FREE COMPLETELY REGULAR SEMIGROUPS

by P. G. TROTTER

(Received 5 May, 1983)

A completely regular semigroup is a semigroup that is a union of groups. The aim here is to provide an alternative characterization of the free completely regular semigroup F_X^{cr} on a set X to that given by J. A. Gerhard in [**3**, **4**].

Although the structure theory for completely regular semigroups was initiated in 1941 [**1**] by A. H. Clifford it was not until 1968 that it was shown by D. B. McAlister [**5**] that F_X^{cr} exists. More recently, in [**7**], M. Petrich demonstrated the existence of F_X^{cr} by showing that completely regular semigroups form a variety of unary semigroups (that is, semigroups with the additional operation of inversion).

In [**2**] Clifford investigated the structure of F_X^{cr} by considering it as a quotient semigroup F_X^u/ρ of the free unary semigroup F_X^u on X. He gave a simple description of Green's relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{D}$ on F_X^u/ρ and showed that F_X^{cr} is a free semilattice of its $\mathscr{D}$-classes. The description of $\mathscr{D}$ was in terms of content of elements while the $\mathscr{L}$-class of an element was described modulo a description of the ρ-class of an element of lesser content. Clifford enabled the ρ-classes of elements of content at most 2 to be explicitly described, by providing a model for F_Z^{cr} when $|Z|=2$.

J. A. Gerhard showed in [**4**] that an $\mathscr{H}$-class of F_X^u/ρ is a free group. The free generators were described modulo solution of the word problem in F_X^u/ρ for words of lesser content. With a given $\mathscr{H}$-class from each $\mathscr{D}$-class and with Petrich's description [**6**] of an arbitrary completely regular semigroup, Gerhard constructed a model for F_X^{cr}.

In this paper we inductively select a unique representative $w\theta \in F_X^u$ for each ρ-class $w\rho \in F_X^u/\rho$. In particular $w\theta$ is uniquely expressed as a product of elements from X and from {segments of $a\theta$; $a \in F_X^u$ has smaller content than w}. It is then shown that the set $\{w\theta;\ w \in F_X^u\}$ with the multiplication $u\theta \cdot v\theta = (u\theta(v\theta))\theta$ is isomorphic to F_X^{cr}. Since they do not appear to shorten the proofs of this paper, the results of [**3**, **4**] are not utilized here.

In section 1 some preliminary information, especially from [**2**], is listed. Some properties of F_X^u/ρ are derived in section 2. In section 3, θ is defined and relevant properties are derived. A model for F_X^{cr} is obtained in the final section.

1. Definitions and preliminaries. Let F_X and F_X^{cr} denote respectively the free semigroup and free completely regular semigroup on a non-empty set X. Let F_X^u denote the *free unary semigroup* on X; that is, the free object on X in the category of semigroups with a unary operation. Let $\bar{X} = X \cup \{(,\ )^{-1}\}$ where $(,\ )^{-1} \notin X$. By [**3**], F_X^u is the smallest subsemigroup of the free semigroup $F_{\bar{X}}$ such that $X \subseteq F_X^u$ and $(w)^{-1} \in F_X^u$ for all $w \in F_X^u$. There is an alternative description of F_X^u in [**2**]. Let $F_{\bar{X}}^1$ and F_X^{u1} denote respectively the semigroups $F_{\bar{X}}$ and F_X^u with identity 1 adjoined.

Define $v \in F_{\bar{X}}$ to be a *segment* of $w \in F_X^u$ if $w = avb$ for some $a, b \in F_{\bar{X}}^1$. The segment v

of w is said to be *maximal with respect to a property P* if and only if v satisfies P and for any final segment c of a and initial segment d of b then cv, vd and cvd do not satisfy P.

If $u = b(a$ or $u = a)^{-1}b$ where $a \in F_X^{u1}$ we say that the occurrence of (or $)^{-1}$ respectively is *unmatched*. Let $\mathbf{v} \in F_X^u$ denote the word obtained from $v \in F_{\bar{X}}$ by successively deleting unmatched occurrences of (and $)^{-1}$.

The *content* of $w \in F_X^1$ is the set

$$C(w) = \{\text{letters of } X \text{ appearing in } w\}.$$

The *left indicator* $L(w)$ of $w \in F_X^u$ is given by $L(w) = \mathbf{a}$ where a is the shortest initial segment of w such that $C(a) = C(w)$. Dually the *right indicator* is $R(w) = \mathbf{b}$ where b is the shortest final segment of w such that $C(w) = C(b)$. Define $\mathbf{c}$ to be an *indicator* of w if there is a segment c of w such that $C(c) = C(w)$ and $L(\mathbf{c}) = R(\mathbf{c}) = \mathbf{c}$.

For $w \in F_X^u$ let $\{c_j; 1 \le j \le r\}$ denote the set of successive segments of w such that $\mathbf{c}_j$ is an indicator and $\mathbf{c}_j$ is not derivable from a proper subsegment of c_j. Let d_j denote the final segment of w beginning with c_j, and let e_j denote the segment beginning and ending with c_j and c_{j+1} respectively. Define $I_j(w) = \mathbf{c}_j$, $W_j(w) = \mathbf{d}_j$ and $M_j(w) = \mathbf{e}_j$ to be respectively the *jth indicator, jth remainder* and *jth link* of w.

EXAMPLE 1.1. Let $w = x_1(x_2((x_4x_1)^{-1}x_3)^{-1}(x_2)^{-1}x_1x_4x_3)^{-1}$. Then

$$L(w) = x_1x_2(x_4x_1)^{-1}x_3, \qquad R(w) = x_2x_1x_4x_3, \qquad I_1(w) = x_2(x_4x_1)^{-1}x_3,$$

$$I_2(w) = x_4x_1x_3x_2, \qquad I_3(w) = x_3(x_2)^{-1}x_1x_4, \qquad I_4(w) = x_2x_1x_4x_3,$$

$$M_1(w) = x_2((x_4x_1)^{-1}x_3)^{-1}x_2, \qquad M_2(w) = x_4x_1x_3(x_2)^{-1}x_1x_4, \qquad M_3(w) = x_3(x_2)^{-1}x_1x_4x_3,$$

$$W_1(w) = x_2((x_4x_1)^{-1}x_3)^{-1}(x_2)^{-1}x_1x_4x_3, \ldots, W_4(w) = x_2x_1x_4x_3.$$

Note that $I_j(w) = L(W_j(w)) = L(M_j(w)) = R(M_{j-1}(w))$ (if the links exist) and $I_1(w) = R(L(w))$.

We next provide a simple characterization of indicators and links. Suppose $v \in F_X^u$ and $C(v) = Y$. For any $u \in F_X^{u1}$ and $x, y \in Y \setminus C(u)$ define v to be

(i) a *left* [*right*] *Y-indicator* if $v = uy$ [$v = xu$],

(ii) a *Y-indicator* if $v = xuy$, $x \neq y$ (or if $v = x$ when $|Y| = 1$), or

(iii) a *Y-link* if $v = xux$.

LEMMA 1.2. *Let $v \in F_X^u$ and $C(v) = Y$. Then v is a left or right Y-indicator, Y-indicator or Y-link if and only if v is a left or right indicator, indicator or link respectively of some $w \in F_X^u$.*

Proof. Let $v = xux$ be a Y-link as in the definition. Since $C(u) \neq Y$ and $x \in Y \setminus C(u)$ then $L(v)$ and $R(v)$ are successive indicators of v, so v is a link of itself. Conversely let M be the jth link of w where $C(w) = Y$. So M has exactly two indicators, namely $I_j(w) = xa$ and $I_{j+1}(w) = by$ for some $a, b \in F_X^{u1}$, $x \in Y \setminus C(a)$ and $y \in Y \setminus C(b)$. So $M = xcy$ for some $c \in F_X^{u1}$. If $C(c) = C(w)$ then $R(L(c))$ and $L(R(c))$ are indicators of w. Hence since $I_j(w)$ and $I_{j+1}(w)$ are successive indicators then $C(c) \neq C(w)$ while $C(xc) = C(w) = C(cy)$. But then, since $x \in C(cy)$, if $x \neq y$ we get $x \in C(c)$ and $C(c) = C(w)$, a contradiction. Thus $x = y$ and M is a Y-link. The other cases follow directly from the definitions.

Let ρ be the least congruence on F_X^u containing $(w(w)^{-1}w, w)$, $(w(w)^{-1}, (w)^{-1}w)$ and $(((w)^{-1})^{-1}, w)$ for all $w \in F_X^u$.

THEOREM 1.3. (Clifford [2]). *Let $u, v \in F_X^u$. Then*

(i) $F_X^{cr} \cong F_X^u/\rho$,

(ii) $u\rho\,\mathcal{D}\,v\rho$ *if and only if* $C(u) = C(v)$, *and*

(iii) $u\rho\,\mathcal{R}\,v\rho$ *if and only if* $L(u) = ax$ *and* $L(v) = bx$ *for some* $x \in X$ *and* $a, b \in F_X^{u1}$ *such that* $a\,\rho\,b$ *or* $a = b = 1$.

COROLLARY 1.4. *Suppose $u, v \in F_X^u$ and p is an initial segment of u. If $u\,\rho\,v$ then v has an initial segment q such that $L(\mathbf{p})\,\rho\,L(\mathbf{q})$.*

Proof. Assume $1 \neq |C(u)| \neq |C(p)|$: otherwise the result follows directly from the theorem. Let u, v, a, b be as in Theorem 1.3(iii). So $a\,\rho\,b$ and by [2, Lemma 5.1], $a\,\rho\,\mathbf{p}a_1$ for some $a_1 \in F_X^{u1}$. Since $|C(a)| < |C(u)|$ the result follows by induction on $|C(u)|$.

NOTATION. For $w \in F_X^u$ define w^n to be the product of n copies of w. Define $w^{-1} = (w)^{-1}$, w^{-n} to be the product of n copies of w^{-1} and $w^0 = ww^{-2}w$.

Throughout the paper assume that X is a well ordered set. We will always denote by Y the subset $Y = \{x_1, \ldots, x_n\}$ of X where $x_i < x_j$ in X if and only if $i < j$. Define

$$\hat{f} = x_1 \ldots x_n \text{ and } f = \hat{f}^0. \tag{1}$$

The symbol $\subset$ denotes proper inclusion of sets.

2. Some ρ-relationships. In this section we determine some relationships in F_X^u/ρ and review Clifford's models for $|X| \leq 2$.

LEMMA 2.1. *Let $w \in F_X^u$ and $u, v \in F_X^{u1}$ such that $C(uv) \subseteq C(w)$. Let $a = L(w)u(vR(w)wL(w)u)^{-1}vR(w)$. Then $w^{-1}\,\rho\,a$.*

Proof. Clearly $(aw)\rho$ is an idempotent. By Theorem 1.3(iii) and its dual $a\rho\,\mathcal{H}\,w\rho\,\mathcal{H}\,(aw)\rho$. So $awa\,\rho\,a$ and $waw\,\rho\,w$. By Theorem 1.3(i), $w^{-1}\rho$ is the unique $\mathcal{H}$-related inverse of $w\rho$ so $a\,\rho\,w^{-1}$.

LEMMA 2.2. *Let $w \in F_X^u$ have an initial segment u. Then $\mathbf{u}\mathbf{u}^{-1}w\,\rho\,w$.*

Proof. By [2, Lemma 5.1], $w\,\rho\,\mathbf{u}v$ for some $v \in F_X^{u1}$.

The next lemma is the major step towards a decomposition of elements of F_X^u in terms of their left and right indicators, indicators and links.

LEMMA 2.3. *Suppose $w \in F_X^u$ has no segment u^{-1} such that $u \in F_X^u$ and $C(u) = C(w)$. Let $I_j = I_j(w)$ and $M_h = M_h(w)$, $1 \leq j \leq r$, $1 \leq h < r$, be respectively the indicators and links of w. Then*

$$w\,\rho\,L(w)I_1^{-1}M_1 \ldots I_{r-1}^{-1}M_{r-1}I_r^{-1}R(w).$$

Proof. Let $W_j = W_j(w)$ be the jth remainder of w. By the definitions of section 1, $L(W_j) = I_j$ and since $L(R(w)) = I_r$ then $W_r = R(w)$. Furthermore since $W_j = \mathbf{W}_j$ then there is a $z_j \in X$ and $w_j \in F_X^{u1}$ such that $W_j = z_j w_j$. Then $I_{j+1} = R(L(w_j))$ and $M_j = z_j L(w_j)$ (if they exist).

If $w = L(w)a$ for some a then since $I_1 = R(L(w))$ we get $W_1 = I_1 a$ and, by the dual of Lemma 2.2, $w\,\rho\,(L(w)I_1^{-1}I_1)a = L(w)I_1^{-1}W_1$. Alternatively $w = cd^{-1}e$ for some $c, d, e \in F_X^{u1}$ where $C(d) \neq C(w)$ and $L(w) = c\mathbf{g}$ for some inital segment g of d^{-1}. Since $I_1 = R(L(w))$ and $C(w) \neq C(d) \supseteq C(g)$ then $I_1 = h\mathbf{g}$ and $W_1 = hd^{-1}e$ for some h. So by Lemma 2.2 and its dual

$$w\,\rho\,c(\mathbf{g}\mathbf{g}^{-1}d^{-1}e) = L(w)\mathbf{g}^{-1}d^{-1}e\,\rho\,(L(w)I_1^{-1}I_1)\mathbf{g}^{-1}d^{-1}e$$
$$= L(w)I_1^{-1}h(\mathbf{g}\mathbf{g}^{-1}d^{-1})e\,\rho\,L(w)I_1^{-1}hd^{-1}e = L(w)I_1^{-1}W_1.$$

Hence we have $w\,\rho\,L(w)I_1^{-1}W_1$. If $r > 1$ then by applying the argument to w_1 and using the initial comments of the proof, $w\,\rho\,L(w)I_1^{-1}M_1I_2^{-1}W_2$. Since $W_r = R(w)$ we get the result by repeating the argument for $w_2, \ldots, w_{r-1}$.

Recall the definition of Y and f in section 1.

COROLLARY 2.4. *Let w be as in Lemma 2.3 with $C(w) = Y$. Then*

$$w\,\rho\,L(w)f(fI_1f)^{-1}fM_1f\ldots(fI_{r-1}f)^{-1}fM_{r-1}f(fI_rf)^{-1}fR(w).$$

Proof. Since $I_1 = R(L(w))$ then by Theorem 1.3(iii) the idempotent $(f(fI_1f)^{-1}fI_1)\rho$ is $\mathscr{L}$-related to $(L(w))\rho$. Hence since $I_1 = L(M_1)$ then by Lemma 2.2

$$L(w)I_1^{-1}M_1\,\rho\,L(w)f(fI_1f)^{-1}fI_1I_1^{-1}M_1\,\rho\,L(w)f(fI_1f)^{-1}fM_1.$$

Likewise $I_j = R(M_{j-1}) = L(M_j)$ if $j \neq r$ so

$$M_{j-1}I_j^{-1}M_j\,\rho\,M_{j-1}f(fI_jf)^{-1}fM_j$$

and since $I_r = R(M_{r-1}) = L(R(w))$ then

$$M_{r-1}I_r^{-1}R(w)\,\rho\,M_{r-1}f(fI_rf)^{-1}fR(w).$$

The result is now a consequence of Lemma 2.3.

Notice that if $w \in F_X^u$ and $C(w) = Y$ then by Lemma 2.1

$$w^{-1}\,\rho\,L(w)f(fR(w)wL(w)f)^{-1}fR(w). \tag{2}$$

Hence by Corollary 2.4 any $w \in F_X^u$ where $C(w) = Y$ can be expressed modulo ρ as a product of left and right Y-indicators, Y-indicators, Y-links and $\hat{f}$.

LEMMA 2.5. *Suppose $u, v, w \in F_X^{u1}$, $C(w) = Y$ and $1 < i \leq n$, $1 \leq j < n$.*
(i) *If $w = ux_1\ldots x_j$ and $R(w) = vx_1\ldots x_j$ then $w\,\rho\,uf(fvf)^{-1}fR(w)$.*
(ii) *If $w = x_i\ldots x_nux_1\ldots x_j$ then $w\,\rho\,x_i\ldots x_nuf(fuf)^{-1}fux_1\ldots x_j$.*

Proof. (i) By Theorem 1.3 the idempotent $(x_{j+1}\ldots x_n\hat{f}^{-1}(fvf)^{-1}fR(w))\rho$ is $\mathscr{L}$-related

https://doi.org/10.1017/S0017089500005668 Published online by Cambridge University Press

to $w\rho$. So

$$w \,\rho\, wx_{j+1} \ldots x_n \hat{f}^{-1}(fvf)^{-1} fR(w) \,\rho\, uf(fvf)^{-1} fR(w).$$

(ii) Let $R(w) = vx_1 \ldots x_j$. Since $R(fux_1 \ldots x_j) = R(w)$ then by (i) $w \,\rho\, x_i \ldots x_n uf(fvf)^{-1} fR(w)$ and $fux_1 \ldots x_j \,\rho\, fuf(fvf)^{-1} fR(w)$. The result follows.

DEFINITION 2.6. (i) A segment u of $w \in F_X^u$ is *v-excluded*, for $v \in F_{\bar{X}}$, if and only if v is not a subsegment of u.

(ii) An $\hat{f}$-excluded segment u of $w \in F_X^u$ is *$\hat{f}$-bounded* if and only if either $w = u$, $w = a\hat{f}u$, $w = u\hat{f}b$ or $w = a\hat{f}u\hat{f}b$ for some $a, b \in F_X^{u1}$.

(iii) Denote by G_Y the free group freely generated by

$$\{fuf, \hat{f};\, u \in F_X^u,\, C(u) \subseteq Y \text{ and } u \text{ is } \hat{f}\text{-excluded}\}$$

where $(fuf)^{-1}$ and $\hat{f}^{-1}$ denote respectively the inverses of fuf and $\hat{f}$, and f is the identity. If $Y = \{x\}$ then $\hat{f} = x$, so there exists no $\hat{f}$-excluded $u \in F_X^u$ such that $C(u) \subseteq Y$. Hence $G_{\{x\}}$ is the free group on $\{x\}$. Let G_f denote the subgroup of G_Y generated by $\hat{f}$. *We will regard $v \in F_X^u$ as an alternative expression for $u \in G_Y$ if and only if a common expression can be obtained by replacing segments of u and v that are words of G_f by their reduced forms.* For example $(fa\hat{f})^{-1} fb\hat{f}^{-1}$ denotes $(fa f\hat{f})^{-1} fbf\hat{f}^{-1} = \hat{f}^{-1}(faf)^{-1} fbf\hat{f}^{-1}$ in G_Y.

EXAMPLE 2.7. (i) Let $X = \{x\}$. By [2] $F_X^{cr} \cong G_X$. Let $w\theta$ be the reduced form in G_X of $w \in F_X^u$. By [2], for any $u, v \in F_X^u$, $u \,\rho\, v$ if and only if $u\theta = v\theta$. So $G_X = \{w\theta, w \in F_X^u\}$ with multiplication given by $u\theta \cdot v\theta = (u\theta(v\theta))\theta$.

(ii) Let $X = Y = \{x, y\}$, so $\hat{f} = xy$. Let $A = \{x^i, y^j;\, j \text{ is a non-zero integer}\}$ and H_Y be the subgroup of G_Y freely generated by $\{fuf, \hat{f};\, u \in A\}$. Let $D_x = G_{\{x\}}$, $D_y = G_{\{y\}}$ and

$$D_{xy} = \{pfhfq;\, p \in A \cup \{y^0, 1\},\, q \in A \cup \{x^0, 1\},\, h \in H_Y\}.$$

Let $S = D_x \cup D_y \cup D_{xy}$. Note that $x^0 f \,\rho\, f \,\rho\, fy^0$ and for integers i, j that $x^i y^j \,\rho\, x^{i-1} f\hat{f} f y^{j-1}$ and by Lemma 2.5(ii) $y^i x^j \,\rho\, y^i f f x^j$. With these relations we can uniquely choose $w\theta \in S$ such that $w \,\rho\, w\theta$ for all $w \in F_X^u$. It follows easily from [**2**, section 6] that for $u, v \in F_X^u$, $u \,\rho\, v$ if and only if $u\theta = v\theta$ and $F_X^{cr} \cong S = \{u\theta;\, u \in F_X^u\}$ with multiplication $u\theta \cdot v\theta = (u\theta(v\theta))\theta$. The $\mathscr{D}$-classes of S are D_x, D_y and D_{xy} and H_Y is the $\mathscr{H}$-class of f.

3. θ-forms. An element $w\theta \in F_X^u$ will be constructed from any $w \in F_X^u$. It will be shown that $w\theta \,\rho\, w$, $w\theta\theta = w\theta$ and for $u, v \in F_X^u$ that $(uv)\theta = (u\theta(v\theta))\theta$. These properties will be used in the next section to show that $S = \{w\theta;\, w \in F_X^u\}$, under the multiplication $u\theta \cdot v\theta = (u\theta(v\theta))\theta$, is a semigroup isomorphic to F_X^{cr}, and that $w\theta$ is a unique representative of the ρ-class $w\rho$.

The construction of $w\theta$ will depend on the following assumption. Recall the definition of Y and f from section 1.

ASSUMPTION 3.1. In the remainder of the paper assume for each $w \in F_X^u$ where $C(w) \subset Y$ that a unique representative $w\theta$ of the ρ-class $w\rho$ has been constructed. In particular if $|C(w)| \leq 2$ let $w\theta$ be as in Examples 2.7.

The following definition is needed for our selection of representatives of the ρ-classes fw, wf and fwf, where $C(w)\subset Y$.

DEFINITION 3.2. Define $x_0=x_{n+1}=1$. For $w\in F_X^u$ where $C(w)\subset Y$ let $i(w)$ and $j(w)$ be respectively the least and greatest integer such that whenever $0\le i\le j(w)$ or $i(w)\le i\le n+1$ then $x_i\in C(w)\cup\{1\}$. Define $w_L\in F_X^{u1}$ to be the shortest initial segment of $(w(x_0\dots x_{j(w)})^0)\theta$ such that $wx_0\dots x_{j(w)}\,\rho\, w_Lx_0\dots x_{j(w)}$. Define $w_R\in F_X^{u1}$ dually to be the shortest final segment of $((x_{i(w)}\dots x_{n+1})^0w)\theta$ such that $x_{i(w)}\dots x_{n+1}\,w\,\rho\, x_{i(w)}\dots x_{n+1}\,w_R$. Define

$$w_l=[(x_{i(w)}\dots x_{n+1})^0w]_L,\qquad w_r=[w(x_0\dots x_{j(w)})^0]_R\quad\text{and}\quad w_M=w_{lr}.$$

The next lemma indicates the need for Definition 3.2. To facilitate its proof we make another assumption.

ASSUMPTION 3.3. Suppose $v\in F_X^u$, $u=(x_i,\dots x_{n+1})^0v$ for some i and $Y\supset C(u)\supset C(v)$. Assume that v_R is a final segment of $u\theta$. Dually assume that if $u=v(x_0\dots x_j)^0$ for some j then v_L is an initial segment of $u\theta$.

The assumption can be seen to be valid, by Examples 2.7, if $|C(w)|\le 2$. We will define θ such that the assumption will be valid when $C(u)=Y$ (see comment after Lemma 3.11).

LEMMA 3.4. *Suppose $v, w\in F_X^{u1}$ and $C(x_i\dots x_{n+1}wx_0\dots x_j)\subset Y$ for some i, j. Then*

(i) *$wf\,\rho\, w_Lf$, $fw\,\rho\, fw_R$ and $fwf\,\rho\, fw_Mf$;*

(ii) *if $wx_0\dots x_j\,\rho\, vx_0\dots x_j$ then $w_L=v_L$;*

(iii) *if $x_i\dots x_{n+1}wx_0\dots x_j\,\rho\, x_i\dots x_{n+1}vx_0\dots x_j$ then $w_M=v_M$;*

(iv) *$w_{LL}=w_L$, $w_{LM}=w_{RM}=w_{MM}=w_M$.*

Proof. (i) Since $(x_0\dots x_{j(w)})^0f\,\rho\, f$ and $w(x_0\dots x_{j(w)})^0\,\rho\, w_L(x_0\dots x_{j(w)})^0$ then $wf\,\rho\, w_Lf$ and dually $fw\,\rho\, fw_R$. Hence (with duals) $fw_lf\,\rho\, f(x_{i(w)}\dots x_{n+1})^0wf\,\rho\, fwf$, so $fw_Mf=fw_{lr}f\,\rho\, fw_lf\,\rho\, fwf$.

(ii) Since $C(x_i\dots x_{n+1}wx_0\dots x_j)\subset Y$ then $0\le j<i\le n+1$, $i\ne 1$ and $j\ne n$. Suppose $vx_0\dots x_j\,\rho\, wx_0\dots x_j$. Assume $j(w)\ge j(v)$. Also assume $j\ge j(w)$; otherwise multiply both sides of the relation by suitable elements of Y. Since $x_{j(w)+1}\notin C(w)$ then $wx_0\dots x_{j(w)+1}=L(wx_0\dots x_{j(w)+1})$. If $j>j(w)$ then by Corollary 1.4 there is a segment a of $vx_0\dots x_j$ such that $L(\mathbf{a})=\mathbf{a}\,\rho\, wx_0\dots x_{j(w)+1}$. Since $x_{j(v)+1}\notin C(v)$ it follows by Theorem 1.3(iii) that $\mathbf{a}=vx_0\dots x_{j(w)+1}$ and $wx_0\dots x_{j(w)}\,\rho\, vx_0\dots x_{j(w)}$. So assume $j=j(w)$. The result is then immediate if $j(w)=j(v)$, so assume $j(w)>j(v)$. Then $C(v)\subset C(w)$ and by Assumption 3.3, v_L is an initial segment of $(w(x_0\dots x_j)^0)\theta$. But $v_Lx_0\dots x_j\,\rho\, vx_0\dots x_j\,\rho\, wx_0\dots x_j$, so w_L is an initial segment of v_L. Since $C(v)\supseteq C(v_L)$ then $j(w_L)\le j(v_L)\le j(v)$. Since $vx_0\dots x_j\,\rho\, w_Lx_0\dots x_j$ then as above v_L and hence w_L are initial segments of w_{LL}. By the definition of w_{LL} this is possible only if $w_L=w_{LL}$. So $v_L=w_L$.

(iii) Let $\mathbf{U}=\{u;\, x_i\dots x_{n+1}ux_0\dots x_j\,\rho\, x_i\dots x_{n+1}wx_0\dots x_j\}$. Select $a, b\in\mathbf{U}$ such that $i(a)\ge i(u)$ and $j(b)\le j(u)$ for all $u\in\mathbf{U}$. We first prove the existence of $d\in\mathbf{U}$ such that $i(d)=i(a)$, $j(d)=j(b)$. Suppose $j(a)>j(b)$; otherwise put $d=a$. So $x_{j(b)+1}\in C(a)$ and a

has a shortest initial segment p that includes $x_{j(b)+1}$. So $\mathbf{p}=cx_{j(b)+1}=L(\mathbf{p})$ for some $c\in F_X^{u1}$. By Corollary 1.4 and Theorem 1.3(iii) applied to $x_i\ldots x_{n+1}ax_0\ldots x_j$ and $x_i\ldots x_{n+1}bx_0\ldots x_j$ we get $x_i\ldots x_{n+1}c\,\rho\,x_i\ldots x_{n+1}bx_0\ldots x_{j(b)}$. It follows that with $d=c(x_0\ldots x_{j(b)})^{-1}$ then $d\in\mathbf{U}$. We have $i(c)\geq i(a)\geq i>j\geq j(b)\geq j(c)$ by the choice of $i(a)$ and $j(b)$, hence since $d\in\mathbf{U}$ then $i(d)=i(a)$ and $j(d)=j(b)$.

It suffices to prove $w_M=d_M$; by the same proof $v_M=d_M$. As in the proof of (ii) we assume $j=j(w)$ and likewise $i=i(w)$. Since $(x_i\ldots x_{n+1})^0wx_0\ldots x_j\,\rho\,(x_i\ldots x_{n+1})^0dx_0\ldots x_j$ then by (ii), $w_l=((x_i\ldots x_{n+1})^0d)_L$. So $j(d)=j((x_i\ldots x_{n+1})^0d)\geq j(w_l)$. Since $w_l\in\mathbf{U}$ then by the choice of d, $j(d)=j(w_l)$. By these observations

$$\begin{aligned}x_i\ldots x_{n+1}w_l(x_0\ldots x_{j(d)})^0&\\ =x_i\ldots x_{n+1}&((x_i\ldots x_{n+1})^0d)_L(x_0\ldots x_{j(d)})^0\,\rho\,x_i\ldots x_{n+1}(x_i\ldots x_{n+1})^0d(x_0\ldots x_{j(d)})^0\\ &\rho\,x_i\ldots x_{n+1}(x_{i(d)}\ldots x_{n+1})^0d(x_0\ldots x_{j(d)})^0\,\rho\,x_i\ldots x_{n+1}d_l(x_0\ldots x_{j(d)})^0.\end{aligned}$$

Since $w_M=(w_l(x_0\ldots x_{j(d)})^0)_R$ and similarly for d_M, then by the dual of (ii), $w_M=d_M$.

(iv) We have $w_Lx_0\ldots x_{j(w)}\,\rho\,wx_0\ldots x_{j(w)}$ and for $z=w_L$, w_R or w_M then $x_{i(w)}\ldots x_{n+1}zx_0\ldots x_{j(w)}\,\rho\,x_{i(w)}\ldots x_{n+1}wx_0\ldots x_{j(w)}$. The result follows by (ii) and (iii).

We now extend Definition 3.2 to include some cases where $C(w)=Y$.

Definition 3.5. Let u be a segment of $w\in F_X^u$ where $C(u)=C(w)=Y$. Let $v\in F_X^{u1}$ and $x, y\in Y\setminus C(v)$. Define

$u_L=v\theta y$ if $u=vy$ is a left Y-indicator,

$u_R=x(v\theta)$ if $u=xv$ is a right Y-indicator, and

$$u_M=\begin{cases}x(v\theta)y & \text{if } u=xvy \text{ is a } Y\text{-indicator or } Y\text{-link},\\ v_Ry & \text{if } u=vy \text{ is a left (not right) } Y\text{-indicator},\\ xv_L & \text{if } u=xv \text{ is a right (not left) } Y\text{-indicator}.\end{cases}$$

By Lemma 3.4 and Assumption 3.1 we easily get the following.

Lemma 3.6. *If they exist* $u_{LL}=u_L$, $u_{RR}=u_R$ *and* $u_{MMM}=u_{MM}$ *for any* $u\in F_X^u$, $C(u)\subseteq Y$.

Suppose $w\in F_X^u$ and $C(w)=Y$. The following operations will be used in the selection of $w\theta$. Recall the definitions of section 1 and Definitions 2.6, 3.2 and 3.5.

($\theta1$) Construct $w\theta_1$ from w by replacing each segment that is maximal with respect to being a word in $G_{\{x\}}$, for any $x\in X$, by its reduced form in $G_{\{x\}}$. Clearly $w\theta_1\,\rho\,w$.

($\theta2$) Construct $w\theta_2$ from $w\theta_1$ by replacing each segment u^{-1}, where $C(u)=Y$ and $\hat{f}^i\neq u\neq faf$ for any i and a, by $L(u)f(fR(u)uL(u)f)^{-1}fR(u)$. Likewise replace segments v^{-1} of u where $C(v)=Y$, and so on. If $u=\hat{f}^i$ or faf and u^{-1} is not preceded and succeeded by $\hat{f}$ in the spelling of w then replace u^{-1} by $fu^{-1}f$.

By (2), $w\theta_2\,\rho\,w$. Note that if u^{-1} is a segment of $w\theta_2$ and $C(u)=Y$ then $u^{-1}\in G_Y$ and $\hat{f}u^{-1}\hat{f}$ is also a segment of $w\theta_2$.

($\theta3$) Construct $w\theta_3$ from $w\theta_2$ by replacing each $\hat{f}$-bounded segment u, where

https://doi.org/10.1017/S0017089500005668 Published online by Cambridge University Press

$C(u)=Y$ and u has r Y-indicators, by

$$L(u)f(fI_1(u)f)^{-1}fM_1(u)f\ldots(fI_{r-1}(u)f)^{-1}fM_{r-1}(u)f(fI_r(u)f)^{-1}fR(u).$$

By Corollary 2.4, $w\theta_3\,\rho\,w$. Note that $w\theta_3$ is a product in F_X^u of $\hat{f}$ and $\hat{f}$-bounded segments. We see by Definitions 3.2 and 3.5 that u_M exists for any $\hat{f}$-bounded segment u of $w\theta_3$, where u is not an initial or final segment; otherwise u_L or u_R respectively exist. This property is invariant under the operations on $w\theta_3$ that follow.

($\theta 4$) Construct $w\theta_4$ from $w\theta_3$ by replacing each $\hat{f}$-bounded segment $u=x_ia$, where $C(u)=Y$, a has initial segment p such that $\mathbf{p}\,\rho\,x_{i+1}\ldots x_{n+1}$ and $L(u)=x_ib\neq u$ for some i, by $L(u)f(f\mathbf{p}^{-1}bf)^{-1}f\mathbf{p}^{-1}a$. Since $u\,\rho\,x_i\ldots x_n\mathbf{p}^{-1}a$ then by the dual of Lemma 2.5(i), $w\theta_4\,\rho\,w$.

($\theta 5$) Construct $w\theta_5$ from $w\theta_4$ by replacing each $\hat{f}$-bounded segment $u=ax_j$, where $C(u)=Y$, a has final segment q such that $\mathbf{q}\,\rho\,x_0\ldots x_{j-1}$ and $R(u)=bx_j\neq u$ for some j, by $a\mathbf{q}^{-1}f(fb\mathbf{q}^{-1}f)^{-1}fR(u)$. Then $w\theta_5\,\rho\,w$.

($\theta 6$) Construct $w\theta_6$ from $w\theta_5$ by replacing each $\hat{f}$-bounded segment $u=x_iax_j$, where a has initial and final segments p and q respectively such that $\mathbf{p}\,\rho\,x_{i+1}\ldots x_{n+1}$ and $\mathbf{q}\,\rho\,x_0\ldots x_{j-1}$ for some i and j, by $x_ia\mathbf{q}^{-1}f(f\mathbf{p}^{-1}a\mathbf{q}^{-1}f)f\mathbf{p}^{-1}ax_j$. By Lemma 2.5(ii), $w\theta_6\,\rho\,w$.

($\theta 7$) Construct $w\theta_7$ from $w\theta_6$ by replacing each $\hat{f}$-bounded segment u by u_L, u_R or u_{MM} according as u is an initial, final or other type of segment. By Lemma 3.4(i) and Definition 3.5, $w\theta_7\,\rho\,w$.

(ϕ) For $w\in F_X^u$ where $C(w)=Y$, construct $w\phi$ from w by replacing the segment of w that is maximal with respect to being a word of G_Y by its reduced form in G_Y.

(θ) Define $w\theta=w\theta_7\phi$. Then $w\theta\,\rho\,w$.

Notice that each $\hat{f}$-bounded segment of $w\theta$ of content Y is a left Y-, right Y- or Y-indicator or Y-link. Furthermore $w\theta$ is a product in F_X^u of $\hat{f}$ and $\hat{f}$-bounded segments. By ($\theta 3$), $w\theta$ has a segment that is a word in G_Y. We have $w\theta=phq$ where $h\in G_Y$, $p=1$ or $p=uf$ and $q=1$ or $q=fv$ where u and v are respectively the $\hat{f}$-bounded initial and final segments (if they exist) of $w\theta$.

The next result follows easily from the definitions.

Lemma 3.7. *Let $w=phkq$ where $h,k\in G_Y$, $p,q\in F_X^{u1}$ and $C(w)=Y$. Then $w\theta=((ph)\theta(kq)\theta)\phi$.*

Lemma 3.8. *If $w\in F_X^u$ then $w\theta\theta=w\theta$.*

Proof. The result is immediate by Assumption 3.1 if $C(w)\subset Y$ and it is easy to check that $h\theta=h$ for any $h\in G_f$ (see Definition 2.6(iii)) and that $u^{-1}\theta=(u\theta)^{-1}$ for any $u\in G_Y$. Clearly $w\theta\theta_2=w\theta$. Assume $C(w)=Y$ and v is an $\hat{f}$-bounded segment of $w\theta$. By ($\theta 7$), $v=u_L$, u_R or u_{MM} for some $u\in F_X^u$ and by Lemma 3.6 $v_L=v$, $v_R=v$ or $v_M=v$ respectively. Assume $w\theta=vf$ or fvf: by duality and Lemma 3.7 we need only prove the result in these cases. If $C(v)\subset Y$ then we easily see $w\theta\theta=w\theta$. So assume $C(v)=Y$. Since ($\theta 4$), ($\theta 5$) and ($\theta 6$) are used in the construction of v it can be easily checked that if $v=x_ipa$ for some i, where $\mathbf{p}\,\rho\,x_{i+1}\ldots x_{n+1}$ then $L(v)=v$. Hence if v is a segment of $w\theta\theta_3$ then v is not modified by ($\theta 4$); similarly v is invariant under ($\theta 5$) and ($\theta 6$). If $w\theta=vf$ then

$v = L(v)$, so $w\theta\theta_3 = vf(fR(v)f)^{-1}fR(v)f$ and since v is invariant under $(\theta 4)$, $(\theta 5)$, $(\theta 6)$ and $(\theta 7)$ then $w\theta\theta = w\theta$. Now suppose $w = fvf$. If v is a left Y-, right Y-, or Y-indicator then we likewise get $w\theta\theta = w\theta$. Alternatively if v is a Y-link then $w\theta\theta_3 = fL(v)f(fL(v)f)^{-1}fvf(fR(v)f)^{-1}fR(v)f$ and as above we get the result.

Reasoning in a similar way we get the following.

Corollary 3.9. *If* $w \in F_X^u$ *and* $C(w) = Y$ *then* $w\theta_i\theta = w\theta$ for $1 \leq i \leq 7$.

The last three results will be used several times without comment in the following lemmas. The next result is like Corollary 2.4 but without restrictions on inverses.

Lemma 3.10. *Let* $w \in F_X^u$ *where* $C(w) = Y$ *and* $w = w\theta_2$. *Then* $w\theta = (L(w)f(fI_1(w)f)^{-1}fW_1(w))\theta$ *and* $(fw)\theta = (fL(w)f(fI_1(w)f)^{-1}fW_1(w))\theta$. *Furthermore if* $W_1(w) = w \neq R(w)$ *and* $\hat{f}$ *is not an initial segment of* w *then* $(fW_1(w))\theta = (fM_1(w)f(fI_2(w)f)^{-1}fW_2(w))\theta$.

Proof. Let b be the $\hat{f}$-bounded initial segment of w; if no such segment exists then, by $(\theta 2)$, w has initial segment $\hat{f}$ and the results follow. If $L(b) = L(w)$ then the expressions for $w\theta$ and $(fw)\theta$ are consequences of $(\theta 3)$. Suppose $L(b) \neq L(w)$. So $w = b\hat{f}c$ for some $b, c \in F_X^u$ such that $C(b) \subset Y$. Then $L(w) = bqx_j$ for some $j < n$, $q = x_0 \ldots x_{j-1}$ and $R(L(w)) = I_1(w) = dqx_j$ for some d. By $(\theta 5)$ and (ϕ), $(L(w)f)\theta = (bqq^{-1}f(fdqq^{-1}f)^{-1}fdqx_jf)\theta$. Since $(bqq^{-1})_L = b_L$ and $(dqq^{-1})_M = d_M$ by Lemma 3.4(ii), (iii) then by $(\theta 7)$ $(L(w)f)\theta = (bf(fdf)^{-1}fI_1(w)f)\theta$. We have $(fW_1(w))\theta = (fd\hat{f}c)\theta = (fdf\hat{f}c)\theta$. So by (ϕ), $(L(w)f(fI_1(w)f)^{-1}fW_1(w))\theta = (bf\hat{f}c)\theta = w\theta$. To get the second equality pre-multiply by f throughout the proof.

With the additional restrictions b and $I_2(w)$ exist. To prove the result for $(fW_1(w))\theta$ proceed as above, using $(\theta 3)$ if $I_2(w)$ is an indicator of b or $(\theta 5)$ applied to $M_1(w)$ otherwise.

We now deduce a result for θ like Lemma 2.5(i).

Lemma 3.11. *Let* $w = x_ia \in F_X^u$ *where* $C(w) = Y$, $L(w) = x_ib$ *and* a *has initial segment* p *such that* $x_ip \neq \hat{f}$ *and* $\mathbf{p}\,\rho\, x_{i+1} \ldots x_{n+1}$ *for some* i. *Then*

(i) $w\theta = (L(w)f(f\mathbf{p}^{-1}bf)^{-1}f\mathbf{p}^{-1}a)\theta$ *and* $(fw)\theta = (fL(w)f(f\mathbf{p}^{-1}bf)^{-1}f\mathbf{p}^{-1}a)\theta$ *and*

(ii) *if* $r = x_i \ldots x_{n+1}$ *and* $w = rs$ *then* $(fr^{-1}w)\theta = (fs)\theta$.

Proof. (i) We may assume $w = w\theta_2$, since $w\theta_2\theta = w\theta$, $L(w\theta_2) = L(w)$ and $(f\mathbf{p}^{-1}(a\theta_2))\theta = (f\mathbf{p}^{-1}a)\theta$. By Lemma 3.10(i) $w\theta = (L(w)f(fI_1(w)f)^{-1}fW_1(w))\theta$. If $W_1(w) \neq w$ then $L(w) \neq I_1(w)$ so $C(b) = Y$ and $L(\mathbf{p}^{-1}a) = \mathbf{p}^{-1}b$. Hence by Lemma 3.10(i) $(f\mathbf{p}^{-1}a)\theta = (f\mathbf{p}^{-1}bf(fI_1(w)f)^{-1}fW_1(w))\theta$ and the result follows, using (ϕ). Suppose $W_1(w) = w$, so $L(w) = I_1(w)$. If $w = R(w)$ then w is $\hat{f}$-excluded so by $(\theta 3)$ and $(\theta 4)$ (acting in particular on $R(w)$) $w\theta_4 = L(w)f(fI_1(w)f)^{-1}fL(w)f(f\mathbf{p}^{-1}bf)^{-1}f\mathbf{p}^{-1}a$ and the result follows. If $w \neq R(w)$ then by Lemma 3.10

$$w\theta = (L(w)f(fI_1(w)f)^{-1}fM_1(w)f(fI_2(w)f)^{-1}fW_2(w))\theta.$$

Let $M_1(w) = x_ic$. Then by $(\theta 4)$, $(fM_1(w)f)\theta = (fI_1(w)f(f\mathbf{p}^{-1}bf)^{-1}f\mathbf{p}^{-1}cf)\theta$. Since

$L(\mathbf{p}^{-1}a) = \mathbf{p}^{-1}c$ then by Lemma 3.10(i) $(f\mathbf{p}^{-1}a)\theta = (f\mathbf{p}^{-1}cf(fI_2(w)f)^{-1}fW_2(w))\theta$. Combining these expressions we get the result.

(ii) Let $L(w) = rt$. We first prove that $(fr^{-1}rtf)\theta = (ftf)\theta$. Since $L(r^{-1}rt) = r^{-1}rt$ then by $(\theta 3)$, (ϕ) and $(\theta 7)$, we need to show $(r^{-1}rt)_{MM} = t_{MM}$. We have $t = dx$ for some d and $x \in Y \setminus C(d)$ so $(r^{-1}rdx)_M = (r^{-1}rd)_R x = d_R x$ by Definition 3.5 and the dual of Lemma 3.4(ii). If $C(dx) \subset Y$ then by Lemma 3.4(iii) $(dx)_M = (d_R x)_M$. If $C(dx) = Y$ and $R(dx) \neq dx$ then by Definition 3.5 $(dx)_M = d_R x$. Suppose $C(dx) = Y$ and $R(dx) = dx$. We have $x_{i(d)} \ldots x_{n+1} d_R \,\rho\, x_{i(d)} \ldots x_{n+1} d$. If $C(d_R) \subset C(d)$ then by comparing right indicators we get $x_k \ldots x_n d_R \,\rho\, d$ for some $k \leq n$. But then $x_{i(d)} \ldots x_{n+1} d_R \,\rho\, x_{i(d)} \ldots x_{k-1}(x_k \ldots x_n)^2 d_R$. This is not possible since there exists a homomorphism from F_X^u/ρ onto the free cyclic group $G_{\{x\}}$ taking generators to x. Hence $C(d_R) = C(d)$ and $R(d_R) \,\rho\, R(d) = d$ so by its definition $d_R = R(d_R)$. Thus $d_R x$ and dx are ρ-related Y-indicators and by Theorem 1.3(iii) and Definition 3.5 $(d_R x)_M = (dx)_M$. Thus in all cases $(r^{-1}rt)_{MM} = t_{MM}$.

We have $(fr^{-1}w)\theta = (fs)\theta$ when $i = 1$ or n, by (ϕ) and $(\theta 1)$. Assume the result for $i > j > 1$. Let $i = j$ and proceed by induction. Since $r = x_i p$ then by comparing expressions for $w\theta$ from (i) and Lemma 3.10(i) we get $((fI_1(w)f)^{-1}fW_1(w))\theta = ((fp^{-1}bf)^{-1}fp^{-1}a)\theta = ((ftf)^{-1}fs)\theta$, by Lemma 3.7 and induction. So by Lemma 3.10(i) and the first part of the proof

$$(fr^{-1}w)\theta = (fr^{-1}rtf(fI_1(w)f)^{-1}fW_1(w))\theta = (ftf(ftf)^{-1}fs)\theta = (fs)\theta.$$

Recall Assumption 3.3. We can now see that it is valid when $C(u) = Y$. Say $u = (x_i \ldots x_{n+1})^0 v$ where $C(u) \supset C(v)$. By Lemmas 3.10 and 3.11 $u\theta = (L(u)f(ftf)^{-1}fv)\theta$ where $L(u) = (x_i \ldots x_{n+1})^0 t$. Since $C(v) \subset Y$ then $(fv)\theta = fv_R$, so $u\theta$ has final segment v_R.

LEMMA 3.12. *Suppose $a, b \in F_X^u$, $x, y \in Y \setminus C(a)$ and $a \,\rho\, b$. If xay is a Y-indicator or Y-link then $(fxayf)\theta = (fxbyf)\theta$. If ay is a left Y-indicator then $(ayf)\theta = (byf)\theta$.*

Proof. If $(\theta 4)$, $(\theta 5)$ and $(\theta 6)$ do not vary the segments xay and xby then the result is easy to check, using Definition 3.5. A similar statement applies for left Y-indicators. Let xay be a Y-indicator. Since $(\theta 4)$ and $(\theta 5)$ do not vary Y-indicators assume $x = x_i$, $y = x_j$ and $a = pcq$ where $\mathbf{p} \,\rho\, x_{i+1} \ldots x_{n+1}$ and $\mathbf{q} \,\rho\, x_0 \ldots x_{j-1}$ for some i and j. Since xay is a Y-indicator then $i > j$ and xay is $\hat{f}$-excluded so by $(\theta 6)$, $(\theta 7)$ and (ϕ), $(fxayf)\theta = f(xa\mathbf{q}^{-1})_M f(f(\mathbf{p}^{-1}a\mathbf{q}^{-1})_M f)^{-1} f(\mathbf{p}^{-1}ay)_M f$. But by Corollary 1.4 $b = rds$ for some r, d, s such that $\mathbf{r} \,\rho\, \mathbf{p}$ and $\mathbf{s} \,\rho\, \mathbf{q}$ (with $r = 1$ or $s = 1$ if and only if $p = 1$ or $q = 1$ respectively). By Lemma 3.4(iii) then $(xa\mathbf{q}^{-1})_M = (xb\mathbf{s}^{-1})_M$, $(\mathbf{p}^{-1}a\mathbf{q}^{-1})_M = (\mathbf{r}^{-1}b\mathbf{s}^{-1})_M$ and $(\mathbf{p}^{-1}ay)_M = (\mathbf{r}^{-1}by)_M$ so $(fxayf)\theta = (fxbyf)\theta$.

Now let xay be a Y-link with $x = y = x_i$ for some i, where p is an initial segment of a, $\mathbf{p} \,\rho\, x_{i+1} \ldots x_{n+1}$ and $L(xay) = xd$. By Lemma 3.11(i) $(fxayf)\theta = (fxdf(f\mathbf{p}^{-1}df)^{-1}f\mathbf{p}^{-1}ayf)\theta$. This equation still holds if $xp = \hat{f}$, by Lemma 3.11(ii) and (ϕ). We have $xby = xqe$ and $L(xby) = xg$ where $\mathbf{p} \,\rho\, \mathbf{q}$. Since xd and xg are ρ-related Y-indicators (so $d \,\rho\, g$) and $C(\mathbf{p}^{-1}d) \subset Y$ we need only show that $(f\mathbf{p}^{-1}ayf)\theta = (f\mathbf{q}^{-1}byf)\theta$. This follows since $(\mathbf{p}^{-1}ay)_M = (\mathbf{p}^{-1}a)_R y = (\mathbf{q}^{-1}b)_R y = (\mathbf{q}^{-1}by)_M$ by the dual of Lemma 3.4(ii). By an analysis similar to the first paragraph we get the left Y-indicator result.

LEMMA 3.13. *Let* $w \in F_X^u$ *and* $x \in Y$. *Then*
(i) $(xw)\theta = (x(w\theta))\theta$ *and* (ii) $(fw)\theta = (f(w\theta))\theta$.

Proof. (i) We may assume $w = w\theta_2$. Suppose $C(xw) = Y$: otherwise the result follows by Assumption 3.1. By Theorem 1.3(iii) and Lemma 3.12 $(fL(xw)f(fI_1(xw)f)^{-1})\theta = (fL(x(w\theta))f(fI_1(x(w\theta))f)^{-1})\theta$. So by Lemma 3.10 we need to show $(fW_1(xw))\theta = (fW_1(x(w\theta)))\theta$. If $W_1(xw) \neq xw$ then $I_1(xw) = I_1(w)$. Using Theorem 1.3(iii) it is easy to check, since $w\theta \, \rho \, w$ that $I_1(x(w\theta)) = I_1(w\theta)$. Then $W_1(xw) = W_1(w)$ and $W_1(x(w\theta)) = W_1(w\theta)$. Equating the expressions for $w\theta$ and $w\theta\theta$ from Lemma 3.10(i) we get $(fW_1(w))\theta = (fW_1(w\theta))\theta$ and hence the result. If $W_1(xw) = xw = R(xw)$ we get the result by the dual of Lemma 3.12.

Suppose $W_1(xw) = xw \neq R(xw)$. If $\hat{f}$ is not an initial segment of xw then by Lemma 3.10(ii), $(fW_1(xw))\theta = (fM_1(xw)f(fI_2(xw)f)^{-1}fW_2(xw))\theta$. This equation also holds if $xw = \hat{f}a$. To see this let $M_1(xw) = \hat{f}b$. If $W_2(xw) = x_i \ldots x_n a$ and $I_2(xw) = x_i \ldots x_n b$ for some $i \leq n$ then $(fW_2(xw))\theta = (fI_2(xw)f(fbf)^{-1}fa)\theta$ by Lemma 3.11. Alternatively if $W_2(xw)$ is a segment of a, by Lemma 3.10(i) $(fa)\theta = (fbf(fI_2(xw)f)^{-1}fW_2(xw))\theta$, and since $(fW_1(xw))\theta = (\hat{f}fa)\theta$ we get the equation. Observe that since $W_1(xw) = xw$ then $I_2(xw) = I_1(w)$ so by Lemma 3.10(i), $w\theta = (L(w)f(fI_1(w)f)^{-1}fW_2(xw))\theta$. Likewise $w\theta\theta = L(w\theta)f(fI_1(w\theta)f)^{-1}fW_2(x(w\theta)))\theta$. Since $w\theta = w\theta\theta$ then by Lemma 3.12 $(fW_2(xw))\theta = (fW_2(x(w\theta)))\theta$. Hence since $M_1(xw) = xL(w) \, \rho \, xL(w\theta) = M_1(x(w\theta))$, then by Lemma 3.12, $(fW_1(xw))\theta = (fW_1(x(w\theta)))\theta$.

(ii) It follows by straightforward induction, based on (i), that $(\hat{f}w)\theta = (x_1 \ldots x_n w)\theta = (x_1 \ldots x_n(w\theta))\theta = (\hat{f}(w\theta))\theta$. It is easily seen that $(\hat{f}^{-1})\theta = \hat{f}^{-1}$. So by Lemma 3.7

$$(f(w\theta))\theta = (\hat{f}^{-1}\hat{f}(w\theta))\theta = (\hat{f}^{-1}(\hat{f}(w\theta))\theta)\phi = (\hat{f}^{-1}(\hat{f}w)\theta)\phi = (\hat{f}^{-1}\hat{f}w)\theta = (fw)\theta.$$

The next result is the key lemma of the paper. It will be used to show that $\{w\theta;\, w \in F_X^u\}$ with multiplication $u\theta \cdot v\theta = (u\theta(v\theta))\theta$ is a semigroup.

LEMMA 3.14. *Let* $u, v \in F_X^u$ *where* $C(uv) \subseteq Y$. *Then* $(uv)\theta = (u\theta(v\theta))\theta$.

Proof. The result is immediate if $C(uv) \subset Y$ (by Assumption 3.1), or if $|C(u)| = 1$ (by Lemma 3.13(i)) since then $u\theta = u$ by Assumption 3.1. Assume the result for $C(u) \subset \mathbf{U}$, some $\mathbf{U} \subseteq Y$, and proceed by induction. Suppose $C(u) = \mathbf{U}$ and $C(uv) = Y$.

Suppose $C(u) \subset Y$. Then $L(uv) = uv_1$ and $L(u\theta(v\theta)) = u\theta v_2$ where $v_1 \, \rho \, v_2$ by Corollary 1.4. Either $I_1(uv) = RL(uv) = u_1 v_1$ and $I_1(u\theta(v\theta)) = u_2 v_2$ where $u_1 \, \rho \, u_2$ by Corollary 1.4 or $I_1(uv) = I_1(v) \, \rho \, I_1(v\theta) = I_1(u\theta(v\theta))$ (see Theorem 1.3(iii)). By Lemmas 3.10(i) and 3.12 we need to show $(fW_1(uv))\theta = (fW_1(u\theta(v\theta)))\theta$. If $I_1(uv) = u_1 v_1$ then $W_1(uv) = u_1 v$ and $W_1(u\theta(v\theta)) = u_2(v\theta)$; also $u_1 = xa_1$, $u_2 = xa_2$ for some $x \in Y \setminus C(a_1)$ and $a_1\theta = a_2\theta$ by Assumption 3.1 and Theorem 1.3(iii). By Lemmas 3.13, 3.8 and the induction assumption $(fW_1(uv))\theta = (fxa_1 v)\theta = (f(xa_1 v)\theta)\theta = (f(x(a_1 v)\theta)\theta)\theta = (f(x(a_1\theta(v\theta))\theta)\theta)\theta = (f(x(a_2\theta(v\theta))\theta)\theta)\theta = (f(x(a_2(v\theta))\theta)\theta)\theta = (f(xa_2(v\theta))\theta)\theta = (fxa_2(v\theta))\theta = (fW_1(u\theta(v\theta)))\theta$. Alternatively $I_1(uv) = I_1(v)$, so $W_1(uv) = W_1(v)$. Since $L(v) \, \rho \, L(v\theta)$ and $I_1(v) \, \rho \, I_1(v\theta)$,

https://doi.org/10.1017/S0017089500005668 Published online by Cambridge University Press

then equating the expressions from Lemma 3.10(i) for $v\theta$ and $v\theta\theta$, using Lemma 3.12, we get $(fW_1(uv))\theta=(fW_1(u\theta(v\theta)))\theta$.

Now suppose $C(u)=Y$ and $u\theta_3=a\hat{f}w$ where w is an $\hat{f}$-excluded segment or $w=1$. Clearly $(uv)\theta=(u\theta_3(v\theta_3))\theta$, so by Lemmas 3.7 and 3.9 we need only prove $(\hat{f}wv)\theta=((\hat{f}w)\theta(v\theta))\theta$, or equivalently $(fwv)\theta=((fw)\theta(v\theta))\theta$. If $C(w)\subset Y$ then, with $p=x_{i(w)}\ldots x_{n+1}$, we get $(fwv)\theta=(fp^{-1}pwv)\theta=(f(p^{-1}pwv)\theta)\theta=(f((p^{-1}pw)\theta(v\theta))\theta)\theta=(f((p^{-1}pw_R)\theta(v\theta))\theta)\theta=(f(p^{-1}pw_R(v\theta))\theta)\theta=(fp^{-1}pw_R(v\theta))\theta=(fw_R(v\theta))\theta=((fw)\theta(v\theta))\theta$ by Lemmas 3.11(ii), 3.13(ii), 3.8, the induction assumption, Definition 3.2 and ($\theta 7$). Now suppose $C(w)=Y$, so $w=R(w)=xb$ for some $x\in Y\setminus C(b)$. If ($\theta 4$), ($\theta 5$) and ($\theta 6$) do not vary w then $(fwv)\theta=(f(xbv)\theta)\theta=(f(x((b\theta)(v\theta))\theta)\theta)\theta=(f(x(b\theta)(v\theta))\theta)\theta=(fx(b\theta)(v\theta))\theta=((fw)\theta(v\theta))\theta$ by Lemma 3.13, the induction assumption, Definition 3.5 and ($\theta 7$). Finally suppose $w=x_ib$, $L(w)=x_ic$, and p is an initial segment of b such that $\mathbf{p}\,\rho\,x_{i+1}\ldots x_{n+1}$ for some i. By Lemma 3.11 $(fwv)\theta=(fx_icf(f\mathbf{p}^{-1}cf)^{-1}f\mathbf{p}^{-1}bv)\theta$. Since $C(\mathbf{p}^{-1}b)\subset Y$ then by the above $(f\mathbf{p}^{-1}bv)\theta=((f\mathbf{p}^{-1}b)\theta(v\theta))\theta$. But by Lemma 3.11(i), $(fw)\theta=(fx_icf(f\mathbf{p}^{-1}cf)^{-1}f\mathbf{p}^{-1}b)\theta$ so $(fwv)\theta=((fw)\theta(v\theta))\theta$.

Remark. We have not yet shown, for $C(w)=Y$, that $w\theta$ is a unique representative of the class $w\rho$. This will follow from Theorem 4.1.

4. A model for F_X^{cr}.

Theorem 4.1. *Let $S=\{w\theta;\,w\in F_X^u\}$ with a binary operation defined by $u\theta\cdot v\theta=(u\theta(v\theta))\theta$. Then $S\cong F_X^{cr}$.*

Proof. For any $u,v,w\in F_X^u$ we have by Lemma 3.14 that $u\theta\cdot v\theta=(uv)\theta$ so $(u\theta\cdot v\theta)\cdot w\theta=(uv)\theta\cdot w\theta=(uvw)\theta=u\theta\cdot(vw)\theta=u\theta\cdot(v\theta\cdot w\theta)$. Hence S is a semigroup.

We will now check that S is completely regular. For any $u,v\in F_X^u$, since $u\theta\,\rho\,u$ then by Theorem 1.3(iii) $u\theta\,\mathscr{L}\,v\theta$ only if $L(u)\,\rho\,L(v)$. Conversely suppose $L(u)\,\rho\,L(v)$. Since $L(u\theta)\,\rho\,L(u)$, assume $u=u\theta$ and $v=v\theta$. By Lemma 3.10(i) $u\theta=(L(u)f(fI_1(u)f)^{-1}fW_1(u))\theta$, by Lemma 3.12 $(L(u)f)\theta=(L(v)f)\theta$ and by ($\theta 2$) and Lemma 3.9 $(fW_1(u)(fW_1(u)f)^{-1})\theta=(fW_1(u)f(fW_1(u)f)^{-1}f)\theta=f$. So with $a=(fW_1(u)f)^{-1}fI_1(u)f(fI_1(v)f)^{-1}fW_1(v)$ we get by (ϕ) and Lemma 3.10(i) that $u\theta\cdot a\theta=(ua)\theta=v\theta$. Hence $u\theta\,\mathscr{L}\,v\theta$ if and only if $L(u)\,\rho\,L(v)$. There is a dual result for $\mathscr{R}$. But then $u\theta\,\mathscr{H}\,(L(u)f(fR(u)L(u)f)^{-1}fR(u))\theta$, which is an idempotent. So S is a union of groups. We have $u^{-1}\theta=(u\theta)^{-1}$ in S by Theorem 1.3(iii) and ($\theta 2$).

By Lemma 3.14 S is generated by $\{x\theta;\,x\in X\}$. So by the free property of $F_X^u/\rho\cong F_X^{cr}$ there is a surjective homomorphism $\alpha\colon F_X^u/\rho\to S$ given by $(x\rho)\alpha=x\theta$ for all $x\in X$. By the definition of multiplication in S and Lemma 3.14 then $(w\rho)\alpha=w\theta$ for all $w\in F_X^u$. Since $w\theta\,\rho\,w$ then α is injective, so α is an isomorphism.

Notice that since α in this proof is an isomorphism then, for each $w\in F_X^u$, $w\theta$ is a unique representative of $w\rho$. This is in accordance with Assumption 3.1.

Some properties of the model S for F_X^{cr} can be easily deduced. Recall the definitions of section 1 and Definitions 2.6, 3.2 and 3.5. We first characterize the $\hat{f}$-bounded segments of an element of S of content Y.

Define $a \in F_X^u$ to be *Y-basic* if and only if a satisfies the following properties.

(i) $C(a) \subseteq Y$, $a = a_{MM}$ and a is $\hat{f}$-excluded.

(ii) Suppose $p, q \in F_X^u$ where $(\mathbf{p}, x_i \dots x_n) \in \rho$ and $(\mathbf{q}, x_1 \dots x_j) \in \rho$ for some $i \geq 1$ and $j \leq n$. If $C(a) = Y$ and $a = pb$ or $a = bq$ for some b then a is a left or right Y-indicator respectively. If $C(a) = Y$ then $a \neq pdq$ for any d.

Define $a \in F_X^{u1}$ to be *left* (*right*) *Y-basic* if and only if a satisfies (ii) and

(i′) $C(a) \subseteq Y$, $a = a_L$ (respectively a_R), and a is $\hat{f}$-excluded.

Define H_Y to be the subgroup of G_Y freely generated by $\{faf, \hat{f};\ a$ is Y-basic$\}$.

Define $D_Y = \{ufhfv;\ u$ and v are respectively left Y- and right Y-basic and $h \in H_Y\}$.

COROLLARY 4.2. *Let $w \in F_X^u$ and $C(w) = Y$. Then there is a unique left Y-basic u, a unique right Y-basic v and a unique $h \in H_Y$ such that $w\theta = ufhfv$. In S the $\mathscr{D}$-class of $w\theta$ is $\{r\theta; C(r) = C(w)\} = D_Y$, the $\mathscr{R}$-class of $w\theta$ is $\{r\theta; L(r\theta) = L(w\theta)\}$, the $\mathscr{L}$-class of $w\theta$ is $\{r\theta; R(r\theta) = R(w\theta)\}$, and the $\mathscr{H}$-class of $w\theta$ is the free group ufH_Yfv.*

Proof. The expression for $w\theta$ follows from its definition; u is the $\hat{f}$-bounded initial segment of $w\theta$ if it exists, otherwise $u = 1$ (there is a dual statement for v). By Theorem 1.3(ii), $\{r\theta; C(r) = C(w)\}$ is the $\mathscr{D}$-class of $w\theta$. It can be directly checked that the free generators of H_Y are in S (by a proof along the lines of that for Lemma 3.8), as are uf and fv for any left Y-basic u and right Y-basic v. So by Lemma 3.7, $D_Y \subset S$ and by Theorem 1.3(ii), D_Y is the $\mathscr{D}$-class of $w\theta$. The $\mathscr{L}$ and $\mathscr{R}$-class characterizations are by Lemmas 3.10(i) and 3.12. The $\mathscr{H}$-class characterization then follows by the definition of D_Y.

Notice that by Theorem 4.1, the construction of $w\theta$ from $w \in F_X^u$ may be simplified by replacing w by an alternative ρ-related element.

We observe that the representative $w\theta$ of the ρ-class of $w \in F_X^u$ is uniquely defined modulo the choice of $u\theta$ for all $u \in F_X^u$ where $C(u) \subset C(w)$. To see this first note that the operations $(\theta 1), \dots, (\theta 6)$ and (ϕ) just manipulate the spelling of w. By Definitions 3.2 and 3.5 the application of $(\theta 7)$ requires knowledge of the spelling of $u\theta$ for some $u \in F_X^u$, $C(u) \subset C(w)$.

As mentioned in the introduction our characterization of F_X^{cr} is different from that of Gerhard [**4**]. He determines a set of free generators of $(H_Y)\rho$ that are unique up to solution of the word problem in F_X^u/ρ for words of content less than Y. By this approach he gets many expressions of the form $(faf)\rho$, $faf \in F_X^u$, for a generator. It is difficult to determine, using the solution to the word problem for words of content less than Y, whether two of these expressions denote the same generator. Gerhard's model for F_X^{cr}, based on Petrich's structure theorem for completely regular semigroups [**6**, Theorem 3], is a union of Rees matrix semigroups. The Rees matrix semigroup corresponding to the $\mathscr{D}$-class of elements of content Y has structure group $(H_Y)\rho$.

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