Canad. Math. Bull. Vol. 17 (2), 1974.

ON THE INEQUALITY

$$\sum_{i=1}^{n} p_i \frac{f(p_i)}{f(q_i)} \leq 1$$

BY
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1. In this article, we are concerned with the following inequality

$$\sum_{i=1}^{n} p_i \frac{f(p_i)}{f(q_i)} \leq 1 \tag{1}$$

where $0<p_i<1$, $0<q_i<1$, $(i=1, 2, \ldots, n) \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, n is a fixed positive integer, $n \geq 2$ and $f(p) \neq 0$ for $0<p<1$.

This inequality was first considered by A. Renyi, who gave the general differentiable solution of (1) for $n \geq 3$, [1]. With the help of this inequality one can characterize Renyi's entropy [2].

We shall state later the Renyi's result, which will be a special case of the Theorem 3.

These inequalities, the so-called quasi-linear inequalities are the subject of some recent articles [4], [5], [6].

2. Let us denote by $A_1 \backslash A_2$ the following set

$$A_1 \backslash A_2 = \{X : X = X_1 - X_2,\ X_1 \in A_1 \quad \text{and} \quad X_2 \in A_2\},\ m \text{ (resp. } m)$$

denotes the Lebesque measure (resp. inner Lebesque measure), $\bar{A}$ denotes the closure of A and f/A denotes the restriction of f to A.

In this section, we shall give the general positive and the general continuous solution of the inequality (1) for $n \geq 3$.

Let us consider the inequality (1) in that special case when $n=2$, i.e.

$$p \frac{f(p)}{f(q)} + (1-p) \frac{f(1-p)}{f(1-q)} \leq 1 \tag{2}$$

where $0<p<1$, $0<q<1$, and $f(p) \neq 0$ for $0<p<1$.

THEOREM 1. *The general positive solution of the inequality* (2) *is monotone decreasing and continuous.*

Proof. The inequality (2) can be written in the following form:

$$p \frac{f(p)-f(q)}{f(q)} \leq (1-p) \frac{f(1-q)-f(1-p)}{f(1-q)} \tag{3}$$

https://doi.org/10.4153/CMB-1974-039-8 Published online by Cambridge University Press

that is if $f(p)>f(q)$ then $f(1-q)>f(1-p)$, and from (3) we obtain, by interchanging p and q and adding

$$\left[\frac{p}{f(q)}-\frac{q}{f(p)}\right][f(p)-f(q)] \le \left[\frac{1-p}{f(1-q)}-\frac{1-q}{f(1-p)}\right][f(1-q)-f(1-p)] \tag{4}$$

If for $0<q<p<1$ we had $f(q)<f(p)$, then we would have $f(1-q)>f(1-p)$, in which case $f(p)$ does not satisfy (4).

To prove the continuity of the solution of the inequality (2) let us write it in the form

$$q\left[\frac{f(p)}{f(q)}+\frac{f(q)}{f(p)}\right]+(1-p)\left[\frac{f(1-p)}{f(1-q)}+\frac{f(1-q)}{f(1-p)}\right]+(p-q)\left[\frac{f(p)}{f(q)}+\frac{f(1-q)}{f(1-p)}\right] \le 2 \tag{5}$$

If $q\nearrow p_0$ and $p\searrow p_0$, then we have

$$p_0\left[\frac{f(p_0+0)}{f(p_0-0)}+\frac{f(p_0-0)}{f(p_0+0)}\right]+(1-p_0)\left[\frac{f(1-p_0-0)}{f(1-p_0+0)}+\frac{f(1-p_0+0)}{f(1-p_0-0)}\right] \le 2 \tag{6}$$

Since $\min(x+1/x)=2$ occurs for $x=1$ it follows that $f(p_0+0)=f(p_0-0)$ and $f(1-p_0-0)=f(1-p_0+0)$ for every p, $0<p<1$, that is, f is continuous. Conversely, if f is continuous it does not change its sign, so it is monotone.

Exactly the same manner we can prove the following two generalizations of the former theorem.

Proposition 1. *The general positive solution of the inequality*

$$* \qquad p_1\frac{f(p_1)}{f(q_1)}+p_2\frac{f(p_2)}{f(q_2)} \le p_1+p_2$$

where p_1, p_2, q_1 and q_2 are positive, p_1+p_2 is fixed $p_1+p_2=q_1+q_2$ and $f(p)\neq 0$ for $0<p<p_1+p_2$ is monotone decreasing and continuous.

Proposition 2. *The general solution of the inequality* 2 *in the interval* $(p, 1-p)$ *where $0<p<1/2$ is monotone decreasing and continuous under the hypothetis that $f(p)$ is positive in the interval* (p, $1-p$).

Let us assume that $f(p)>0$ for $0<p<1$.

Theorem 2. *If the solution of the inequality* (2) *is differentiable at the point p then it is differentiable at $1-p$ and the following relation is valid:*

$$\frac{p}{f(p)}f'(p) = \frac{1-p}{f(1-p)}f'(1-p). \tag{7}$$

Proof. From the inequality (3) we obtain the following inequality:

$$\begin{aligned} \frac{p}{1-p}\cdot\frac{f(1-q)}{f(q)}[f(p)-f(q)] &\le [f(1-q)-f(1-p)] \\ &\le \frac{f(1-p)}{f(p)}\cdot\frac{q}{1-q}[f(p)-f(q)] \end{aligned} \tag{8}$$

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and this inequality gives the following two equalities

(9) $$\frac{p}{1-p}\frac{f(1-p)}{f(p)}D^{-}f(p) = D^{+}f(1-p)$$

(10) $$\frac{p}{1-p}\frac{f(1-p)}{f(p)}D^{+}f(p) = D_{-}f(1-p)$$

which proves the theorem.

Exactly the same manner we can prove the following

PROPOSITION 3. *If the solution of the inequality* $*$ *is differentiable at* p_0 *for* $0<p_0<p_1+p_2$, *then it is differentiable at the point* $p_1+p_2-p_0$ *and the following relation is valid*

$$\frac{p_0}{f(p_0)}f'(p_0) = \frac{p_1+p_2-p_0}{f(p_1+p_2-p_0)}f'(p_1+p_2-p_0)$$

under the assumption that $f(p)$ *is positive in the interval* $(0, p_1+p_2)$.

THEOREM 3. *The general positive solution of the inequality* (1) *for* $n\geq 3$ *is differentiable, and it has the following form:*

$$f(p) = dp^c \quad \textit{where} \quad d>0 \quad \textit{and} \quad -1\leq c\leq 0.$$

Proof. If in the inequality (1) we set $p_3=q_3, \ldots, p_n=q_n$, then it follows that the solution of (1) must satisfy the inequality $*$ in the interval $(0, p_1+p_2)$, that is, it is monotone decreasing in the interval $(0, p_1+p_2)$, and since it is monotone decreasing for every $0<p_1+p_2<1$ the solution of (1) is monotone decreasing in (0, 1). On the other part, from Theorem (2) and Proposition 3 we see that if the solution is differentiable at the point p, $0<p<(p_1+p_2)$, then it is also differentiable at the point (p_1+p_2-p). If there existed a point p at which f is non-differentiable, then we can find a set of points of positive measure at which f is not differentiable, but this is impossible after a Theorem of Lebesgue. Thus the solution of the inequality (1) must satisfy the following relation:

(11) $$\frac{p}{f(p)}f'(p) = C$$

where C is non-positive, since $f(p)$ is monotone decreasing. Thus if $f(p)$ is a solution it has to have the form

(12) $$f(p) = dp^C$$

where $d>0$ and $C\leq 0$.

We prove now that (12) is a solution of (1) if and only if $-1\leq C\leq 0$. It is easy to prove that (12) does not satisfy (1) if $C<-1$, and that it satisfies (1) if $C=-1$ or $C=0$. Let us assume that $-1<C<0$. We have to prove that

(13) $$\sum_{i=1}^{n} p_i^{1-|C|}q_i^{|C|} \leq 1$$

By virtue of the Hölder inequality we have

$$\sum_{i=1}^{n} p_i^{1-|C|} q_i^{|C|} \le \left(\sum_{i=1}^{n} p_i^{1-|C|1/1-|C|}\right)^{1-|C|} \left(\sum_{i=1}^{n} q_i^{|C|1/|C|}\right)^{|C|} = 1.$$

Therefore, the general positive solution of the inequality is

$$f(p) = dp^C \tag{14}$$

where $d>0$ and $-1 \le C \le 0$.

REMARK. As special case we have the general differentiable solution of the inequality (2) for $n \ge 3$. This result was obtained by A. Renyi.

3. If $f(p)$ changes its sign then the solution of the inequality (1) is not necessarily measurable. Namely if $f(p)>0$ and $f(p)$ is a solution of the inequality (1), then $e(p)f(p)$ also satisfies (1), where $e(p)$ is any real solution of the $x^2(p)=1$

In this section we want to give the general monotone decreasing solution of the inequality (2) with non-constant sign.

Let $A=\{p:f(p)>0\}$, $B=\{p:f(p)<0\}$. We prove the following:

LEMMA 1. *If f is a monotone decreasing solution of the inequality* (2) *with non-constant sign, then f is constant in the intervals* $(0, a)$ *and* $(1-a, 1)$ *where* $a=\min(m(A), m(B))$.

Proof. Let $0<p<q<a$. If $f(p)>f(q)$, then we have a contradiction by the inequality (3); and for the same reason it is impossible that for $1-a<q<p<1, f(q)$ is greater than $f(p)$.

If $m(B) \ge m(A)$; then the only monotone decreasing solution is

$$f(p) = \begin{cases} C_1 > 0 & \text{for } p \in A \\ C_2 < 0 & \text{for } p \in B \end{cases}$$

if $m(B)>m(A)$, then $f(p)<0$ for $m(A)<p<1-m(A)$ so according to Proposition 2 it is monotone increasing. If $m(B)<m(A)$, then the general monotone decreasing solution has the following form:

$$f(p) = \begin{cases} C & \text{for } 0 < p < m(B), \qquad C > 0 \\ f(p) & \text{for } m(B) < p < 1-m(B) \\ -d & \text{for } 1-m(B) < p < 1, \qquad d > 0 \end{cases}$$

where $f(p)>0$ in the interval $(m(B), 1-m(B))$ and f satisfies in this interval the inequality (2), and moreover,

$$m(B)\frac{C}{f(1-m(B)-0)} \le 1 + \frac{d}{f(m(B)+0)}(1-m(B))$$

4. In the following, we do not assume that $f(p)>0$ for every $p \in (0, 1)$. Let us define by

$$A_1 = \{p:f(p) > 0, 0 < p \le \tfrac{1}{2}\}, \qquad A = \{p:f(p) > 0, 0 < p < 1\}$$
$$B_1 = \{p:f(p) < 0, 0 < p \le \tfrac{1}{2}\} \qquad B = \{p:f(p) < 0, 0 < p < 1\}$$

LEMMA 2. *If f is a solution of the inequality* (1) *for $n \geq 3$ then f is monotone decreasing in A_1 and f is monotone increasing in B_1. Furthermore, f/A (resp. f/B) is continuous on A_1 (resp. B_1).*

We omit the proof of this Lemma. We shall prove the following

THEOREM 4. *If $m_*(A \cap (0, x))>0$ for every positive x, and if $\bar{A} \supset [\frac{1}{2}, 1]$ then f/A is monotone decreasing and continuous.*

Proof. If the result were not true, then we can find p_1 and q_1 such that $f(p_1)>0$, $f(q_1)>0$, $0<p_1<q_1, f(p_1)<f(q_1)$ and $\frac{1}{2}<q_1$. It is then easy to see that for every $\delta>0$, we can find in the interval $[p_1, q_1]$ p_1' and q_1' such that $p_1'<q_1'>\frac{1}{2}p_1'+\delta>q_1'$ and $0<f(p_1')<f(q_1')$. If we choose δ such that $0<\delta<1-q_1$ and δ less than the length of the interval contained in $A \cap (0, 1-q_1)/A \cap (0, 1-q_1)$ with left endpoint $x=0$, then it is easy to see, that

$$q_1' \frac{f(q_1')}{f(p_1')} + q_2 \frac{f(q_2)}{f(p_2)} + (1-q_1'-q_2) \leq 1$$

is impossible where $f(q_2)>0$, $f(p_2)>0$, because $\frac{1}{2}>p_2>q_2$ and it was shown that for these values f is monotone decreasing. To prove the continuity, it suffices to prove first that for every $p/2<p<1, f/A(p-0)=f/A(p+0)$. Let us assume that for a fixed $p/2<p<1, f/A(p-0)>f/A(p+0)$ and let

$$\frac{f/A(p-0)}{f/A(p+0)} = 1+\varepsilon \qquad \text{where } \varepsilon > 0,$$

let p_m be a monotone increasing sequence and q_m be a monotone decreasing

$$\tfrac{1}{2} < p_m < p \quad \text{and} \quad p < q_m < 1$$

$$\lim p_m = p, \lim q_m = p \quad \text{and} \quad p_m \in m \qquad (m = 1, 2, \ldots).$$

Thus if $m \geq N_0$ we can find $p_m' \in A$ and $q_m' \in A$ such that $q_m'-p_m'=q_m-p_m$ and $\lim p_m'=\lim q_m'=0$.

$$p_m \frac{f(p_m)}{f(q_m)} + p_m' \frac{f(p_m')}{f(q_m')} + (1-p_m-p_m') \leq 1$$

And if $m \to \infty$ we obtain that

$$p(1+\varepsilon)+(1-p) \leq 1$$

Furthermore it is easy to see that $f/A(\frac{1}{2}-0)=f/A(\frac{1}{2}+0)$ if $f/A(\frac{1}{2}-0)$ exists, which proves the theorem. In the same manner one can prove the following

PROPOSITION 4. *If $A \cap (0, x)$ has the Baire property and if it is a set of second category for every positive x, and if furthermore $\bar{A} \supset [\frac{1}{2}, 1]$, then f/A is monotone decreasing and continuous.*

THEOREM 5. *Let $f(x)$ be a solution of the inequality* (1) *for $n \geq 3$ such that* $\lim_{x=+0} x f(x)=0$. *If $\bar{A} \supset [\frac{1}{2}-\varepsilon, \frac{1}{2}]$, where ε is an arbitrary small positive number then f is monotone decreasing on A.*

***Proof.** We can assume that in (*) $\lim_{p=+0} pf(p)=0$ then we obtain (since this limit can be performed holding p_1 fixed) $f(p_1)/f(q_1) \leq 1$ whenever $0<q_1 \leq p_1<1$, $p_1-q_1<\frac{1}{2}$, $p_1 \in A$, $q_1 \in A_1$ since from the Lemma 2 follows that

$$\liminf_{q_2 \to p_1 - q_1 + 0} |f(q_2)| > 0.$$

Let $p_1' \in A$, $q_1' \in A$ and $0<q_1'<p_1'<1$, where $p_1'-q_1'>\frac{1}{2}$ then we can find a p_2', such that

$$q_1' = p_3' < p_2' < p_1'$$

and such that $p_2' \in A$ and

$$p_2'-p_3' < \tfrac{1}{2} \quad \text{and} \quad p_1'-p_2' < \tfrac{1}{2}$$

In this case

$$\frac{f(p')}{f(q_1')} = \frac{f(p_1')}{f(p_2')}\frac{f(p_2')}{f(p_3')} \leq 1$$

which proves the Theorem.

Finally, we shall prove the following theorem.

THEOREM 6. *Let $f(x)$ be a solution of the inequality* (1) *for $n \geq 3$. Let $\bar{A} \supset [\frac{1}{2}, 1]$, and let us assume that there exists an interval $(0, p_0)$ such that for every $p \in (0, p_0)$ p is a point of maximal density of A, and that for every positive x $m(A \cap (0, x))>0$. Then the function*

$$f^*(p) = \begin{cases} f(p) & \text{for } p \in A \\ \text{continuous on } (0, 1) \end{cases} \tag{15}$$

satisfies the inequality (*) *on the interval $(0, p_0)$.*

Proof. Let p_1, p_2 be arbitrary positive numbers such that $p_1+p_2<p_0$. As f satisfies the inequality (1), it satisfies the following relation:

$$p_1\frac{f(p_1)}{f(q_1)}+p_2\frac{f(p_2)}{f(q_2)}+(1-p_1-p_2) \leq 1$$

Let $p_1^{(m)}$, $p_2^{(m)}$ be sequences such that $p_1^{(m)} \to p_1$, $p_2^{(m)} \to p_2$, $p_1^{(m)}+p_2^{(m)}=p_1+p_2$, and $f(p_1^{(m)})>0$, $f(p_2^{(m)})>0$. The existence of such sequences follows from the following result. Let A_1 and A_2 be sets of positive measures, let X_1 (resp. X_2) be a point of maximal density with respect to A_1 (resp. A_2) where X_1 does not necessarily belong to A_1 (resp. A_2). Then there exist sequences X_m^1 and X_m^{11} such that $X_m^1 \in A_1$, $X_m^{11} \in A_2$

* We wish to express our thanks to the Referee for the simplification of the proof of this Theorem and for his very helpful suggestions.

$(m=1, 2, \ldots)$ $X_m^1 \to X_1$, $X_m^{11} \to X_2$ and $X_m^1 - X_m^{11} = X_1 - X_2$. (This result slightly generalizes a theorem of Kemperman [7].) Since

$$p_1^{(m)} \frac{f(p_1^{(m)})}{f(q_1)} + p_2^{(m)} \frac{f(p_2^{(m)})}{f(q_2)} + (1 - p_1 - p_2) \leq 1 \tag{16}$$

we have from the continuity and the monotoncity of f/A that

$$p_1 \frac{f^*(p_1)}{f(q_1)} + p_2 \frac{f^*(p_2)}{f(q_2)} + (1 - p_1 - p_2) \leq 1.$$

On the other hand, let $q_1^{(m)}$, $q_2^{(m)}$ be sequences such that

$$q_1^{(m)} \to q_1 q_2^{(m)} \to q_2, \qquad q_1^{(m)} + q_2^{(m)} = q_1 + q_2$$

and $f(q_1^{(m)}) > 0$, $f(q_2^{(m)}) > 0$. Since

$$p_1 \frac{f^*(p_1)}{f(q^{(m)})} + p_2 \frac{f^*(p_2)}{f(q_2^{(m)})} \leq p_1 + p_2$$

we have

$$p_1 \frac{f^*(p_1)}{f^*(q_1)} + p_2 \frac{f^*(p_2)}{f^*(q_2)} \leq p_1 + p_2$$

As a corollary of Theorem 3, it can be proved that if f^* satisfies the inequality (17) for every $0 < p_1 + p_2 < p_0$, then it has the form

$$f^*(p) = dp^c \quad \text{for} \quad 0 < p < p_0$$

where $d > 0$, and $-1 \leq C \leq 0$.

References

1. A. Renyi, Unpublished.
2. A. Renyi, *On the Foundations of Information Theory*, Rev. Inst. Internat. Stat. **33** (1965), 1–14.
3. J. Aczel and G. Pfanzagl, *Remarks on the Measurement of Subjective Probability and Information*, Metrika II (1966), 91–105.
4. J. Aczel and Z. Daroczy, *Measures of Information and their Characterizations*, Academic Press, New York, in preparation.
5. P. Fischer, *On the inequality* $\Sigma p_i f(p_i) \geq \Sigma p_i f(q_i)$, Metrika, Vol. 18 1972, 199–208.
6. P. Fischer, *On the inequality* $\Sigma g(p_i) f(p_i) \geq \Sigma g(p_i) f(q_i)$, in print, Aequation. Math.
7. J. M. B. Kemperman, *A general functional equation Transactions of the American Math. Society*, 1957, pp. 28–56, 1957.

This work was supported in part by N.R.C. Grant A 8040.