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A GENERAL INTEGRAL INEQUALITY FOR THE DERIVATIVE OF AN EQUIMEASURABLE REARRANGEMENT

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1. Introduction. The theory of non-increasing (decreasing) equimeasurable rearrangements of functions was introduced by Hardy and Littlewood [**6**] in connection with their studies of fractional integrals and integral operators. Elementary properties of equimeasurable decreasing rearrangements are given in the monograph [**7**] of Hardy, Littlewood, and Polya on inequalities, while a more recent treatment is Okikiolu [**9**, § 5.4].

The rearrangement operation has the property of being variation-reducing. A study of this property for infinite sequences and functions defined on infinite domains was made in [**3**], a typical result being the reduction by rearrangement of the pth power integral $\int |f'(x)|^p dx$. Strengthened versions of these inequalities were also derived in [**4**] using the index or multiplicity functions $n(x)$ in one dimension and $S(f)$ in several dimensions. The results for several variables are related to symmetrization and the minima of Dirichlet integrals as studied by Polya and Szegö [**12**].

The primary object of this paper is to extend such rearrangement inequalities to integrals containing an arbitrary function of the rearrangement, the function being restricted only by a convexity condition. The method is based on that of [**4**] in both the single and multiple variable cases. Examples and an application to the calculation of extreme cases of Sobolev embeddings are given.

2. One dimensional rearrangements. For a real valued measurable function f on the domain $[0, b]$ the equimeasurable decreasing rearrangement f^* of f is defined as a function μ^{-1} inverse to μ, where $\mu(y)$ is the measure of the set $\{x | f(x) > y\}$. Since f^* is monotonic $f^{*\prime}$ is defined almost everywhere on $[0, b]$. The *multiplicity* $n(y)$ of f at the level y is the number of roots $x_k = x_k(y)$, $k = 1, \ldots, n(y)$ of the equation $y = f(x)$, in $[0, b]$. If this number of roots is infinite, we set $n(y) = \infty$.

The basic relation connecting the derivatives of $f \in C'[0, b]$ can now be derived [**4**] and a derivation using differentials will be sketched here. If y varies by dy, then x_k and x^* will vary by amounts dx_k, dx^* respectively. From the equimeasurable property it follows that

$$|dx^*| = \sum_{k=1}^{n} |dx_k|$$

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Divide this relation by dy, and let $dy \to 0$. By the definition of derivative we obtain in the limit the basic relation

$$\frac{1}{|f^{*\prime}(x)|} = \sum_{k=1}^{n} \frac{1}{|f'(x_k)|}.$$

For interpretation of the cases where n is infinite or one or more zero values appears, reference is made to [**4**, p. 410].

The basic relation is local, in the sense that values in the neighbourhood of one range point only are involved. Moreover the inequalities of [**4**] of the type

$$\int \chi(f^*(x))|f^{*\prime}(x)|^p dx \leqq \int \chi(f) \left| \frac{f'(x)}{n(f(x))} \right|^p dx,$$

where $\chi(f)$ is an arbitrary positive function, show that such inequalities can be taken over an arbitrary subset of the range, with arbitrary weighting.

Focussing attention therefore on the values of the terms in the basic relation at one independent range point, one is led to enquire whether inequalities involving functions of these values, subject to some convexity condition, can be established.

3. The general inequality. The integrals to be considered have the form

$$\int G(|f'(x)|)dx$$

where the domain D is, in the one-dimensional case, an interval $[0, b]$. It will appear that the natural condition on G is the convexity of $G(x)$, as will be shown in the following lemma and theorem.

LEMMA 1. *If $G(x)$ is convex for $x > 0$, then $H(x) = xG(1/x)$ is also convex for $x > 0$.*

Proof. Let $0 \leqq \lambda \leqq 1$ and for given positive x, y define

$$p = \frac{\lambda y}{(1 - \lambda)x + \lambda y}$$

Evidently $0 \leqq p \leqq 1$ with $p = 0$ when $\lambda = 0$ and $p = 1$ when $\lambda = 1$. Since

$$\lambda = \frac{px}{(1 - p)y + px}$$

it follows that λ can be specified for a given value of p.

Given p, where $0 \leqq p \leqq 1$, we have

$$\begin{aligned}\frac{1}{(1-\lambda)x+\lambda y} &= \frac{1-\lambda}{(1-\lambda)x+\lambda y}+\frac{\lambda}{(1-\lambda)x+\lambda y}\\ &= \frac{(1-\lambda)x}{x\{(1-\lambda)x+\lambda y\}}+\frac{\lambda y}{y\{(1-\lambda)x+\lambda y\}}\\ &= \frac{1}{x}\left(1-\frac{y\lambda}{(1-\lambda)x+\lambda y}\right)+\frac{\lambda y}{y\{(1-\lambda)x+\lambda y\}}\\ &= \frac{1-p}{x}+\frac{p}{y}.\end{aligned}$$

Therefore by the convexity of $G(x)$ we find

$$\begin{aligned}G\left(\frac{1}{(1-\lambda)x+\lambda y}\right) &= G\left(\frac{1-p}{x}+\frac{p}{y}\right)\\ &\leqq (1-p)G(1/x)+pG(1/y)\\ &= \frac{(1-\lambda)xG(1/x)+\lambda yG(1/y)}{(1-\lambda)x+\lambda y}.\end{aligned}$$

Multiplying by the form $(1-\lambda)x+\lambda y$, we obtain

$$\{(1-\lambda)x+\lambda y\}\, G\left(\frac{1}{(1-\lambda)x+\lambda y}\right) \leqq (1-\lambda)xG(1/x)+\lambda yG(1/y)$$

and this shows directly that $xG(1/x)$ is convex.

A result of this type is stated for twice differentiable positive convex functions in [**9**, p. 97, Theorem 120]. The transformation or functional operation $H(x) = xG(1/x)$ is involutory, since $G(x) = x\,H(1/x)$.

THEOREM 1. *Let f be differentiable almost everywhere in* $[0, b]$ *and let* $G(y)$ *be a function convex for* $y \geqq 0$. *Then*

$$\int_0^b G(|f^{*\prime}(x)|)dx \leqq \int_0^b G\left(\frac{|f'(x)|}{n(f(x))}\right)dx$$

Proof. Using x^* as independent variable for the rearranged function f^*, we have

$$|f^{*\prime}(x^*)| = |dy/dx^*|,$$

hence from the basic relation

$$dx^* = \frac{dy}{|f^{*\prime}(x^*)|} = \sum_{k=1}^{n}\frac{dy}{|f'(x_k)|} = \sum_{k=1}^{n} dx_k.$$

Therefore

$$\begin{aligned}G(|f^{*\prime}|)dx^* &= G\left(\left(\sum_{k=1}^{n}(|f'(x_k)|)^{-1}\right)^{-1}\right)\sum_{k=1}^{n}(|f'(x_k)|)^{-1}dy\\ &= H\left(\sum_{k=1}^{n}(|f'(x_k)|)^{-1}\right)dy\end{aligned}$$

By Lemma 1, $H(x) = xG(1/x)$ is convex. We now introduce n equal 'probabilities' $p_k = 1/n$ in the convexity condition $H(\sum p_k S_k) \leqq \sum p_k H(S_k)$, where $n = n(f(x))$. Thus we obtain

$$\begin{aligned} G(|f^{*\prime}(x^*)|)dx^* &= H\left(\sum_{k=1}^{n} (1/n)(n/|f'(x_k)|)\right)dy \\ &\leqq \sum_{k=1}^{n} (1/n)H(n/|f'(x_k)|)dy \\ &= \sum_{k=1}^{n} (1/n)(n/|f'(x_k)|)G(|f'(x_k)|/n)dy \\ &= \sum_{k=1}^{n} G(|f'(x_k)|/n)dx_k \end{aligned}$$

using in the last step the relation $dy = |f'(x_k)|\, dx_k$. Integration over the domain $[0, b]$ now yields the stated result, as the integral elements based on the dx_k exactly cover the interval $[0, b]$ once when the summation over all integral elements based on dx^* is performed. This completes the proof of Theorem 1.

Example 1. Choose $G(x) = p(p - 1)\, x^{1-p}$, where $p \in R$. The convexity of $xG(1/x)$ can be directly verified, and this choice yields the three primary inequalities stated separately in Theorem 1 of [**4**] for the ranges $p > 1$, $0 < p < 1$ and $p < 0$. In the case $p = 1$ equality holds.

Example 2. Choose $G(x) = \sqrt{1 + x^2}$. Convexity of $xG(1/x) = \sqrt{1 + x^2}$ can be verified straightforwardly and leads to inequalities for integrals of arc length of the graphs of f^* and f/n. As n is discontinuous, one may construct a continuous function

$$\mathscr{F}(x) = \int^x \frac{df(s)}{n(f(s))}$$

for which the graph consists of segments with n constant and which are congruent to portions of the graph of f, but vertically reduced in the proportion of 1 to n. The theorem then states that the arc length of f^* is less than that of $\mathscr{F}$.

COROLLARY 1.1. *Let $\chi(f)$ and $a(f)$ be arbitrary positive functions on the range of f. Then, assuming the indicated expressions are defined and that $G(x)$ is convex, we have*

$$\int \chi(f^*)\, G(a(f^*)|f^{*\prime}|)dx \leqq \int \chi(f)\, G\left(a(f)\left|\frac{f}{n(f)}\right|\right)dx$$

Proof. The positive function $\chi(f)$ can be inserted in the proof of the theorem simply on multiplication by $\chi(f) = \chi(f^*)$ before the final integration. That the factor $a(f)$ can also be inserted as indicated can be verified by repeating the main calculation. However for positive $a(f)$ the same result can be achieved

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if the theorem is applied to the function

$$F(f) = \int a(f)\, df$$

and its rearrangement $F(f^*)$ which involves the same rearrangement of points of the x axis. Indeed $F(f^*) = F^*$ for positive $a(f)$. From these observations Corollary 1.1 follows easily.

Remarks. Since the multiplicity function n is defined on the range, either $\chi(f)$ or $a(f)$ can be functions of n as well as of f. For example, n itself is a function of f. As this case has independent interest we state it as a separate Corollary when $a(f) = n$ and $\chi(f)$ is taken to be unity.

COROLLARY 1.2. *If $G(x)$ is convex, then*

$$\int G(n(f^*)|f^{*\prime}|)dx \leqq \int G(|f'|)dx$$

Note that the function $F(f)$ in this instance satisfies $dF = ndf^* = -n|df|$ and thus F is essentially the equivariational transform of [**4**, Section 7]. The particular choice $G(x) = \sqrt{1 + x^2}$ then yields Theorem 4 of that section. The present proof is direct and avoids the need of vector constructions for this result on arc length.

4. The m-dimensional case. Recall from [**4,** p. 417] that a function $f(x) = f(x_1, \ldots, x_m)$ has a spherically symmetric equimeasurable decreasing rearrangement which is essentially a function $f^*(x)$ of volume or of radial distance only. Let

$$\mu(z) = \text{meas. } \{(x_1, \ldots, x_n) | f(x_1, \ldots, x_n) > z\},$$

and let

$$f^*(x) = \mu^{-1}(x)$$

The basic relation for $f \in PC^1$ is derived by integration over the level surface $f = f^*$ in the domain D involved. If dn denotes the inward normal differential, and ∇f the gradient, then $|\nabla f| dn = df$. Since

$$\mu(z) = \int_{f \geqq z} dV = \int_{f \geqq z} dn dS$$

we find

$$d\mu = \sum \int dn dS = \sum \int \frac{df}{|\nabla f|} dS$$

The summation runs over all components of the level surface $f = f^*$. However,

$$d\mu = -\frac{df^*}{|f^{*\prime}(x)|}$$

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while $|df| = |df^*|$. Comparing, we obtain the m-dimensional basic relation in the form [**4**, p. 418]

$$\frac{1}{|f^{*\prime}(x)|} = \sum \int_{f=f^*} \frac{dS}{|\nabla f|}.$$

The m-dimensional analogue of the multiplicity function is now the level surface area

$$S = \sum \int_{f=f^*} dS$$

THEOREM 2. *Let $G(x)$ be convex for $x > 0$. Then*

$$\int G(|f^{*\prime}(x)|)dV \leqq \int G\left(\frac{|\nabla f|}{S}\right)dV.$$

Proof. The integral on the left has the differential

$$\begin{aligned} G(|f^{*\prime}(x)|)dV &= G\left(\left(\sum \int (|\nabla f|)^{-1}dS\right)^{-1}\right) \sum \int dSdn \\ &= G\left(\left(\sum \int (|\nabla f|)^{-1}dS\right)^{-1}\right) \sum \int |\nabla f|^{-1}dSdf \\ &= H\left(\sum \int |\nabla f|^{-1}dS\right)df \\ &= H\left((1/S) \sum \int (S/|\nabla f|)dS\right)df \end{aligned}$$

For use in the convexity relation $h(\sum px) \leqq \sum pH(x)$ we take the "probability differential" $dp = dS/S$. Hence

$$\begin{aligned} G(|f^{*\prime}(x)|)dV &\leqq (1/S) \sum \int_{f=f^*} H(S/|\nabla f|)dSdf \\ &= (1/S) \sum \int_{f=f^*} (S/|\nabla f|)G(|\nabla f|/S)dSdf \\ &= \sum \int_{f=f^*} G(|\nabla f|/S)dSdn \\ &= \sum \int_{f=f^*} G(|\nabla f|/S)dV \end{aligned}$$

The result now follows, as in the one dimensional case, by integration over the domain D.

Again it is possible to include a positive weight function $\chi(f) = \chi(f^*)$ on the range in the integration, and an arbitrary positive function $a(f)$ in the argument of G. The corollary thus has the form

COROLLARY 2.1. *Let $G(x)$ be convex, and let $\chi(f)$, $a(f)$ be arbitrary positive functions on the range of f. Then*

$$\int \chi(f^*)G(a(f^*)|f^{*\prime}(x)|)dV \leqq \int \chi(f)G\left(\frac{a(f)|f'(x)|}{S}\right)dV.$$

The functions $\chi(f)$ and $a(f)$ may again be functions of S since $S = S(f)$. In contrast to the one-dimensional case, S will generally be a continuous function of f. We note in particular the choice $\chi(f) = 1$ and $a(f) = S$ which yields

$$\int_{D^*} G(S|f^{*\prime}(x)|)dV \leqq \int_D G(|\nabla f|)dV.$$

With the choice $G(x) = \sqrt{1 + x^2}$ this gives a surface area inequality that could be used to motivate the definition of a higher dimensional equivariational transform. However we do not pursue any details here.

The choice for $s(x)$ of the power function $p(p - 1)x^{1-p}$ leads to inequalities derived in Theorem 2 of [**4**]. These in the quadratic case are Dirichlet integrals and the inequality leads to another proof of the decrease of capacity of a domain D under Steiner or Schwarz symmetrization (or their analogues in higher dimension) as described in [**12**, p. 157].

The rearrangement function $f^*(x)$ used in the foregoing has domain $0 \leqq x \leqq$ meas (D) and thus x represents an m-dimensional volume. In preparation for the following section we introduce the "radial" rearrangement $f_m^*(r) = f^*(x)$, where

$$x = \mu = \omega_m r^m/m, \quad r^2 = \sum_{i=1}^{m} x_i^2$$

and ω_m is the surface area of the unit sphere in R^m. We see that

$$f^{*\prime}(x) = \frac{df^*(x)}{dx} = \frac{df_m^*(r)}{\omega_m r^{m-1} dr} = \frac{f_m^{*\prime}(r)}{S^*(r)}$$

where primes as usual denote derivatives with respect to the argument indicated, and $S^*(r)$ is the area of the sphere of radius r in R^m. With this notation Theorem 2 takes the form

$$\int_{D^*} G(|f_m^{*\prime}(r)|/S^*(r))dV \leqq \int_D G(|\nabla f|/S)dV,$$

where $dV = \omega_m r^{m-1} dr\, d\Omega$ and $0 \leqq r \leqq r^*$, where r^* is the radius of a sphere D^* having volume equal to the measure of D.

In the foregoing form of the inequality the comparison of the given surface $z = f(x_1, \ldots, x_m)$ and its symmetrized counterpart $z = f_m(r)$ is made in the most direct and comparable way, the two surfaces entering in the same form and on separate sides of the inequality. For a general choice of the arbitrary functions $\chi(f)$ and $a(f)$ this symmetry of comparison may be lost. For example the choice $\chi(f) = 1$, $a(f) = S(r)$ mentioned above now leads to the form

$$\int_{D^*} G\left(\frac{S(r)}{S^*(r)}\,|f_m{}^{*\prime}(r)|\right)dV \leqq \int_D G(|\nabla f|)dV$$

in which the left side contains the term $S(r)$ referring to the given surface and not the symmetrized surface. In the concluding section which follows, still another variant of this type will be employed.

For functions $G(x)$ which are increasing as well as convex, we may derive a simpler inequality from which the surface area factors have been removed. Since $S^* \leqq S$ by the "isoperimetric" inequality for surfaces enclosing the same volume, we deduce

$$\int_{D^*} G(|f_m{}^{*\prime}(r)|)dV \leqq \int_{D^*} G\left(\frac{S(r)}{S^*(r)}\,|f_m{}^{*\prime}(r)|\right)dV$$
$$\leqq \int_D G(|\nabla f|)dV$$

This inequality was established in a different way by Polya and Szegö [**12**, p. 154] also for convex increasing functions G.

5. Calculation of embedding constants. The inequality of the preceding section will now be used in a particular case to calculate the least value of the constant C such that the inequality

$$\left(\int_D |u|^q dV\right)^{1/q} \leqq C\left(\int_D |\nabla u|^p dV\right)^{1/p}$$

holds for functions $u \in C^1$ which vanish on the boundary of the domain D.

We begin by constructing the inequality of Corollary 2.1 for $G(x) = x^p$, $p > 1$ and with the radial rearrangement function $f_m{}^*(r)$,

$$\int_{D^*} \chi(f^*)\left|\frac{f_m{}^{*\prime}(r)}{S^*(r)}\right|^p dV \leqq \int_D \chi(f)\left|\frac{\nabla f}{S}\right|^p dV.$$

We then take $\chi(f^*) = S^*(r)^p$ and use the isoperimetric property $S^* \leqq S$, where S is the surface area at level f enclosing the equal volume μ. Thus

$$\int_{D^*} |f_m{}^{*\prime}(r)|^p dV \leqq \int_D \frac{S^{*p}}{S^p}\,|\nabla f|^p dV \leqq \int_D |\nabla f|^p dV.$$

Setting

$$||f||_q = \left(\int |f|^q dV\right)^{1/q},$$

we observe that $||f_m{}^*||_q = ||f||_q$ by the definition of equimeasurable rearrange-

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ment. Hence

$$\frac{||\nabla f^*||_p}{||f^*||_q} \leq \frac{||\nabla f||_p}{||f||_q},$$

so that if f is any function of class PC^1 on a domain of m-dimensional measure equal to that of D, the variational quotient above has value at least equal to the rearranged quotient on the left. Denoting the minimum of this quotient by C^{-1}, we observe that

$$||f||_q \leqq C\,||\nabla f||_p,$$

so that C is a Sobolev embedding constant [**13**].

To determine C we shall consider the following alternative formulation: for all functions f^* with $||f^*||_q = 1$, find the minimum of $||\nabla f^*||_p = ||f^{*\prime}(r)||_p$. That is, we seek the free minimum of

$$I = ||\nabla f||_p^{\,p} - \lambda ||f||_q^{\,q} = \int \{|df/dr|^p - \lambda |f|^q\} dV,$$

where $f = f^*(r)$ is a function of r only.

Variation of f produces the expression

$$\delta I = \int \{p(df/dr)^{p-1}\delta\,(df/dr) - \lambda q f^{q-1}\delta f\} r^{m-1} dr d\Omega$$

$$= - \int \{p\, d/dr\,((df/dr)^{p-1} r^{m-1}) + \lambda q f^{q-1} r^{m-1}\}\, \delta f dr d\Omega.$$

The vanishing of the first variation δI at the extremal leads to the ordinary nonlinear differential equation for f:

$$p\, d/dr\,(r^{m-1}\,(df/dr)^{p-1}) + \lambda q r^{m-1} f^{q-1} = 0.$$

The solutions are functions of r on the interval $0 \leqq r \leqq r^*$ where $\omega_m\, r^{*m}/m =$ meas (D).

For smooth functions f the quotient vanishes at a maximum, and correspondingly the derivative $df_m{}^*(r)/dr$ will vanish at the origin. Hence the boundary conditions are the corresponding variational free boundary condition

$$\lim_{r\to 0} r^{m-1}\,(df/dr)^{p-1} = 0 \quad \text{at } r = 0,$$

and

$$f = 0 \quad \text{at } r = r^*.$$

If meas (D) is infinite a condition $f \to 0$ as $r \to \infty$ applies.

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If $m = 3$ and $p = 2$ the differential equation becomes the Lane-Emden equation of astrophysics [**2**, Ch. 4] which describes the embedding or equilibrium in space of a mass of gas subject to the polytropic equation of state. For $1 < q < 6$ the Lane-Emden equation has a solution with derivative $f'(r) = 0$ at $r = 0$, and vanishing at a finite value for r. For $q = 6$, however, the zero of this solution reaches infinity and for $q > 6$ there is no such solution. An explicit solution $f(r) = (1 + \frac{1}{3}r^2)^{-1/2}$ is known when $q = 6$ and this expression approaches zero as r approaches infinity. For further information in this particular case, we refer to [**2**] and [**5**].

The explicit solution can also be constructed in the general critical case

$$1/p = 1/q + 1/m, \quad p > 1,$$

of the embedding inequalities. Indeed, this solution is

$$f_e(r) = (e + r^{p/(p-1)})^{-p/(q-p)}$$

where e is an arbitrary constant giving rise to a homology family of solutions. Thus

$$||f_e(r)||_q{}^q = \int\int_0^\infty |f_e(r)|^q r^{m-1} dr d\Omega = \omega_m \int_0^\infty r^{m-1}(e + r^{p/(p-1)})^{-m} dr,$$

where use has been made of the relation $pq/(q-p) = m$, and

$$\omega_m = 2\pi^{m/2}/\Gamma(m/2).$$

Setting $x = r^{p/(p-1)}$, $dx = (p/(p-1))r^{1/(p-1)}dr$ we find, using [**10**, p. 285, Formula 3] that

$$\begin{aligned} 1 = ||f_e||_q{}^q &= \omega_m((p-1)/p) \int_0^\infty x^{(m-1-1/(p-1))(p-1)/p}(e+x)^{-m}dx \\ &= \omega_m((p-1)/p) \int_0^\infty z^{(pm-p-m)/p}(1+z)^{-m}dz \cdot e^{-m/p} \\ &= \omega_m((p-1)/p)e^{-m/p}(\Gamma(m-(m/p))\Gamma(m/p)/\Gamma(m)) \end{aligned}$$

Similarly, we find

$$\begin{aligned} ||\nabla f_e||_p{}^p &= \int_0^\infty r^{m-1}|f_e'(r)|^p dr \cdot \omega_m \\ &= \omega_m[p^2/((q-p)(p-1))]^p \int_0^\infty r^{m+1/(p-1)}(e + r^{p/(p-1)})^{-m} dr \end{aligned}$$

where use is made of the formula

$$f_e(r) = \frac{df_e}{dr} = \frac{-p^2}{(q-p)(p-1)} \cdot \frac{r^{1/(p-1)}}{(e + r^{p/(p-1)})^{q/(q-p)}}$$

Again setting $x = r^{p/(p-1)}$ we find

$$||\nabla f_e||_p^p = \omega_m \left[\frac{p^2}{(q-p)(p-1)}\right]^p \int_0^\infty \frac{x^{m(p-1)/p} dx \cdot (p-1)/p}{(e+x)^m}$$

$$= \omega_m \frac{p-1}{p} \left[\frac{p^2}{(q-p)(p-1)}\right]^p \int_0^\infty \frac{(x/e)^{m(p-1)/p} d(x/e) e^{1-m/p}}{(1+x/e)^m}$$

$$= \omega_m \frac{p-1}{p} \left[\frac{p^2}{(q-p)(p-1)}\right]^p \int_0^\infty \frac{z^{m(p-1)/p} dz \cdot e^{1-m/p}}{(1+z)^m}$$

$$= \omega_m \frac{p-1}{p} \left[\frac{p^2}{(q-p)(p-1)}\right]^p \frac{\Gamma(m-m/p+1)\Gamma(m/p-1)e^{1-m/p}}{\Gamma(m)}$$

It follows that

$$C = \frac{||f_e||_q}{||f_e'||_p} = \omega_m^{1/q-1/p} \left(\frac{p-1}{p}\right)^{1/q-1/p} \frac{(q-p)(p-1)}{p^2}$$

$$\times \left(\frac{\Gamma(m-m/p)\Gamma(m/p)}{\Gamma(m)}\right)^{1/q} \left(\frac{\Gamma(m-m/p+1)\Gamma(m/p-1)}{\Gamma(m)}\right)^{-1/p}$$

$$= \left[\omega_m \frac{p-1}{p} \frac{\Gamma(m-m/p+1)\Gamma(m/p-1)}{\Gamma(m)}\right]^{-1/m}$$

$$\times \left(\frac{m-p}{m(p-1)}\right)^{1/q} \frac{(q-p)(p-1)}{p^2}$$

$$= \omega_m^{-1/m} \left(\frac{p-1}{p}\right)^{(p-1)/p} q^{(q-1)/q} \frac{1}{m} \left[\frac{\Gamma(m)}{\Gamma(m-m/p+1)\Gamma(m/p-1)}\right]^{1/m}$$

For example if $m = 3$, $p = 2$ and $q = 6$ then $C = 2^{2/3}3^{-1/2}\pi^{-2/3} = 0.427 \ldots$

This spherically symmetric form of the inequality is equivalent under a change of variable to an inequality studied by Hardy and Littlewood [**7**], and Bliss [**1**]. From the viewpoint of the calculus of variations this problem is irregular with infinite domain and singular values of the factor r^{m-1}. By a special proof, Bliss showed that the particular solution here denoted by $f_e(r)$ attains a global minimum. It follows that C is the best possible constant in the Sobolev inequality.

A detailed study of the best constant has also been made recently by Talenti [**14**], who describes connections of the formula with geometric measure theory. E. Rodemich wrote a manuscript on this in 1966.

The formulas of this and the preceding section permit one other extension of the "best constant" in Sobolev's inequality. We have

$$||f||_q = ||f^*||_q \leqq C||\nabla f^*||_p \leqq C||S^*/S \nabla f||_p \leqq C \text{ ess max } (S^*/S)||\nabla f||_p,$$

and for certain configurations the factor S^*/S may be shown to have essential maximum less than unity.

Example 1. Let f be smooth in R^n with support in the spherical annulus $r_1 \leqq r \leqq r_2 = \alpha r_1$, where $\alpha > 1$, and let f be non-negative with maximum taken on a surface encircling the inner sphere of radius r_1. Then every intermediate value of f is taken on a surface twice encircling the inner sphere, so that $S \geqq 2\omega_n r_1^{n-1}$. If max S^* is given by $\omega_n R^{n-1}$, where $V = n^{-1}\omega_n(r_2^n - r_1^n) =$

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$n^{-1}\omega_n r_1{}^n(\alpha^n - 1) = n^{-1}\omega_n R^n$, then

$$\operatorname{ess\,max} S^*/S \leqq \frac{\omega_n R^{n-1}}{2\omega_n r_1{}^{n-1}} = \tfrac{1}{2}(R/r_1)^{n-1} = \tfrac{1}{2}(\alpha^n - 1)^{(n-1)/n}$$

For α sufficiently close to 1, this expression is itself less than unity.

Example 2. Let f be smooth in R^n with compact support of the form $f(\rho)$, where $f(\rho)$ is monotonic and

$$\rho^2 = \sum_{k=1}^{n} C_k x_k{}^2, \quad C_k > 0$$

Then S^*/S is a constant less than unity being the ratio of surface areas of an ellipsoid and a sphere having equal volume.

These examples can be combined for a function with support in an ellipsiodal annulus. The product of two geometric factors of the above types will appear.

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