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POLYNOMIAL NEAR-RINGS IN k INDETERMINATES

ENOCH K.S. LEE AND NICO J. GROENEWALD

Polynomial near-rings in k-commuting indeterminates are our object of study. We illustrate our work for $k = 2$, that is, $N[x,y]$ as an extension to $N[x]$, while the case for arbitrarily k follows easily. Our approach is different from the recursive definition $N[x][y]$. However, it can be shown that $N[x,y]$ is isomorphic to $N[x][y]$. Several important tools such as the degree, the least degree, et cetera are defined with respect to $N[x,y]$. We also clarify some notations involved in defining polynomial near-rings.

1. INTRODUCTION

Polynomial near-rings in one indeterminate have been defined by Bagley [**1, 2**] and further developed by Farag [**3, 4, 5**] and Lee [**6**]. In view of its development, it is not surprising to see that the object of polynomial near-rings shares many technical similarities as well as difficulties with the object of matrix near-rings, since the notion of polynomial near-rings evolves from that of matrix near-rings developed by Meldrum and van der Walt [**7**]. It is well-known that in the ring case, polynomial rings of multiple commuting indeterminates can be defined recursively from the single indeterminate notion (that is, $R[x,y] = R[x][y]$). Moreover the matrix rings analog of this result can be described as the following, namely the $n \times n$ matrix ring over the $m \times m$ matrix ring over R is isomorphic to the $mn \times mn$ matrix ring over R, that is, $M_{mn}(R) \cong M_n\big(M_m(R)\big)$. In 1997, Meyer [**8**] verifies this counterpart for matrix near-rings. The situation is not exactly the same for polynomial near-rings however. There are two issues in studying multiple commuting indeterminates polynomial near-rings. Firstly the meaning of the object $N[x,y]$ is not clear, at least it is not an immediate consequence from the definition of polynomial near-rings. Secondly we need to illustrate that this notion is equivalent to $\big(N[x]\big)[y]$ and hence $\big(N[y]\big)[x]$.

We investigate the meaning of polynomial near-rings of two commuting indeterminates and their representations, while the approach to arbitrarily finite number of commuting indeterminates follows similarly. There are many important tools aiding the investigation of polynomial near-rings (at least for the case of one commuting indeterminate) as defined in [**1, 2**] such as the degree and the least degree of a polynomial, the

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ideal $\mathbb{S}[x]$ where $\mathbb{S}$ is a sequence of non-decreasing left ideals of N, the ideal I^* where I is an ideal of N, and the symbolically zero polynomials and the associated ideal "sym_0". As a final remark, we explore the extension of these concepts from one indeterminate to multiple ones and re-define these important tools.

2. Preliminaries

In the following, we use R to denote a generic ring with identity while N a right zero-symmetric near-ring with identity. The reader should refer to [**9, 10**] for more about near-rings. Let us begin with a polynomial near-ring in one indeterminate over N. Let K be the set of non-negative integers. We denote N^K the direct product of $|K|$ copies of the group $(N,+)$. For simplicity, let

$$M(N^K) = \{f : N^K \to N^K \mid f(\overline{0}) = \overline{0}\}$$

(since we are considering zero-symmetric near-rings only). Furthermore we let L_a be a mapping from N^K to N^K for any $a \in N$ such that $L_a(c_0, c_1, \ldots) = (ac_0, ac_1, \ldots)$. We also let $x : N^K \to N^K$ such that $x(c_0, c_1, \cdots) = (0, c_0, c_1, \ldots)$, for all $(c_0, c_1, \ldots) \in N^K$. Obviously L_a and x are in $M(N^K)$. Then we can define $N[x]$ to be the subnear-ring of $M(N^K)$ generated by $\{L_a\}_{a \in N} \cup \{x^0, x^1, \ldots\}$. Note in [**1**], $N[x]$ is defined to be the subnear-ring of

$$M_N(N^K) = \{f \in M(N^K) \mid f\rho_a = \rho_a f, \forall a \in N\}$$

where $\rho_a(c_0, c_1, \ldots) = (c_0 a, c_1 a, \ldots)$ for all $(c_0, c_1, \ldots) \in N^K$. It is not difficult to verify this definition is in fact equivalent to our simplified version. In other words, the centraliser condition that $f\rho_a = \rho_a f$ is redundant.

Since a near-ring polynomial is defined as a function on N^K, the extended structure "$\big(N[x]\big)[y]$" becomes quite complicated. Furthermore, it is not immediately clear that $\big(N[x]\big)[y]$ could be identified as $\big(N[y]\big)[x]$, and hence "$N[x,y]$" (if this even makes sense) as if in their ring counterpart.

Notice that $\big(N[x]\big)[y]$ is the subnear-ring of $M\Big(\big(N[x]\big)^K\Big)$ generated by $\{L_f\}_{f \in N[x]}$ and $\{y^0, y^1, \ldots\}$. For any $(h_0, h_1, \ldots) \in \big(N[x]\big)^K$, we have

$$L_f(h_0, h_1, \ldots) = (fh_0, fh_1, \ldots), \text{ and } y^i(h_0, h_1, \ldots) = (\underbrace{0, \ldots, 0}_{i}, h_0, h_1, \ldots).$$

In the ring case, usually one can identify $L_a \in N[x]$ as $a \in N$. However one runs into trouble when discussing $\big(N[x]\big)[y]$. Similarly, the function x behaves quite differently under different contexts, namely in $N[x]$ or in $\big(N[x]\big)[y]$. In order to make our discussion precisely, we distinguish elements of N, $N[x]$, and $\big(N[x]\big)[y]$, and denote the elements differently until stated otherwise.

The following table indicates our terminology and should be self-explanatory.

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N	$N[x]$	$(N[x])[y]$
a	L_a	$\lambda_a(= L_{L_a})$
	f (for example x)	L_f (for example L_x)
		F (for example y)

Let us review some basic properties of some near-ring polynomials.

(1) In $N[x]$, $L_a(c_0, c_1, \dots) = (ac_0, ac_1, \dots)$, where $a \in N$ and $(c_0, c_1, \dots) \in N^K$.

(2) In $N[x]$, $x(c_0, c_1, \dots) = (0, c_0, c_1, \dots)$, where $a \in N$ and $(c_0, c_1, \dots) \in N^K$.

(3) In $(N[x])[y]$, $L_f(f_0, f_1, \dots) = (ff_0, ff_1, \dots)$, where $f \in N[x]$ and $(f_0, f_1, \dots) \in (N[x])^K$. In particular, we have $\lambda_a(f_0, f_1, \dots) = (L_a f_0, L_a f_1, \dots), a \in N$ and $L_x(f_0, f_1, \dots) = (xf_0, xf_1, \dots)$.

(4) In $(N[x])[y]$, $y(f_0, f_1, \dots) = (0, f_0, f_1, \dots)$, where $(f_0, f_1, \dots) \in (N[x])^K$.

It is well-known that $N[x] \cong N_{\text{finite}}[x]$ where $N_{\text{finite}}[x]$ is the subnear-ring generated by $\{L_a\}_{a\in N} \cup \{x^0, x^1, \dots\}$ of $M(N^K_{\text{finite}})$ and N^K_{finite} is the direct sum of $|K|$ copies of $(N, +)$, that is, $= \{\overline{c} \in N^K \mid \overline{c}$ has finitely many non-zero entries$\}$. We shall use this equivalence in our discussion. To further facilitate our discussion, we introduce the following notations. For convenience, we denote an element of $N^{K^2}_{\text{finite}}$ by an infinite square matrix such as $[c_{ij}]_{ij}$ or just $[c_{ij}]$ for $c_{ij} \in N$ and $i, j \in K$, where only finitely many c_{ij}'s are non-zero. Let

$$B_N = \left\{\beta_a : N^{K^2}_{\text{finite}} \to N^{K^2}_{\text{finite}} \mid a \in N \text{ such that } \beta_a([c_{ij}]) = [ac_{ij}]\right\}.$$

Let $\overline{X} = \{\overline{x}^i \mid i \in K\}$ where $\overline{x} : N^{K^2}_{\text{finite}} \to N^{K^2}_{\text{finite}}$ such that $\overline{x}([c_{ij}]) = [d_{ij}]$ and $d_{i0} = 0$, $d_{i1} = c_{i0}$, $d_{i2} = c_{i1}, \dots$. Similarly let $\overline{Y} = \{\overline{y}^i \mid i \in K\}$ where $\overline{y} : N^{K^2}_{\text{finite}} \to N^{K^2}_{\text{finite}}$ such that $\overline{y}([c_{ij}]) = [d_{ij}]$ and $d_{0j} = 0$, $d_{1j} = c_{0j}$, $d_{2j} = c_{1j}, \dots$. Clearly $B_N \cup \overline{X} \cup \overline{Y} \subseteq M(N^{K^2}_{\text{finite}})$. Now let $N_{\text{finite}}[\overline{x}, \overline{y}]$ be a subnear-ring of $M(N^{K^2}_{\text{finite}})$ generated by $B_N \cup \overline{X} \cup \overline{Y}$. It should be noted that the notion of $\overline{x}$ and $\overline{y}$ are symmetric to each other, therefore, the use of the notation $N_{\text{finite}}[\overline{x}, \overline{y}]$ makes sense. Furthermore, denote by $\overline{\varepsilon}_0 = (1, 0, 0, \dots) \in N^K_{\text{finite}}$. Note that the cardinality of K and K^2 are of course equal, one can actually use K to define the object $N_{\text{finite}}[\overline{x}, \overline{y}]$ instead of K^2 with a different set of generating functions in our quest, yet yielding the same results. However the choice of K^2 here makes our discussion more transparent and more coherent to that of $N[x]$. The object $N[\overline{x}, \overline{y}]$ is defined similarly by using N^{K^2} instead of $N^{K^2}_{\text{finite}}$.

We remark the following observation about $N[x]$. Note that $N[x]$ is a (left) N-module. A tedious however routine argument shows that $N[x]$ is indeed a left N-submodule of $M(N^K)$ generated by $\{L_a\}_{a\in N} \cup \{x^0, x^1, \dots\}$. Using this fact, one can obtain all polynomials by the following construction method. (A more detailed account on this construction method can be found in [**6**].) Let C be a subset of $N[x]$ such that

(1) $\{L_a\}_{a\in N} \cup \{x^0, x^1, \dots\} \subseteq C$;

(2) $af \in C$ if $a \in N$ and $f \in C$;

(3) $f + g \in C$ if f and $g \in C$.

The collection, C, of polynomials yielded by the above method is actually $N[x]$ itself. This simple yet useful observation provides some sort of "standardised" polynomial representation, although not quite yet a unique representation as for ring polynomials. We shall use this observation in the sequel.

3. Multiple indeterminates

We want to show $(N[x])[y]$ and $N_{\text{finite}}[\overline{x}, \overline{y}]$ are isomorphic to each other. Define a mapping $\phi : (N[x])[y] \to M(N^{K^2})$ such that $\phi(F)([c_{ij}]) = [d_{ij}] \in N^{K^2}$ where $F \in (N[x])[y]$, $[c_{ij}] \in N^{K^2}_{\text{finite}}$ with $d_{ij} = \pi_j f_i(\overline{\varepsilon}_0)$ and $f_i = \pi_i F\left(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\right)$ where $l \in K$. Since $[c_{ij}] \in N^{K^2}_{\text{finite}}$, we have $\sum_l L_{c_{il}} x^l$ actually a finite sum, hence belongs to $N[x]$. Furthermore the sequence $\left\{\sum_l L_{c_{il}} x^l\right\}_i$ contains only finitely many non-zero terms. Therefore ϕ is indeed a well-defined mapping from $\phi : (N[x])[y]$ to $M(N^{K^2}_{\text{finite}})$.

Before we proceed, let us examine the images of some polynomials under ϕ. Suppose $[c_{ij}] \in N^{K^2}_{\text{finite}}$.

(1) Let $F = \lambda_a \in (N[x])[y], a \in N$. By definition, we have:

$$f_i = \pi_i \lambda_a\left(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\right) = L_a \sum_l L_{c_{il}} x^l = \sum_l L_{ac_{il}} x^l.$$

This implies that $\pi_j f_i(\overline{\varepsilon}_0) = ac_{ij}$. Thus $\phi(\lambda_a)([c_{ij}]) = [ac_{ij}]$.

(2) Let $F = L_x$. Then we have:

$$f_i = \pi_i L_x\left(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\right) = \pi_i\left(x \sum_l L_{c_{0l}} x^l, x \sum_l L_{c_{1l}} x^l, \ldots\right) = \sum_l L_{c_{il}} x^{l+1}.$$

This implies

$$\phi(L_x)([c_{ij}]) = \pi_j f_i(\overline{\varepsilon}_0) = \pi_j\left(\sum_l L_{c_{il}} x^{l+1}\right)(\overline{\varepsilon}_0) = \pi_j(0, c_{i0}, \ldots) = \begin{cases} 0, & \text{if } j < 1, \\ c_{i,j-1}, & \text{otherwise.} \end{cases}$$

(3) Let $F = y$. Thus we have:

$$f_i = \pi_i y\left(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\right) = \pi_i\left(0, \sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\right).$$

This implies

$$\phi(y)([c_{ij}]) = \pi_j f_i(\overline{\varepsilon}_0) = \begin{cases} 0 & \text{if } i < 1, \\ c_{i-1,j} & \text{otherwise.} \end{cases}$$

From these examples, we see at once that $\phi(\lambda_a) = \beta_a, \phi(L_x) = \overline{x}, \phi(y) = \overline{y}$. In a moment, we show ϕ is a bijective mapping and also it preserves addition and multiplication. As a consequence ϕ maps $(N[x])[y]$ onto $N_{\text{finite}}[\overline{x}, \overline{y}] \subseteq M(N^{K^2}_{\text{finite}})$ and is actually an isomorphism.

The following technical result is essential to our quest.

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Lemma 1. *Let $(g_0, g_1, \ldots) \in (N[x])^K$ and $[c_{ij}] \in N_{\text{finite}}^{K^2}$ such that there exists $\overline{d} \in N_{\text{finite}}^K$, for which $g_i(\overline{d}) = (c_{i0}, c_{i1}, \ldots)$ for any $i \in K$. Then for any $F \in (N[x])[y]$, we have*

$$\pi_j\big(\pi_i F(g_0, g_1, \ldots)\big)(\overline{d}) = \pi_j\Big(\pi_i\Big(F\Big(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\Big)\Big)\Big)(\overline{\varepsilon}_0), \forall i, j \in K.$$

Proof: We use the construction technique remarked at the end of the last section to prove this lemma. Let $F = y^m, m \in K$. We have

$$\pi_j\Big(\pi_i y^m\Big(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\Big)\Big)(\overline{\varepsilon}_0) = \begin{cases} 0 & \text{if } i < m, \\ c_{i-m,j} & \text{otherwise.} \end{cases}$$

However $\pi_j\big(\pi_i y^m(g_0, g_1, \ldots)\big)(\overline{d}) = \pi_j\big(g_{i-m}(\overline{d})\big) = \begin{cases} 0 & \text{if } i < m, \\ c_{i-m,j} & \text{otherwise.} \end{cases}$ This shows the claim for $F = y^m$. Now suppose $F = L_f, f \in N[x]$. We have

$$\pi_j\big(\pi_i L_f(g_0, g_1, \ldots)\big)(\overline{d}) = \pi_j\big(\pi_i(f g_0, f g_1, \ldots)\big)(\overline{d}) = \pi_j f g_i(\overline{d}) = \pi_j f(c_{i0}, c_{i1}, \ldots).$$

However,

$$\pi_j\Big(\pi_i L_f\Big(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\Big)\Big)(\overline{\varepsilon}_0) = \pi_j\Big(f\Big(\sum_l L_{c_{il}} x^l\Big)\Big)(\overline{\varepsilon}_0) = \pi_j f(c_{i0}, c_{i1}, \ldots).$$

Thus the claim is also true for $F = L_f, f \in N[x]$. It is quite easy to see that $F + G$ also satisfies the claim if both F and $G \in (N[x])[y]$ satisfy the claim.

It remains to show that if $F \in (N[x])[y]$ satisfies the claim and $f \in N[x]$, then $L_f F$ satisfies the claim. In fact, we have

$$\pi_j\Big(\pi_i\big(L_f F(g_0, g_1, \ldots)\big)\Big)(\overline{d}) = \pi_j\Big(f\big(\pi_i F(g_0, g_1, \ldots)\big)\Big)(\overline{d}) = \pi_j\big(f(h_0, h_1, \ldots)\big),$$

where $h_k = \pi_k\big(\pi_i F(g_0, g_1, \ldots)\big)(\overline{d}) \in N$. However, by the assumption that F satisfies the claim, we have $h_k = \pi_k\Big(\pi_i F\Big(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\Big)\Big)(\overline{\varepsilon}_0)$. Therefore this implies that $\pi_j\Big(\pi_i\big(L_f F(g_0, g_1, \ldots)\big)\Big)(\overline{d})$ and $\pi_j f\Big(\pi_i F\Big(\sum_l L_{c_{0l}} x^l, \sum_l L_{c_{1l}} x^l, \ldots\Big)\Big)(\overline{\varepsilon}_0)$ are equal to each other. Hence $L_f F$ satisfies the claim and the lemma follows. $\square$

We are ready to prove our main result.

Theorem 2. $(N[x])[y] \cong N_{\text{finite}}[\overline{x}, \overline{y}]$ *(and hence* $\cong \big(N_{\text{finite}}[x]\big)_{\text{finite}}[y]$*).*

Proof: We show the function ϕ defined earlier is the desired isomorphism. Firstly, we show ϕ preserves addition and multiplication.

It is quite obvious that it is true for the "addition" case, that is, $\phi(F + G) = \phi(F) + \phi(G)$. We proceed to verify the "multiplication" case. Suppose $\phi(FG)\big([c_{ij}]\big) = [d_{ij}]$.

By definition, we have

$$\begin{aligned} d_{ij} &= \pi_j\Bigl(\pi_i(FG)\Bigl(\sum\nolimits_l L_{c_{0l}}x^l, \sum\nolimits_l L_{c_{1l}}x^l, \dots\Bigr)\Bigr)(\bar{\varepsilon}_0) \\ &= \pi_j\bigl(\pi_i F(g_0, g_1, \dots)\bigr)(\bar{\varepsilon}_0), \text{where } g_k = \pi_k G\Bigl(\sum\nolimits_l L_{c_{0l}}x^l, \sum\nolimits_l L_{c_{1l}}x^l, \dots\Bigr). \end{aligned}$$

If $g_k(\bar{\varepsilon}_0) = (b_{k0}, b_{k1}, \dots)$, then by the previous lemma we have

$$d_{ij} = \pi_j\Bigl(\pi_i F\Bigl(\sum\nolimits_l L_{b_{0l}}x^l, \sum\nolimits_l L_{b_{1l}}x^l, \dots\Bigr)\Bigr)(\bar{\varepsilon}_0).$$

Therefore $\phi(FG) = \phi(F)\bigl([b_{ij}]\bigr)$. However $\phi(G)\bigl([c_{ij}]\bigr) = \bigl[\pi_j g_i(\bar{\varepsilon}_0)\bigr] = [b_{ij}]$. Hence $\phi(FG) = \phi(F)\phi(G)$.

From the remark that precedes the previous lemma and the fact that ϕ preserves addition and multiplication, we see that the mapping ϕ from $\bigl(N[x]\bigr)[y]$ to $N[\bar{x}, \bar{y}]$ is an onto homomorphism. Hence it remains to show ϕ is injective. However to show it is injective, one just has to notice the following equivalence (in sequence) by applying the previous lemma:

(1) $\phi(F) = 0$.

(2) $\pi_j\Bigl(\pi_i F\Bigl(\sum_l L_{c_{0l}}x^l, \sum_l L_{c_{1l}}x^l, \dots\Bigr)\Bigr)(\bar{\varepsilon}_0) = 0, \forall\ [c_{hk}] \in N^{K^2}$ and $i, j \in K$.

(3) $\pi_j\bigl(\pi_i F(g_0, g_1, \dots)\bigr)(\bar{d}) = 0, \forall\ (g_0, g_1, \dots) \in \bigl(N[x]\bigr)^K$ and $\bar{d} \in N^K$.

(4) $F = 0$.

As a consequence, ϕ is an isomorphism. ∎

As seen so far, we have shown that $\bigl(N[x]\bigr)[y] \cong N_{\text{finite}}[\bar{x}, \bar{y}]$. Therefore we can identify x and y with $\bar{x}$ and $\bar{y}$, respectively. Hence we can use the notation $N_{\text{finite}}[x, y]$ without creating any confusion. Moreover it can be shown that we can drop the subscript "finite". We begin our verification by introducing the following useful notation and result.

Suppose $[c_{ij}]$ and $[d_{ij}] \in N^{K^2}$. For any pair $m, n \in K$, we define a relation on N^{K^2} such that $[c_{ij}] \sim_{(m,n)} [d_{ij}]$ if and only if $c_{ij} = d_{ij}, \forall 0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n$. Note that this is an obvious extension to $\sim_n$ introduced in [**1, 2**] for $N[x]$.

Furthermore, we write $\pi_{hk}[c_{ij}] = c_{hk}$ where c_{hk} is the (h, k)-entry of $[c_{ij}]$. Then a routine computation shows the following.

Lemma 3. *Suppose $F \in N[x, y]$. If $[c_{ij}] \sim_{(m,n)} [d_{ij}]$, then $F\bigl([c_{ij}]\bigr) \sim_{(m,n)} F\bigl([d_{ij}]\bigr)$.*

Theorem 4. *$N[x, y]$ is isomorphic to $N_{\text{finite}}[x, y]$.*

Proof: Define a mapping $\psi : N[x, y] \to N_{\text{finite}}[x, y]$ such that $\psi(F) =$ the restriction of F on $N_{\text{finite}}^{K^2}$. Clearly the mapping ψ is well-defined. Furthermore, by definition, ψ preserves addition and multiplication. It remains to show the mapping is one-to-one and onto.

Suppose $\psi(F) = 0$. We have $F([c_{ij}]) = 0$ for any $[c_{ij}] \in N^{K^2}_{\text{finite}}$. We want to show $F = 0$. Assume $F([c_{ij}]) \neq 0$ for some $[c_{ij}] \in N^{K^2}$. There exist $m, n \in K$ such that $\pi_{mn}F([c_{ij}]) \neq 0$. Let us define $[d_{ij}]$ such that

$$d_{ij} = \begin{cases} c_{ij} & \text{if } 0 \leqslant i \leqslant m \text{ and } 0 \leqslant j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

In fact $[d_{ij}] \in N^{K^2}_{\text{finite}}$ and $[c_{ij}] \sim_{(m,n)} [d_{ij}]$. Hence we have $F([c_{ij}]) \sim_{(m,n)} F([d_{ij}])$. But $F([d_{ij}]) = \psi(F)([d_{ij}]) = 0$ by assumption. Thus $0 = \pi_{mn}F([d_{ij}]) = \pi_{mn}F([c_{ij}]) \neq 0$, a contradiction. In other words, ψ is injective. Finally observe that ψ maps β_a, x, and y of $N[x, y]$ onto β_a, x, and y of $N_{\text{finite}}[x, y]$ naturally. Since ψ is a homomorphism, we have ψ a surjective mapping, hence an isomorphism. $\square$

Hereby we shall not distinguish $N[x, y]$ from $N_{\text{finite}}[x, y]$. As a final remark in this section, we note that $N[x, y]$ can be viewed as an N-submodule of $M(N^{K^2})$ generated by $B_N \cup \overline{X} \cup \overline{Y}$.

4. Remarks on degree, least degree, et cetera

In the previous section, we define the object $N[x, y]$ and identify it with $(N[x])[y]$. There are many basic tools to facilitate the investigation of polynomial near-rings such as the degree, the least degree, the symbolically zero ideal "sym_0", I^*, et cetera. Since the notion of $N[x, y]$ does not come from a straightforward iterative definition of $(N[x])[y]$, it does not allow us to import the various important tools simply from one indeterminate to multiple indeterminates without careful examinations. In this section, we explore the equivalent concepts of some important tools. It is beneficial to "re-define" some of the important tools for the record. However most of the verifications can be obtained by appropriated modifications of the original ones.

(1) Least Degree: Let $f \in N[x, y]$. We denote the least degree of f by

$$\min\left\{h + k \mid \pi_{hk}f([c_{ij}]) \neq 0 \;\text{ for some }\; [c_{ij}] \in N^{K^2}\right\}.$$

A routine computation shows this notion is consistent with the one defined for $N[x]$ by Bagley [**1**]. In fact the least degree of $g \in N[x]$ and that of the natural correspondence in $N[x, y]$ are equal as expected.

(2) $\mathbb{S}[x, y]$: An important type of structure for polynomial near-rings is

$$\mathbb{S}[x, y] = \left\{f \in N[x, y] \mid \forall [c_{ij}] \in N^{K^2}, \pi_{hk}f([c_{ihj}]) \in S_{hk},\ \forall h, k \in K\right\}$$

where $\mathbb{S} = \{S_{ij} \mid i, j \in K\}$ is a collection of left ideals of N with $S_{ij} \subseteq S_{hk}$ whenever $i \leqslant h$ and $j \leqslant k$. It follows that $\mathbb{S}[x, y]$ is indeed an ideal of $N[x, y]$.

Of particular interest is $\mathbb{S}_n = \{S_{ij} \mid S_{ij} = 0 \text{ if } i + j < n \text{ and } S_{ij} = N \text{ otherwise}\}$ for $n \geqslant 0$. Clearly we have $\mathbb{S}_0[x, y] = N[x, y]$. Furthermore $\mathbb{S}_n[x, y] = I_n$ where

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$I_n = \{f \in N[x,y] \mid \text{least degree of } f \geqslant n\}$. In fact, if $f \in I_n$, we have $\pi_{hk}(f[c_{ij}]) = 0$ if $h+k < n$, and then $f \in \mathbb{S}_n[x,y]$. That is $I_n \subseteq \mathbb{S}_n[x,y]$. Hence $I_n = \mathbb{S}_n[x,y]$ follows from the definition.

(3) sym_0: The symbolically zero polynomials and the associated ideal "sym_0" are important and useful tools. The ideal sym_0 can be defined as follows.

Let $W = \{x^i y^j \mid 0 \leqslant i,j\}$. For simplicity, we can write $W = \{w_l \mid 0 \leqslant l\}$ where $w_l = x^i y^j$ and $l = \big((i+j)(i+j-1)\big)/2 + j$. From basic number theory, we know l and the pair (i,j) are in a one-to-one correspondence. Let $P(n)$ denote the set of all permutations of $\{0,\dots,n\}$. We can now define sym_0 as the ideal of $N[x,y]$ generated by the set

$$\big\{a(b_0w_0 + b_1w_1 + \cdots + b_nw_n) - ab_{\sigma(n)}w_{\sigma(n)} - \cdots - ab_{\sigma(0)}w_{\sigma(0)} \mid \forall 0 \leqslant n,\ \forall\, \sigma \in P(n) \text{ and } a, b_i \in N\big\}.$$

As an immediate consequence we have:

Theorem 5. *Suppose $f \in N[x,y]$. There exist $f_0, f_1, \cdots, f_n \in N$ for some $n \geqslant 0$ such that $f \equiv f_0w_0 + f_1w_1 + \cdots + f_nw_n \pmod{\text{sym}_0}$.*

(4) The ideal I^*: Let I be an ideal of N. Then

$$I^* = \Big\{f \in N[x,y] \mid \pi_{hk}f([c_{ij}]) \in I, \forall h,k \text{ and } \forall [c_{ij}]\Big\}$$

is an ideal of $N[x,y]$. We have $N[x,y]/I^* \cong (N/I)[x,y]$. If $\mathcal{D}(N)$ denotes the distributor ideal of N, then it can be easily proved that $\mathcal{D}(N[x,y]) = (\mathcal{D}(N))^*$.

(5) Degree : The last concept we explore is the notion of the degree of a polynomial. First of all, for $\overline{c} = [c_{ij}] \in N^{K^2}_{finiite}$, we let $|\overline{c}| = \max\{i+j \mid c_{ij} \neq 0\}$. The degree of $f \in N[x,y]$ is defined as the $\max\Big\{\big|f(\overline{c})\big| - |\overline{c}| \,\Big|\, \overline{c} \in N^{K^2}_{\text{finite}}\Big\}$. Following the convention, the degree of $f = 0$ is negative infinity. One can check that this notion is well-defined, namely each $0 \neq f \in N[x,y]$ has a degree $< \infty$. Furthermore it is not difficult to show that the degree of a non-zero polynomial is non-negative, that is, for some $\overline{c}$ we have $\big|f(\overline{c})\big| \geqslant |\overline{c}|$ if $f \neq 0$. Assume for purpose of contradiction that $\big|f(\overline{c})\big| < |\overline{c}|, \forall \overline{c}$. Therefore for all $n \geqslant 0$ if $|\overline{c}| = n$, we have $\pi_{hk}f(\overline{c}) = 0$ for $h+k = n$. In other words, we have that if $\overline{d} \in N^{K^2}$ and $h+k = n \geqslant 0$ there is a $\overline{c} \in N^{K^2}_{\text{finite}}$ with $\overline{d} \sim_{(h,k)} \overline{c}$ with $|\overline{c}| = n$. Since $f(\overline{d}) \sim_{(h,k)} f(\overline{c})$, we have $0 = \pi_{hk}f(\overline{d}) = \pi_{hk}f(\overline{c})$ whenever $h+k = n$. Note this is true for all $n \geqslant 0$ and $\overline{d}$, hence $f = 0$ a contradiction to our assumption.

References

[1] S.W. Bagley, *Polynomial near-rings, distributor and J_2 ideals of generalized centralizer near-rings*, (Doctoral Dissertation) (Texas A&M University, Texas, 1993).

[2] S.W. Bagley, 'Polynomial near-rings: Polynomials with coefficients from a near-ring', in *Nearrings, Nearfields and Loops*, (Saad and Thomsen, Editors) (Kluwer Academic Publishers, Netherlands, 1997), pp. 179–190.

[3] M. Farag, *On the structure of polynomial near-rings*, (Doctoral Dissertation) (Texas A&M University, Texas, 1999).

[4] M. Farag, 'A new generalization of the center of a near-ring with applications to polynomial near-rings', *Comm. Algebra* **29** (2001), 2377–2387.

[5] M. Farag, 'Ideals in polynomial near-rings', *Algebra Collo.* **9** (2002), 219–232.

[6] E.K.S. Lee, 'Theory of polynomial near-rings', *Comm. Algebra* **32** (2004), 1619–1635.

[7] J.D.P. Meldrum and A.P.J. van der Walt, 'Matrix near-rings', *Arch. Math.* **47** (1986), 312–319.

[8] J.H. Meyer, 'On the near-ring counterpart of the matrix ring isomorphism $M_{mn}(R) \cong M_n(M_m(R))$', *Rocky Mountain J. Math.* **27** (1997), 231–240.

[9] J.D.P. Meldrum, *Near-rings and their links with groups*, Research Notes in Mathematics **134** (Pitman, Marshfield, M.A., 1985).

[10] G. Pilz, *Near-rings* (North-Holland/American Elsevier, Amsterdam, 1983).

Mathematics Department
Ferris State University
Big Rapids, MI 49307
United States of America
e-mail: leee@ferris.edu

Mathematics Department
University of Port Elizabeth
P.O. Box 1600
Port Elizabeth
South Africa
e-mail: nico.groenewald@upe.ac.za