+1}}{2n+3}\right] \quad (n \geqslant 1), \tag{18}$$

$$E_n = \frac{2(\sqrt{2})\,e^{-(n+\frac{1}{2})\alpha}}{\sinh(n+\frac{1}{2})\alpha} + k_n\left[\frac{(n-1)nA_{n-1}}{2n-1} - \frac{(n+1)(n+2)A_{n+1}}{2n+3}\right] \quad (n \geqslant 0), \tag{19}$$

$$G_n = -k_n\left[\frac{A_{n-1}}{2n-1} - \frac{A_{n+1}}{2n+3}\right] \quad (n \geqslant 2), \tag{20}$$

where

$$k_n = (n+\tfrac{1}{2})\coth(n+\tfrac{1}{2})\alpha - \coth\alpha \quad (n \geqslant 0).$$

5. It can be shown that each of the functions

$$\left[3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right]Q_1, \quad c\frac{\partial U_0}{\partial r}, \quad c\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)U_2, \quad 2c\frac{\partial w_1}{\partial z}$$

is a solution of the equation $L_1{}^2\phi = 0$. Hence, (5) will be satisfied at all points of the fluid if the sums of the coefficients of

$$(\cosh\xi - \mu)^{\frac{1}{2}} \sin\eta \cosh(n+\tfrac{1}{2})\xi\, P_n{}'(\mu) \quad (n \geqslant 1)$$

and

$$(\cosh\xi - \mu)^{\frac{1}{2}} \sin\eta \sinh(n+\tfrac{1}{2})\xi\, P_n{}'(\mu) \quad (n \geqslant 1)$$

in each of the functions is zero. It therefore follows that (5) is satisfied if

$$\begin{aligned}5B_n &- (n-1)B_{n-1} + (n+2)B_{n+1} - D_{n-1} + 2D_n - D_{n+1}\\ &+ (n-2)(n-1)F_{n-1} - 2(n-1)(n+2)F_n + (n+2)(n+3)F_{n+1}\\ &- 2(n-1)A_{n-1} + 2(2n+1)A_n - 2(n+2)A_{n+1} = 0 \quad (n \geqslant 1),\end{aligned} \tag{21}$$

and

$$\begin{aligned}5C_n &- (n-1)C_{n-1} + (n+2)C_{n+1} - E_{n-1} + 2E_n - E_{n+1}\\ &+ (n-2)(n-1)G_{n-1} - 2(n-1)(n+2)G_n + (n+2)(n+3)G_{n+1} = 0 \quad (n \geqslant 1).\end{aligned} \tag{22}$$

The equation of continuity when $\xi=0$ has been used to deduce (13), (14) and (15). Consequently, if B_n, D_n and F_n are substituted by these equations, the set of equations (21) is identically satisfied. When (18), (19) and (20) are used to express C_n, E_n and G_n in terms of A_n, (22) gives

$$[(2n-1)k_{n-1}-(2n-3)k_n]\left[\frac{(n-1)A_{n-1}}{2n-1}-\frac{nA_n}{2n+1}\right]$$
$$-[(2n+5)k_n-(2n+3)k_{n+1}]\left[\frac{(n+1)A_n}{2n+1}-\frac{(n+2)A_{n+1}}{2n+3}\right]$$
$$=(\sqrt{2})[2\coth(n+\tfrac{1}{2})\alpha-\coth(n-\tfrac{1}{2})\alpha-\coth(n+\tfrac{3}{2})\alpha] \quad (n\geqslant 1). \quad (23)$$

The structure of (23) does not permit the direct determination of the coefficients A_n. However, A_n necessarily converges to zero for every non-zero value of α, hence the significant solution of (23) may be derived numerically by a successive approximation technique. Satisfactory solutions have been obtained using an electronic digital computer over a range of values of α between 3·0 and 0·02.

6. The components F_x, F_y, F_z of the total force exerted by the fluid on the sphere are given by

$$F_x=\pi\mu_1 Uc\int_{-1}^{+1}\left[\frac{Q_1}{2c}\frac{\partial r}{\partial\xi}-\frac{r}{2c}\frac{\partial Q_1}{\partial\xi}-\frac{\partial U_0}{\partial\xi}\right]d\mu,$$

and

$$F_y=F_z=0.$$

Defining a non-dimensional force coefficient F^* by

$$F^*=-\frac{F_x}{6\pi\mu_1 Ua},$$

it follows that

$$12F^*\operatorname{cosech}\alpha=\int_{-1}^{+1}\frac{\partial Q_1}{\partial\xi}\frac{\sin\eta\,d\mu}{(\cosh\alpha-\mu)^2}+2\int_{-1}^{+1}\frac{\partial U_0}{\partial\xi}\frac{d\mu}{(\cosh\alpha-\mu)}$$
$$+\sinh\alpha\int_{-1}^{+1}\frac{Q_1\sin\eta\,d\mu}{(\cosh\alpha-\mu)^3}.$$

By use of the relations

$$(2n+1)\int_{-1}^{+1}\frac{\sin^2\eta\,P_n'\,d\mu}{(\cosh\alpha-\mu)^{3/2}}=3\sinh\alpha\int_{-1}^{+1}\frac{\sin^2\eta\,P_n'\,d\mu}{(\cosh\alpha-\mu)^{5/2}}$$
$$=4(\sqrt{2})\,n(n+1)\,e^{-(n+\frac{1}{2})\alpha}, \quad (24)$$

$$(2n+1)\int_{-1}^{+1}\frac{P_n\,d\mu}{(\cosh\alpha-\mu)^{\frac{1}{2}}}=\sinh\alpha\int_{-1}^{+1}\frac{P_n\,d\mu}{(\cosh\alpha-\mu)^{3/2}}$$
$$=2(\sqrt{2})\,e^{-(n+\frac{1}{2})\alpha}, \quad (25)$$

it can be shown that the formula for F^* is

$$F^* = \tfrac{1}{6}(\sqrt{2}) \sinh \alpha \sum_{n=0}^{\infty} [(D_n + E_n) + n(n+1)(B_n + C_n)],$$

which, by (13) and (14), reduces to

$$F^* = \tfrac{1}{6}(\sqrt{2}) \sinh \alpha \sum_{n=0}^{\infty} [E_n + n(n+1)\, C_n]. \tag{26}$$

7. The components G_x, G_y, G_z of the couple exerted on the sphere when moments of the forces acting on its surface are taken about the centre are given by

$$G_x = G_z = 0,$$

and

$$G_y = -\pi\mu_1 U \operatorname{cosech} \alpha \int_{-1}^{+1} \left[\frac{\partial r}{\partial \xi}\frac{\partial}{\partial \xi}(\tfrac{1}{2}zQ_1 + cw_1) - \frac{\partial z}{\partial \xi}\frac{\partial}{\partial \xi}(\tfrac{1}{2}rQ_1 + cU_0)\right] d\mu.$$

Defining a non-dimensional couple coefficient G^* by

$$G^* = \frac{G_y}{8\pi\mu_1 U a^2},$$

it follows that

$$-8G^* \operatorname{cosech} \alpha = \int_{-1}^{+1} \frac{\partial U_0}{\partial \xi} \frac{(\mu \cosh \alpha - 1)}{(\cosh \alpha - \mu)^2}\, d\mu$$

$$- \int_{-1}^{+1} \left[\tfrac{1}{2}\frac{\partial Q_1}{\partial \xi} \cosh \alpha + \frac{\partial w_1}{\partial \xi} \sinh \alpha\right] \frac{\sin \eta \, d\mu}{(\cosh \alpha - \mu)^2}.$$

Therefore, by the use of the relations (24) and (25), together with

$$3 \sinh^2 \alpha \int_{-1}^{+1} \frac{P_n \, d\mu}{(\cosh \alpha - \mu)^{5/2}} = 2n + 1 + 2 \coth \alpha,$$

it may be shown that the formula for G^* is

$$12(\sqrt{2})\, G^* \operatorname{cosech}^2 \alpha$$

$$= \sum_{n=0}^{\infty} [2 + e^{-(2n+1)\alpha}][n(n+1)(2A_n + C_n \coth \alpha) - (2n+1-\coth \alpha)\, E_n]$$

$$+ \sum_{n=0}^{\infty} [2 - e^{-(2n+1)\alpha}][n(n+1)\, B_n \coth \alpha - (2n+1-\coth \alpha)\, D_n]. \tag{27}$$

8. The components F_x', F_y', F_z' of the total force exerted by the fluid on the fixed plane $\xi = 0$ are given by

$$F_x' = \mu_1 \int_0^{2\pi} d\theta \int_0^{\infty} \left(\frac{\partial u}{\partial z}\cos\theta - \frac{\partial v}{\partial z}\sin\theta\right) r\, dr, \quad F_y' = F_z' = 0,$$

https://doi.org/10.1112/S0025579300003508 Published online by Cambridge University Press

which gives

$$\frac{F_x'}{\pi\mu_1 c} = \frac{1}{2}\int_{-1}^{+1} \frac{\partial Q_1}{\partial \xi} \frac{\sin\eta\, d\mu}{(1-\mu)^2} + \int_{-1}^{+1} \frac{\partial U_0}{\partial \xi} \frac{d\mu}{1-\mu},$$

which, by the use of the relations

$$\int_{-1}^{+1} \frac{P_n\, d\mu}{(1-\mu)^{\frac{1}{2}}} = \frac{2\sqrt{2}}{2n+1} \quad (n \geqslant 0), \tag{28}$$

$$\int_{-1}^{+1} \frac{(P_{n-1}-P_{n+1})\, d\mu}{(1-\mu)^{3/2}} = 4\sqrt{2} \quad (n \geqslant 1), \tag{29}$$

can be shown to give

$$F_x' = (\sqrt{2})\,\pi\mu_1 Ua \sinh\alpha \sum_{n=0}^{\infty} [E_n + n(n+1)\, C_n]. \tag{30}$$

The components G_x', G_y', G_z' of the moment about the centre of the sphere of the forces acting on the plane are given by

$$G_x' = G_z' = 0,$$

and

$$G_y' = \int_0^{2\pi} d\theta \int_0^{\infty} \left(-p + 2\mu_1 \frac{\partial w}{\partial z}\right) r^3 \cos\theta\, dr - F_x'\, d,$$

which gives

$$G_y' = 2\pi\mu_1 Uc^2 \int_{-1}^{+1} \frac{\partial w_1}{\partial \xi} \frac{\sin\eta\, d\mu}{(1-\mu)^2} - dF_x',$$

which on using (28) and (29) can be shown to give

$$G_y' = (\sqrt{2})\,\pi\mu_1 Ua^2 \sinh^2\alpha \sum_{n=0}^{\infty} \Big(4n(n+1)\, A_n - [E_n + n(n+1)\, C_n] \coth\alpha\Big). \tag{31}$$

Since the fluid is not acted on by body forces, and its motion is slow and steady, the forces that are exerted on a volume of fluid bounded by the sphere, a large hemisphere of radius R_1, $(R_1 > a+d)$, drawn in the fluid, and the part of the plane $z = 0$, $r \leqslant R_1$, by the sphere, the plane boundary, and the stresses over the hemisphere constitute a system of forces in statical equilibrium. As $R_1 \to \infty$, the force and couple about the centre of the sphere, exerted by the plane boundary, tend to $-F_x'\mathbf{i}$ and $-G_y'\mathbf{j}$ respectively, whilst the force and couple about the centre of the sphere, exerted by the sphere, are $-F_x\mathbf{i}$ and $-G_y\mathbf{j}$ respectively. But (26), (27), (30) and (31) show that

$$F_x = F_x', \quad G_y \neq G_y'.$$

Consequently, the order of magnitude of the stresses at a large distance R from the centre of the sphere is like $1/R^3$ which shows that the effect of the plane in this problem is to diminish the order of magnitude of the stresses (and velocity) at large distances due to the motion of the fluid caused by the sphere. This effect contrasts with that of the previously discussed problem [**3**], in which the corresponding orders of magnitude were found to be unaltered by the presence of the plane.

9. The following table shows the calculated values of the force and couple coefficients F^* and G^* for some of the values of α in the range considered. Subsequent to the publication of [**3**], further values of the force and couple coefficients F and G were evaluated and some of these values are also given here for completeness.

α	d/a	F^*	G^*	F	G
3·0	10·0677	1·0591	0·00001	0·04479	1·0005
2·0	3·7622	1·1738	0·00042	0·12357	1·0090
1·0	1·5431	1·5675	0·01465	0·33190	1·1473
0·5	1·1276	2·1515	0·07372	0·52392	1·5585
0·3	1·0453	2·6475	0·14552	0·63223	1·9960
0·1	1·0050	3·7863	0·34187	0·81018	3·1121
0·08	1·0032	4·0223	0·38496	0·84203	3·3516
0·06	1·0018	4·3275	0·44116	0·88220	3·6629
0·04	1·0008	4·7587	0·52120	0·93770	4·1048
0·03	1·0005	5·0651	0·57834	0·97662	4·4197
0·02	1·0002	5·4973	0·65912	—	—

The numerical values above show consistency with F^*, G tending to unity and F, G^* tending to zero as d tends to infinity with a remaining constant. These limits give the forces and couples appropriate to the motions generated by the sphere in isolation in unbounded fluid. If $d = a$, the sphere touches one point of the plane. The shearing force on an element of the surface of the sphere near the point of contact of area $2\pi a^2 \sin\theta_1\, d\theta_1$, where θ_1 is the angle between the positive z-axis and a radius of the sphere through the element, is approximately

$$\frac{2\pi\mu_1 Ua \sin\theta_1\, d\theta_1}{1+\cos\theta_1}$$

which shows that F^* and G^* should tend to infinity as α tends to zero. This has also been shown to be the case with F and G. The numerical values above show consistency with these conclusions; it is, however, remarkable that for such small values of α as have been considered, the variations in the coefficients from their limiting values as α tends to infinity are so small.

The author acknowledges with gratitude helpful advice given to him by Professor W. R. Dean in the preparation of this paper, and also the assistance of Miss S. M. Burrough and Mr. K. S. Davis in the computational work.

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(*Received on the 5th of March,* 1964.)