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Vojta's inequality and rational and integral points of bounded degree on curves

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Abstract

Let $C \subset C_1 \times C_2$ be a curve of type (d_1, d_2) in the product of the two curves C_1 and C_2. Let ν be a positive integer. We prove that if a certain inequality involving d_1, d_2, ν, and the genera of the curves C_1, C_2, and C is satisfied, then the set of points $\{P \in C(\bar{k}) \mid [k(P):k] \leqslant \nu\}$ is finite for any number field k. We prove a similar result for integral points of bounded degree on C. These results are obtained as consequences of an inequality of Vojta which generalizes the Roth–Wirsing theorem to curves.

1. Introduction

In [Voj92], Vojta proved the following theorem.

Theorem 1 (Vojta). *Let C be a nonsingular curve defined over a number field k. Let X be a regular model for C over the ring of integers of k. Let K be the canonical divisor of C, A an ample divisor on C, and D an effective divisor on C without multiple components. Let S be a finite set of places of k. Let ν be a positive integer and let $\epsilon > 0$. Then*

$$m_S(D,P) + h_K(P) \leqslant d_a(P) + \epsilon h_A(P) + O(1) \tag{1}$$

for all points $P \in C(\bar{k}) \setminus \operatorname{Supp} D$ with $[k(P):k] \leqslant \nu$.

Here h_D is a logarithmic height associated to the divisor D, $m_S(D,P)$ is a proximity function, and $d_a(P)$ is the arithmetic discriminant of [Voj91], whose definition we recall below. We refer the reader to [Lan83], [Voj87], and [Voj92] for definitions and properties of heights and proximity functions.

The inequality (1) is a vast generalization of the theorems of Roth and Wirsing. In particular, it implies Faltings' theorem (Mordell's conjecture). As a consequence of (1), Song and Tucker [ST01] derived the following corollary.

Corollary 1 (Song, Tucker, Vojta). *Let C and C' be nonsingular curves of genus g and g', respectively, defined over a number field k. Let $\phi : C \to C'$ be a dominant k-morphism. If*

$$g - 1 > (\nu + g' - 1)\deg\phi \tag{2}$$

for some positive integer ν, then the set

$$\{P \in C(\bar{k}) \mid [k(P):k] \leqslant \nu \text{ and } k(\phi(P)) = k(P)\} \tag{3}$$

is finite.

Vojta [Voj92] noted the case $C' = \mathbb{P}^1$ of the corollary. Note that the condition $k(\phi(P)) = k(P)$ in Corollary 1 precludes one from deducing a finiteness result for algebraic points on C

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with $[k(P) : k] \leqslant \nu$. Of course, this condition in the corollary is necessary (consider, for example, $\nu = 2$, C a hyperelliptic curve of genus $g > 3$, and $\phi : C \to \mathbb{P}^1$ with $\deg \phi = 2$). If we are given more than one dominant morphism of C to a curve where (2) holds, it is natural to try to prove a finiteness result without the $k(\phi(P)) = k(P)$ condition in (3). Clearly we need the maps to be independent in some sense. More precisely, we assume that we are given a morphism ϕ of C into a product of two curves such that ϕ is birational onto its image. In addition to rational points, we study integral points on C.

Let S be a finite set of places of k and let $\mathcal{O}_{k,S}$ denote the ring of S-integers of k. Let D be an effective divisor on C. If $D \neq 0$, we call a set $T \subset C(\bar{k}) \setminus \operatorname{Supp} D$ a set of (D, S)-integral points on C if there exists an affine embedding $C \setminus \operatorname{Supp} D \subset \mathbb{A}^m$ such that every point $P \in T$ has S-integral coordinates, i.e. each coordinate of P in $\mathbb{A}^m$ lies in the integral closure of $\mathcal{O}_{k,S}$ in $\bar{k}$. If $D = 0$, then we call any subset of $C(\bar{k})$ a set of (D, S)-integral points. Our main theorem can now be stated as follows.

Theorem 2. *Let C, C_1, and C_2 be nonsingular curves of genus g, g_1, and g_2, respectively, all defined over a number field k. Let S be a finite set of places of k. Let $\phi : C \to C_1 \times C_2$ be a morphism defined over k that is birational onto its image. Let π_1 and π_2 be the projections of $C_1 \times C_2$ onto the first and second factors, respectively. Suppose that $\pi_1 \circ \phi$ and $\pi_2 \circ \phi$ are dominant morphisms and let $d_1 = \deg \pi_1 \circ \phi$ and $d_2 = \deg \pi_2 \circ \phi$. Let $D = \sum_{i=1}^{r} P_i$ be an effective divisor on C, defined over k, with $P_1, \ldots, P_r$ distinct points of $C(\bar{k})$. If*

$$2g - 2 + r > \max\{2(\nu + g_1 - 1)d_1, 2(\nu + g_2 - 1)d_2, (\nu + 2g_1 - 2)d_1 + (\nu + 2g_2 - 2)d_2\} \qquad (4)$$

for some positive integer ν, then any set of (D, S)-integral points

$$T \subset \{P \in C(\bar{k}) \mid [k(P) : k] \leqslant \nu\} \qquad (5)$$

is finite. In particular, if (4) is satisfied with $r = 0$, then the set

$$\{P \in C(\bar{k}) \mid [k(P) : k] \leqslant \nu\}$$

is finite.

The invariants g, g_1, g_2, d_1, and d_2 of Theorem 2 are not unrelated. A classical inequality of Castelnuovo [ACGH85, p. 366] states an upper bound on the genus g of C in terms of the other invariants:

$$g \leqslant (d_1 - 1)(d_2 - 1) + d_1 g_1 + d_2 g_2. \qquad (6)$$

This places some restrictions on the applicability of Theorem 2. Nonetheless, the inequality (4) is satisfied in a variety of interesting situations, some of which are discussed in the next section.

The main new tool in the proof of Theorem 2 is an inequality, given in Theorem 7, relating the arithmetic discriminant on a model of C (over $\mathcal{O}_k$) to the arithmetic discriminants on models of C_1 and C_2. This inequality can be viewed as an arithmetic analog of Castelnuovo's inequality (6) (see the comments after Theorem 7). The proof of Theorem 2 is easily described as follows. Combining the new inequality of Theorem 7 with an inequality of Song and Tucker bounding the arithmetic discriminant in terms of certain height functions, one obtains an upper bound for an arithmetic discriminant on C in terms of a height function on C, $[k(P) : k]$, g_1, g_2, d_1, and d_2. Thus, in view of the left-hand side of Vojta's inequality, if g and r are large enough compared to ν, g_1, g_2, d_1, and d_2, then any set T of (D, S)-integral points as in (5) must be finite. A calculation shows that g and r are large enough precisely when (4) holds, yielding Theorem 2.

2. Some examples and corollaries

We first give two examples which show that the inequality (4) is sharp in the sense that Theorem 2 is false if '>' is replaced by '$\geqslant$' in (4).

Example 1. Let C be a nonsingular curve, defined over a number field k, of bidegree (d_1, d_2) on $C_1 \times C_2 = \mathbb{P}^1 \times \mathbb{P}^1$ with $d_1 \geqslant d_2 > 0$. Let $P, Q \in \mathbb{P}^1(k)$ be two points above which $\phi_2 = \pi_2|_C$ is unramified, and let $D = P + Q$. Over sufficiently large number fields k, there are infinitely many k-rational (D, S)-integral points on $\mathbb{P}^1$. Pulling back these integral points by ϕ_2, we obtain infinitely many $(\phi_2^* D, S)$-integral points on C (of degree at most $d_2 = \nu$ over k), where $\phi_2^* D$ is a sum of $r = 2d_2$ distinct points. We have $g = (d_1 - 1)(d_2 - 1)$ and we see that equality holds in (4).

Example 2. Let $C_1 \times C_2 = \mathbb{P}^1 \times E$, where E is an elliptic curve defined over a number field k. Let $d_1 > d_2 + 1 > 2$. Let C be a nonsingular curve, defined over a number field k, of type (d_1, d_2) on $\mathbb{P}^1 \times E$ (i.e. $\deg \pi_1|_C = d_1$ and $\deg \pi_2|_C = d_2$). Then by the adjunction formula, $g = g(C) = d_1(d_2 - 1) + 1$. Let $\nu = d_2$ and $r = 0$. Then a simple calculation shows that equality is achieved in (4), but the set $\{P \in C(\bar{k}) \mid [k(P) : k] \leqslant \nu\}$ is infinite for sufficiently large k since C has a degree $\nu = d_2$ map down to E.

Note that when $C_1 \times C_2 = \mathbb{P}^1 \times \mathbb{P}^1$, the inequality (4) simplifies to

$$2g - 2 + r > \max\{2(\nu - 1)d_1, 2(\nu - 1)d_2\}.$$

As a curve of degree $d \geqslant 2$ in $\mathbb{P}^2$ can be mapped birationally onto a curve of bidegree $(d-1, d-1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, we obtain the following corollary.

Corollary 2. *Let $C \subset \mathbb{P}^2$ be a curve, defined over a number field k, of degree $d \geqslant 2$ and geometric genus g. Let S be a finite set of places of k. Let $D = \sum_{i=1}^r P_i$ be an effective divisor on C, defined over k, with $P_1, \ldots, P_r$ distinct points of $C(\bar{k})$. If*

$$2g - 2 + r > 2(\nu - 1)(d - 1) \tag{7}$$

for some positive integer ν, then any set of (D, S)-integral points

$$T \subset \{P \in C(\bar{k}) \mid [k(P) : k] \leqslant \nu\}$$

is finite. In particular, if $g - 1 > (\nu - 1)(d - 1)$, then the set

$$\{P \in C(\bar{k}) \mid [k(P) : k] \leqslant \nu\}$$

is finite.

By definition, the geometric genus of C is the genus of the normalization of C. For nonsingular plane curves, a better theorem on rational points has been proven by Debarre and Klassen [DK94] using Falting's theorem on rational points on subvarieties of abelian varieties.

Theorem 3 (Debarre, Klassen). *Let $C \subset \mathbb{P}^2$ be a nonsingular curve of degree d, defined over a number field k, that does not admit a map of degree $\leqslant d - 2$ onto a genus one curve (this is automatically satisfied if $d \geqslant 7$). Then the set*

$$\{P \in C(\bar{k}) \mid [k(P) : k] \leqslant d - 2\}$$

is finite.

Recall that a curve is called hyperelliptic (respectively bielliptic) if it admits a map of degree two onto a curve of geometric genus zero (respectively one). Harris and Silverman [HS91] have shown, again using Falting's theorem on subvarieties of abelian varieties, that curves possessing infinitely many quadratic points are either hyperelliptic or bielliptic.

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Theorem 4 (Harris, Silverman). *Let C be a nonsingular curve defined over a number field k. If C is not hyperelliptic or bielliptic, then the set $\{P \in C(\bar{k}) \mid [k(P):k] \leqslant 2\}$ is finite.*

A similar theorem is true for degree three rational points (see [AH91]), but not for degrees four and higher (see [DF93]). For quadratic integral points, there is a result due to Corvaja and Zannier [CZ04].

Theorem 5 (Corvaja, Zannier). *Let C be a nonsingular curve defined over a number field k. Let S be a finite set of places of k. Let $D = \sum_{i=1}^{r} P_i$ be an effective divisor on C, defined over k, with $P_1, \dots, P_r$ distinct points of $C(\bar{k})$. Let $T \subset \{P \in C(\bar{k}) \mid [k(P):k] \leqslant 2\}$ be a set of (D,S)-integral points on C. Then the following statements hold.*

(a) *If $r > 4$, then T is finite.*

(b) *If $r > 3$ and C is not hyperelliptic, then T is finite.*

In addition, in the case C is hyperelliptic and $r = 4$ (where T may be infinite), Corvaja and Zannier show how to parametrize all but finitely many of the quadratic integral points. The proof of Theorem 5 in [CZ04] makes use of an appropriate version of the Schmidt subspace theorem [Sch91, p. 178]. We now show that Corollary 2 implies a slight improvement to Theorem 5. Specifically, we show that the inequality in part (b) can be improved to cover the case $r = 3$.

Theorem 6. *Let C be a nonsingular curve defined over a number field k. Let S be a finite set of places of k. Let $D = \sum_{i=1}^{r} P_i$ be an effective divisor on C, defined over k, with $P_1, \dots, P_r$ distinct points of $C(\bar{k})$. Let $T \subset \{P \in C(\bar{k}) \mid [k(P):k] \leqslant 2\}$ be a set of (D,S)-integral points on C. Then the following statements hold.*

(a) *If $r > 4$, then T is finite.*

(b) *If $r > 2$ and C is not hyperelliptic, then T is finite.*

Proof. By Corollary 2, to prove statement (a) it suffices to show that any curve C of genus g has a birational plane model of degree $g+2$. This is true for $g = 0$, so suppose $g > 0$. Since any divisor of degree $2g+1$ on C is very ample and nonspecial, we obtain an embedding of C as a degree $2g+1$ curve in $\mathbb{P}^{g+1}$. Projecting from the linear span of $g-1$ general points of C, we obtain a map $\phi : C \to \mathbb{P}^2$ birational onto its image with $\deg \phi(C) = g+2$ (see [ACGH85, p. 109]).

Similarly, to prove statement (b) it suffices to show that if C has genus g and is not hyperelliptic, then C has a birational plane model of degree $g+1$. Since C is not hyperelliptic, the canonical embedding realizes C as a curve of degree $2g-2$ in $\mathbb{P}^{g-1}$. Projecting from the linear span of $g-3$ general points of C, we obtain a plane curve of degree $g+1$ birational to C. □

Part (a) of Theorem 6 can also be obtained directly from Vojta's inequality. As noted in [CZ04], Vojta's conjecture predicts that the inequality in part (b) can be improved to $r > 0$. This improved inequality can be proved for certain classes of curves using Theorem 2 (of course, by Theorem 4, only bielliptic curves are of interest here). For instance, Theorem 2 implies that if E is an elliptic curve, one may take $r > 0$ in Theorem 6 for any nonsingular bielliptic curve C of type $(a, 2)$, $a > 3$, on $\mathbb{P}^1 \times E$. The full ramifications of Theorem 2 in this direction remain to be determined.

3. Proofs of results

Let C be a nonsingular curve defined over a number field k. Let R denote the ring of integers of k and let $B = \operatorname{Spec} R$. Let $\pi : X \to B$ be a regular model for C over R. For every complex embedding $\sigma : k \hookrightarrow \mathbb{C}$ we have a canonical volume form on $C_\sigma = C \times_\sigma \mathbb{C}$ and an associated canonical Green's function g_σ. With this data one can define intersections of Arakelov divisors

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(see [Lan88]). Let $P \in C(\bar{k})$ and let E_P denote the horizontal prime divisor on X corresponding to P (we also denote the curve on X corresponding to P by E_P). Let $\omega_{X/B}$ denote the relative dualizing sheaf, with its canonical Arakelov metric [Lan88, ch. 4]. We then define the arithmetic discriminant $d_a(P)$ by

$$d_a(P) = \frac{E_P.(\omega_{X/B} + E_P)}{[k(P) : \mathbb{Q}]}.$$

Of course, contrary to the notation, $d_a(P)$ depends on more data than just P. We can also give an alternative formula for $d_a(P)$. Let $L = k(P)$. Then $E_P = \operatorname{Spec} A$, where A is an order of the number field L. Let

$$W_{A/R} = \{b \in L \mid \operatorname{Tr}_{L/k}(bA) \subset R\}$$

be the Dedekind complementary module. It is a fractional ideal of A containing A. For a fractional ideal $\mathfrak{a}$ of A, we define the fractional ideal

$$\mathfrak{a}^{-1} = \{x \in L \mid x\mathfrak{a} \subset A\}.$$

In arbitrary orders, one may not necessarily have $\mathfrak{a}\mathfrak{a}^{-1} = A$. We now define the Dedekind different (of A over R) as

$$\mathcal{D}_{A/R} = W_{A/R}^{-1}.$$

This is an integral ideal of A. For a nice discussion of the relation between the different, discriminant, and conductor of an order, we refer the reader to the article by Del Corso and Dvornicich [DD00]. Now define

$$d_{A/R} = \frac{\log[A : \mathcal{D}_{A/R}]}{[L : \mathbb{Q}]}.$$

Let S_∞ be the set of archimedean places of k. For $v \in S_\infty$, let

$$E_P \times \mathbb{C}_v = \{P_{v,1}, \dots, P_{v,[L:k]}\}$$

be the set of points in $C_v = C \times \mathbb{C}_v$ into which E_P splits. By the Arakelov adjunction formula [Lan88, Theorem 5.3], we have

$$d_a(P) = d_{A/R} + \frac{1}{[L : \mathbb{Q}]} \sum_{v \in S_\infty} \sum_{i \neq j} N_v \lambda_v(P_{v,i}, P_{v,j}), \tag{8}$$

where $N_v = [k_v : \mathbb{Q}_v]$ and $\lambda_v = \frac{1}{2} g_v$ (with g_v normalized as in [Lan88]). We use that λ_v is a Weil function for the diagonal Δ_v in $C_v \times C_v$, i.e. if the Cartier divisor Δ_v is locally represented by a function f on the open set U, then there exists a continuous function α on U such that

$$\lambda_v(P) = -\log|f(P)| + \alpha(P)$$

for all $P \in U \setminus \Delta_v$.

Theorem 7. *Let C_1, C_2, and C_3 be nonsingular curves defined over k and let X_1, X_2, and X_3 be regular models over R for the respective curves. Let $\psi : X_3 \to X_1 \times X_2$ be a morphism that is birational onto its image and let $\phi : C_3 \to C_1 \times C_2$ be the natural morphism induced by ψ. Let ϕ_1 and ϕ_2 denote ϕ composed with the projection maps of $C_1 \times C_2$ onto the first and second factors, respectively. Then for all $P \in C_3(\bar{k})$,*

$$d_a(P) \leqslant d_a(\phi_1(P)) + d_a(\phi_2(P)) + O(1). \tag{9}$$

The inequality (9) can be viewed as a natural arithmetic analog of Castelnuovo's inequality (6). We identify the points $P_1 = \phi_1(P)$, $P_2 = \phi_2(P)$, and P (or the corresponding arithmetic curves on X_1, X_2, and X_3) with the curves C_1, C_2, and C_3, respectively, of Castelnuovo's inequality. In view of the adjunction formula, it is natural to identify $E_P.(\omega_{X/B} + E_P)$, $E_{P_1}.(\omega_{X/B} + E_{P_1})$, and $E_{P_2}.(\omega_{X/B}+E_{P_2})$ with $2g-2$, $2g_1-2$, and $2g_2-2$, respectively. Furthermore, $[k(P):k(P_1)]$ and $[k(P):k(P_2)]$ correspond to d_1 and d_2, respectively, in Castelnuovo's inequality. Multiplying (9) by $[k(P):\mathbb{Q}]$ and identifying the error term $O([k(P):\mathbb{Q}])$ with $O(d_1d_2)$, under the above correspondences (9) corresponds to the inequality

$$g \leqslant d_1g_1 + d_2g_2 - d_1 - d_2 + 1 + O(d_1d_2).$$

Replacing the term $O(d_1d_2)$ by d_1d_2 gives exactly Castelnuovo's inequality. Since E_P, E_{P_1}, and E_{P_2} are not necessarily regular, a more precise analogy would involve a Castelnuovo-type inequality for (not necessarily nonsingular) curves C, C_1, C_2, and their arithmetic genera.

Our strategy for proving Theorem 7 is to break up d_a into a finite and infinite part as in (8), and then prove the inequality for each part separately. Since there is an $O(1)$ term in (9), we can clearly ignore the finite set Z of $C(\bar{k})$ on which ϕ fails to be invertible. To prove the inequality for the finite part $d_{A/R}$ of (8), we use the following lemma.

Lemma 1. *Let R be the ring of integers of a number field k. Let A_1 and A_2 be R-orders of the number fields L_1 and L_2, respectively (with some fixed embedding in $\bar{k}$). Let $L_3 = L_1L_2$ and let $A_3 = A_1A_2$. If A_1, A_2, and A_3 are Gorenstein rings, then*

$$d_{A_3/R} \leqslant d_{A_1/R} + d_{A_2/R}. \tag{10}$$

Proof. As shown in [DD00], an R-order A is Gorenstein if and only if $\mathcal{D}_{A/R}$ is an invertible ideal of A (see [Bas63] for the many equivalent definitions of a Gorenstein ring). Let A_i' denote the integral closure of A_i in L_i for $i=1,2,3$. For the Gorenstein rings A_1, A_2, and A_3 we have the relations (see [DD00, Proposition 3])

$$\mathcal{D}_{A_i/R}A_i' = \mathcal{C}_{A_i}\mathcal{D}_{A_i'/R}, \quad i=1,2,3, \tag{11}$$

where

$$\mathcal{C}_{A_i} = \{x \in A_i' \mid xA_i' \subset A_i\}$$

is the conductor of A_i. For an invertible ideal $\mathfrak{a}$ of A_3 (see [DD00, Theorem 1]),

$$[A_3 : \mathfrak{a}] = [A_3' : \mathfrak{a}A_3'].$$

Now to prove the lemma, it suffices to show that

$$\mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}A_3' \subset \mathcal{D}_{A_3/R}A_3'.$$

Indeed, this inclusion gives

$$[A_3 : \mathcal{D}_{A_3/R}] = [A_3' : \mathcal{D}_{A_3/R}A_3'] \leqslant [A_3' : \mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}A_3']$$

which is equivalent to (10) as

$$\begin{aligned}[A_3' : \mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}A_3'] &= [A_3' : \mathcal{D}_{A_1/R}A_3'][A_3' : \mathcal{D}_{A_2/R}A_3'] \\ &= [A_1' : \mathcal{D}_{A_1/R}A_1']^{[L_3:L_1]}[A_2' : \mathcal{D}_{A_2/R}A_2']^{[L_3:L_2]} \\ &= [A_1 : \mathcal{D}_{A_1/R}]^{[L_3:L_1]}[A_2 : \mathcal{D}_{A_2/R}]^{[L_3:L_2]}.\end{aligned}$$

We now show that $\mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}A_3' \subset \mathcal{D}_{A_3/R}A_3'$. By (11),

$$\mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}A_3' = \mathcal{C}_{A_1}\mathcal{D}_{A_1'/R}\mathcal{C}_{A_2}\mathcal{D}_{A_2'/R}A_3'$$

and

$$\mathcal{D}_{A_3/R}A_3' = \mathcal{C}_{A_3}\mathcal{D}_{A_3'/R} = \mathcal{C}_{A_3}\mathcal{D}_{A_3'/A_1'}\mathcal{D}_{A_1'/R}.$$

Therefore, we need to show that $\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{D}_{A_2'/R}A_3' \subset \mathcal{C}_{A_3}\mathcal{D}_{A_3'/A_1'}$. It is a standard fact that $\mathcal{D}_{A_2'/R}$ is generated by elements of the form $f'(\alpha)$, where $\alpha \in A_2'$, $k(\alpha) = L_2$, and f is the minimal polynomial of α over k. Let g be the minimal polynomial of α over L_1. Note that $L_1(\alpha) = L_3$ and that $g'(\alpha)$ divides $f'(\alpha)$ in A_3'. It is easily shown that $g'(\alpha)A_3' = \mathcal{C}_{A_1'[\alpha]}\mathcal{D}_{A_3'/A_1'}$. We have

$$\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{C}_{A_1'[\alpha]} \subset \mathcal{C}_{A_3}$$

since

$$\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{C}_{A_1'[\alpha]}A_3' \subset \mathcal{C}_{A_1}\mathcal{C}_{A_2}A_1'[\alpha] \subset \mathcal{C}_{A_1}\mathcal{C}_{A_2}A_1'A_2' \subset A_1A_2 = A_3.$$

Therefore,

$$\mathcal{C}_{A_1}\mathcal{C}_{A_2}f'(\alpha) \subset \mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{C}_{A_1'[\alpha]}\mathcal{D}_{A_3'/A_1'} \subset \mathcal{C}_{A_3}\mathcal{D}_{A_3'/A_1'}.$$

As $D_{A_2'/R}$ was generated by the $f'(\alpha)$, we obtain $\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{D}_{A_2'/R}A_3' \subset \mathcal{C}_{A_3}\mathcal{D}_{A_3'/A_1'}$ as desired. $\square$

Let ψ_1 and ψ_2 denote ψ composed with the projection maps of $X_1 \times X_2$ onto the first and second factors, respectively. Let $E_P = E_3 = \operatorname{Spec} A_3$ be the prime horizontal divisor corresponding to $P \in C(\bar{k}) \setminus Z$, and let $\psi_1(E_P) = E_1 = \operatorname{Spec} A_1$ and $\psi_2(E_P) = E_2 = \operatorname{Spec} A_2$. Note that A_1 and A_2 are naturally subrings of A_3 (via ψ_1 and ψ_2) and $A_3 = A_1A_2$. Indeed, the closed immersion $\psi : E_P \to X_1 \times X_2$ factors through $E_1 \times E_2$, and therefore the natural map $A_1 \otimes A_2 \to A_3$ is surjective. Since X_1, X_2, and X_3 were assumed regular, E_P, E_1, and E_2 are locally complete intersections (they are Cartier divisors). This implies in particular that A_1, A_2, and A_3 are Gorenstein rings. Therefore, using Lemma 1, we have proved the finite part of the inequality (9).

We now consider the archimedean part of (9). With notation as above, let L_1, L_2, and L_3 be the quotient fields of A_1, A_2, and A_3. Let $v \in S_\infty$. Let E_{iv} be the set of points of $E_i \times \mathbb{C}_v$, $i = 1, 2, 3$. Let λ_{Δ_1}, λ_{Δ_2}, and λ_{Δ_3} denote the Weil functions of (8) for C_{1v}, C_{2v}, and C_{3v}, respectively, where $C_{iv} = C_i \times \mathbb{C}_v$ and Δ_i is the diagonal of $C_{iv} \times C_{iv}$. It suffices to prove the following lemma.

Lemma 2. *In the notation above,*

$$\frac{1}{[L_3 : \mathbb{Q}]} \sum_{\substack{P,Q \in E_{3v} \\ P \neq Q}} \lambda_{\Delta_3}(P, Q) \leqslant \frac{1}{[L_1 : \mathbb{Q}]} \sum_{\substack{P,Q \in E_{1v} \\ P \neq Q}} \lambda_{\Delta_1}(P, Q) + \frac{1}{[L_2 : \mathbb{Q}]} \sum_{\substack{P,Q \in E_{2v} \\ P \neq Q}} \lambda_{\Delta_2}(P, Q) + O(1).$$

The lemma will follow easily from the following 'distribution relation' of Silverman [Sil87, Proposition 6.2(b)] (proved by Silverman in greater generality).

Theorem 8 (Silverman). *Let C and C' be nonsingular complex curves. Let $\pi : C \to C'$ be a morphism. Let Δ and Δ' denote the diagonals of $C \times C$ and $C' \times C'$, respectively. Let λ_Δ and $\lambda_{\Delta'}$ be Weil functions associated to Δ and Δ' (under the usual complex absolute value). Then for any $P \in C$ and $q \in C'$ with $\pi(P) \neq q$,*

$$\lambda_{\Delta'}(\pi(P), q) = \sum_{Q \in \pi^{-1}(q)} e_\pi(Q/q)\lambda_\Delta(P, Q) + O(1)$$

where $e_\pi(Q/q)$ is the ramification index of π at Q.

Proof of Lemma 2. Let ϕ, ϕ_1, and ϕ_2 be the maps of Theorem 7 base extended to C_{3v}. For all but finitely many E_{3v}, if $P, Q \in E_{3v}$, $P \neq Q$, then either $\phi_1(P) \neq \phi_1(Q)$ or $\phi_2(P) \neq \phi_2(Q)$. Note also that the maps $E_{3v} \to E_{1v}$ and $E_{3v} \to E_{2v}$ are $[L_3 : L_1]$-to-one and $[L_3 : L_2]$-to-one maps, respectively. Thus, we obtain

$$\begin{aligned}\sum_{\substack{P,Q \in E_{3v} \\ P \neq Q}} \lambda_{\Delta_3}(P, Q) &\leqslant \sum_{\substack{P,Q \in E_{3v} \\ \phi_1(P) \neq \phi_1(Q)}} \lambda_{\Delta_3}(P, Q) + \sum_{\substack{P,Q \in E_{3v} \\ \phi_2(P) \neq \phi_2(Q)}} \lambda_{\Delta_3}(P, Q) + O(1) \\ &\leqslant \sum_{P \in E_{3v}} \sum_{\substack{Q \in \phi_1^{-1}(E_{1v}) \\ \phi_1(P) \neq \phi_1(Q)}} \lambda_{\Delta_3}(P, Q) + \sum_{P \in E_{3v}} \sum_{\substack{Q \in \phi_2^{-1}(E_{2v}) \\ \phi_2(P) \neq \phi_2(Q)}} \lambda_{\Delta_3}(P, Q) + O(1)\end{aligned}$$

$$\leqslant \sum_{P\in E_{3v}} \sum_{\substack{q\in E_{1v}\\ \phi_1(P)\neq q}} \lambda_{\Delta_1}(\phi_1(P),q) + \sum_{P\in E_{3v}} \sum_{\substack{q\in E_{2v}\\ \phi_2(P)\neq q}} \lambda_{\Delta_2}(\phi_2(P),q) + O(1)$$
$$\leqslant [L_3:L_1] \sum_{\substack{p,q\in E_{1v}\\ p\neq q}} \lambda_{\Delta_1}(p,q) + [L_3:L_2] \sum_{\substack{p,q\in E_{2v}\\ p\neq q}} \lambda_{\Delta_2}(p,q) + O(1),$$

where the implied constants in the $O(1)$ terms are independent of E_{1v}, E_{2v}, and E_{3v}. Dividing everything by $[L_3:\mathbb{Q}]$ gives the lemma. □

Theorem 7 now follows from Lemmas 1 and 2. We now prove Theorem 2 from the introduction. We need the following estimate of Song and Tucker (see [ST99] and [ST01]) for $d_a(P)$ on a curve.

LEMMA 3 (Song, Tucker). *Let C be a nonsingular curve defined over a number field k with canonical divisor K. Let X be a regular model for C over the ring of integers of k. Let A be an ample divisor on C and let $\epsilon > 0$. Then for all $P \in C(\bar{k})$,*

$$d_a(P) \leqslant h_K(P) + (2[k(P):k] + \epsilon)h_A(P) + O([k(P):k]).$$

Proof of Theorem 2. Let T be as in the hypotheses of Theorem 2 and suppose that the inequality (4) of Theorem 2 is satisfied. Consider the three sets

$$T_1 = \{P \in T \mid [k(\phi_1(P)):k] = [k(P):k]\},$$
$$T_2 = \{P \in T \mid [k(\phi_2(P)):k] = [k(P):k]\},$$
$$T_3 = \{P \in T \mid [k(\phi_1(P)):k] < [k(P):k], [k(\phi_2(P)):k] < [k(P):k]\}.$$

Clearly $T = T_1 \cup T_2 \cup T_3$. As we assumed $2g-2+r > 2(\nu+g_1-1)d_1$ and $2g-2+r > 2(\nu+g_2-1)d_2$, it follows from a trivial generalization of Corollary 1 that T_1 and T_2 are finite. So we are reduced to showing that if $2g-2+r > (\nu+2g_1-2)d_1 + (\nu+2g_2-2)d_2$, then T_3 is finite. Let K, K_1, and K_2 denote the canonical divisors of C, C_1, and C_2, respectively. Let h, h_1, and h_2 denote heights associated to some degree one divisor on C, C_1, and C_2, respectively. Using Theorem 1, Theorem 7, and Lemma 3, we get, for any $\epsilon > 0$,

$$\begin{aligned} m_S(D,P) + h_K(P) &\leqslant d_a(P) + \epsilon h(P) + O(1)\\ &\leqslant d_a(\phi_1(P)) + d_a(\phi_2(P)) + \epsilon h(P) + O(1)\\ &\leqslant h_{K_1}(\phi_1(P)) + (2[k(\phi_1(P)):k] + \epsilon)h_1(\phi_1(P))\\ &\quad + h_{K_2}(\phi_2(P)) + (2[k(\phi_2(P)):k] + \epsilon)h_2(\phi_2(P)) + O(1). \end{aligned}$$

Note that for $P \in T_3$, $[k(\phi_1(P)):k] \leqslant \nu/2$ and $[k(\phi_2(P)):k] \leqslant \nu/2$, since $k(\phi_1(P))$ and $k(\phi_2(P))$ are both proper subfields of $k(P)$. Since T is a set of (D,S)-integral points, $m_S(D,P) = h_D(P) + O(1)$ for $P \in T$. Using functoriality of heights and quasi-equivalence of heights associated to numerically equivalent divisors, we obtain, for any $\epsilon > 0$,

$$(2g-2+r)h(P) \leqslant ((\nu+2g_1-2)d_1 + (\nu+2g_2-2)d_2 + \epsilon)h(P) + O(1)$$

for $P \in T_3$. Taking $\epsilon < 1$, since there are only finitely many points of bounded degree and bounded height with respect to h, we see that if

$$2g-2+r > (\nu+2g_1-2)d_1 + (\nu+2g_2-2)d_2$$

then T_3, and hence T, must be finite. □

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