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HARDY'S UNCERTAINTY PRINCIPLE ON HYPERBOLIC SPACES

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Hardy's uncertainty principle states that it is impossible for a function and its Fourier transform to be simultaneously very rapidly decreasing. In this paper we prove versions of this principle for the Jacobi transform and for the Fourier transform on real hyperbolic spaces.

1. INTRODUCTION

The uncertainty principles roughly state that a non-zero function f and its Fourier transform $\widehat{f}$ cannot both be sharply localised. This is already evident in the Paley-Wiener theorem; the Fourier transform of a compactly supported smooth function extends to an entire function, hence it cannot have compact support. We also know that the Fourier transform of a rapidly decreasing function, that is, a Schwartz function, is again a rapidly decreasing function. Hardy's uncertainty principle tells us, however, that they cannot both be *very* rapidly decreasing:

THEOREM 1.1. [7], [**5**, pp. 155–157] *Let f be a measurable function on $\mathbb{R}$. If $|f(t)| \leqslant Ae^{-\alpha|t|^2}$ and $|\widehat{f}(\lambda)| \leqslant Be^{-\beta|\lambda|^2}$, where A, B, α, β are positive constants and $\alpha\beta > 1/4$, then $f = 0$ almost everywhere.*

Hardy's uncertainty principle has recently been generalised to the Fourier transform on various families of Lie groups, in particular on semisimple Lie groups, see [**16**] and [**4**]. Further work and generalisations on Riemannian symmetric spaces of the non-compact type were made in [**15**]. Also see [**3**] for similar results and a nice reference list.

The aim of this paper is to prove versions of Hardy's uncertainty principle for the Jacobi transform and for the Fourier transform on the real hyperbolic spaces $SO_o(p,q)/SO_o(p-1,q)$, $p, q \in \mathbb{N}$. The proof of the latter case is based on the observation that the Fourier transform of functions of fixed K-type can be expressed in terms of modified Jacobi functions. This approach can be expanded to cover all hyperbolic spaces and also yields a new proof of Hardy's uncertainty principle for all the Riemannian symmetric spaces of rank 1.

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2. Jacobi functions and the Jacobi transform

Let $a, b, \lambda \in \mathbb{C}$ and $0 < t < \infty$. We consider the differential equation

$$(1) \qquad \left(\Delta^{a,b}(t)\right)^{-1} \frac{d}{dt}\left(\Delta^{a,b}(t)\frac{du(t)}{dt}\right) = -(\lambda^2+\rho^2)u(t),$$

where $\rho = a+b+1$ and $\Delta^{a,b}(t) = (2\sinh t)^{2a+1}(2\cosh t)^{2b+1}$. Using the substitution $x = -\sinh^2 t$, we can rewrite (1) as a hypergeometric differential equation with parameters $(\rho+i\lambda)/2$, $(\rho-i\lambda)/2$ and $a+1$ (see [**6**, 2.1.1]). Let ${}_2F_1$ denote the Gauß hypergeometric function. The Jacobi function (of order (a,b)),

$$\varphi_\lambda^{a,b}(t) := {}_2F_1\left(\frac{1}{2}(\rho+i\lambda), \frac{1}{2}(\rho-i\lambda), a+1; -\sinh^2 t\right),$$

is for $a \notin -\mathbb{N}$ the unique solution to (1) satisfying $\varphi_\lambda^{a,b}(0) = 1$ and $\left.\frac{d}{dt}\right|_{t=0} \varphi_\lambda^{a,b} = 0$. The Jacobi functions satisfy the following growth estimates:

Lemma 2.1. *There exists a constant $C > 0$ such that:*

$$\left|\Gamma(a+1)^{-1}\varphi_\lambda^{a,b}(t)\right| \leqslant C\left(1+|\lambda|\right)^k(1+t)e^{(|\Im\lambda|-\Re\rho)t},$$

for all $a, b \in \mathbb{C}$ and all $t \geqslant 0$, where $k = 0$ if $\Re a > -1/2$ and $k = [1/2 - \Re a]$ if $\Re a \leqslant -1/2$.

Proof: See [**11**, Lemma 2.3]. ∎

Here $[\cdot]$ denotes integer part. We note that $\Gamma(a+1)^{-1}\varphi_\lambda^{a,b}(t)$ is an entire function in the variables a, b and $\lambda \in \mathbb{C}$ (also for $a \in -\mathbb{N}$). The Jacobi transform (of order (a,b)) is defined by:

$$\widehat{f}^{a,b}(\lambda) = \Gamma(a+1)^{-1}\int_0^\infty f(t)\varphi_\lambda^{a,b}(t)\Delta^{a,b}(t)\,dt,$$

for all even functions f and all complex numbers λ for which the right hand side is well-defined. The Paley–Wiener theorem for the Jacobi transform, [**11**, Theorem 3.4], states that the application $f \mapsto \widehat{f}^{a,b}$ is a bijection from $C_c^\infty(\mathbb{R})_{\text{even}}$ onto $\mathcal{H}(\mathbb{C})_{\text{even}}$, the space of even entire rapidly decreasing functions of exponential type, for all $a, b \in \mathbb{C}$.

Define the Jacobi c-functions as:

$$c^{a,b}(\lambda) := 2^{\rho-i\lambda}\frac{\Gamma(i\lambda)}{\Gamma\left((i\lambda+\rho)/2\right)\Gamma\left((i\lambda+a-b+1)/2\right)},$$

for $i\lambda \notin -\mathbb{N}$. Also consider the (Jacobi) functions (of the second kind):

$$\phi_\lambda^{a,b}(t) := (2\cosh t)^{i\lambda-\rho}{}_2F_1\left(\frac{1}{2}(\rho-i\lambda), \frac{1}{2}(a-b+1-i\lambda), 1-i\lambda; \cosh^{-2} t\right).$$

Then $\Gamma(a+1)^{-1}\varphi_\lambda^{a,b} = c^{a,b}(\lambda)\phi_\lambda^{a,b} + c^{a,b}(-\lambda)\phi_{-\lambda}^{a,b}$ as a meromorphic identity. The inversion formula for the Jacobi transform can be written as (for any $\mu \geqslant 0$, $\mu > -\Re(a \pm b+1)$):

$$(2) \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}^{a,b}(\lambda+i\mu)\phi_{\lambda+i\mu}^{a,b}(t)\frac{d\lambda}{c^{a,b}(-\lambda-i\mu)}, \quad (t>0),$$

for $f \in C_c^\infty(\mathbb{R})_{\text{even}}$, see [**12**, Theorem 2.2].

Our proof of Hardy's uncertainty principle for the Jacobi transform is inspired by the proof of the semisimple case, see [**16**]. The following lemma from complex analysis is crucial:

LEMMA 2.2. *Let h be an entire function on $\mathbb{C}$ such that:*

$$\left|h(\lambda)\right| \leqslant Ce^{\gamma|\lambda|^2}, \quad \lambda \in \mathbb{C} \quad \text{and} \quad \left|h(\lambda)\right| \leqslant Ce^{-\gamma|\lambda|^2}, \quad \lambda \in \mathbb{R},$$

for some positive constants γ and C. Then $h(\lambda) = const.\, e^{-\gamma\lambda^2}$, $\lambda \in \mathbb{C}$.

PROOF: See [**16**, Lemma 2.1]. ∎

THEOREM 2.3. [Hardy's uncertainty principle for the Jacobi transform.] *Let $a, b \in \mathbb{C}$, $a \notin -\mathbb{N}$. Let f be an even measurable function on $\mathbb{R}$ satisfying the following growth estimates:*

$$\left|f(t)\right| \leqslant Ae^{-\alpha|t|^2}, \quad t \in \mathbb{R} \quad \text{and} \quad \left|\widehat{f}^{a,b}(\lambda)\right| \leqslant Be^{-\beta|\lambda|^2}, \quad \lambda \in \mathbb{R},$$

for positive constants A, B, α, β. If $\alpha\beta > 1/4$, then $f = 0$ almost everywhere.

PROOF: Let f be an even measurable function satisfying the above growth conditions. The very rapid decay implies that $f \in L^1\left(\mathbb{R}_+, \left|\Delta^{a,b}(t)\right| dt\right) \cap L^2\left(\mathbb{R}_+, \left|\Delta^{a,b}(t)\right| dt\right)$ and that $\widehat{f}^{a,b}(\lambda)$ defines an analytic function in $\lambda \in \mathbb{C}$ for all $a, b \in \mathbb{C}$. Choose numbers α' and α'' such that $0 < \alpha'' < \alpha' < \alpha$ and $\alpha'\beta > \alpha''\beta > 1/4$. Using Lemma 2.1, we get the following estimates on $\widehat{f}^{a,b}(\lambda)$ (for different constants $C > 0$):

$$\begin{aligned}
\left|\widehat{f}^{a,b}(\lambda)\right| &\leqslant C\int_0^\infty e^{-\alpha t^2}\left(1+|\lambda|\right)^k(1+t)e^{(|\Im\lambda|-\Re\rho)t}\,dt \\
&\leqslant C\left(1+|\lambda|\right)^k\int_0^\infty e^{-\alpha' t^2}e^{(|\Im\lambda|)t}\,dt \\
&= C\left(1+|\lambda|\right)^k e^{|\Im\lambda|^2/4\alpha'}\int_0^\infty e^{-\alpha'(t-|\Im\lambda|/2\alpha')^2}\,dt \\
&\leqslant Ce^{|\lambda|^2/4\alpha''}\int_{-\infty}^\infty e^{-\alpha' t^2}\,dt \leqslant Ce^{|\lambda|^2/4\alpha''},
\end{aligned}$$

for $\lambda \in \mathbb{C}$, using translation invariance of dt and the inequality $|\Im\lambda| \leqslant |\lambda|$.

Since $-\beta < -1/4\alpha''$, we also have $\left|\widehat{f}^{a,b}(\lambda)\right| \leqslant Be^{-|\lambda|^2/4\alpha''}$ for $\lambda \in \mathbb{R}$, whence by Lemma 2.2:

$$\widehat{f}^{a,b}(\lambda) = const.e^{-\lambda^2/4\alpha''} \leqslant Be^{-\beta|\lambda|^2},$$

for $\lambda \in \mathbb{R}$, which is impossible unless the constant in the middle is zero. We conclude that $\widehat{f}^{a,b}$ is identically zero on $\mathbb{C}$ and hence that f is zero almost everywhere by the inversion formula (2). See [2] for more details. ∎

3. The Fourier transform on real hyperbolic spaces

Let $p \geqslant 1$ and $q \geqslant 2$ be two integers and consider the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{p+q}$ given by

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}, \quad x, y \in \mathbb{R}^{p+q}.$$

Let $G = SO_o(p,q)$ denote the connected group of $(p+q) \times (p+q)$ matrices preserving $\langle \cdot, \cdot \rangle$ and let $H = SO_o(p-1,q) \subset G$ denote the isotropy subgroup of the point $(1,0,\dots,0) \in \mathbb{R}^{p+q}$. Let $K = SO(p) \times SO(q) \subset G$ be the (maximal compact) subgroup of elements fixed by the classical Cartan involution on G: $\theta(g) = (g^*)^{-1}$.

The space $\mathbb{X} := G/H$ is a semisimple symmetric space (an involution τ of G fixing H is given by $\tau(g) = JgJ$, where J is the diagonal matrix with entries $(1,-1,\dots,-1)$). The map $g \mapsto g \cdot (1,0,\dots,0)$ induces an embedding of $\mathbb{X}$ in $\mathbb{R}^{p+q}$ as the hypersurface (with $x_1 > 0$ if $p = 1$):

$$\mathbb{X} = \left\{ x \in \mathbb{R}^{p+q} \mid \langle x, x \rangle = 1 \right\}.$$

Let $\mathbb{Y} := \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$. We introduce spherical coordinates on $\mathbb{X}$ as:

$$x(t,y) = (v \cosh t, w \sinh t),\ t \in \mathbb{R}_+,\ y = (v,w) \in \mathbb{Y}.$$

The map is injective, continuous and maps onto a dense subset of $\mathbb{X}$. The (K-invariant) metric distance from $x \in \mathbb{X}$ to the origin is given by $|x| = \left|x(t,y)\right| = |t|$.

The unique (up to a constant) G-invariant measure on $\mathbb{X}$ is in spherical coordinates given by:

$$\int_{\mathbb{X}} f(x)\, dx = \int_{\mathbb{R}_+ \times \mathbb{Y}} f\big(x(t,y)\big) J(t)\, dt\, dy,$$

see for example, [8, Part II, Example 2.3], where $J(t) = \cosh^{p-1} t \sinh^{q-1} t$ is the Jacobian, dt the Lebesgue measure on $\mathbb{R}$ and dy an invariant measure on $\mathbb{Y}$, normalised such that $\int_{\mathbb{Y}} 1\, dy = 1$.

The action of $SO(p)$ on $C^\infty(\mathbb{S}^{p-1})$ decomposes into irreducible representations $\mathcal{H}^l$ of spherical harmonics of degree $|l|$, see for example, [9, Introduction], characterised as the eigenfunctions of the Laplace–Beltrami operator Δ_p on $\mathbb{S}^{p-1}$ with eigenvalue $-l(l+p-2)$. Here $l = 0$ if $p = 1$, $l \in \mathbb{Z}$ for $p = 2$ and $l \in \mathbb{N} \cup \{0\}$ for $p > 2$.

Let $\mathcal{H}^{l,m} = \mathcal{H}^l \otimes \mathcal{H}^m$ and denote the representation of K on $\mathcal{H}^{l,m}$ by $\delta_{l,m}$. Let $d_{l,m} = \dim \mathcal{H}^{l,m}$ and $\chi_{l,m}$ denote the dimension and the character of $\delta_{l,m}$. A function in $L^2(\mathbb{X})$ is said to be of K-type (l,m) if its translates under the left regular action of K span a vector space which is equivalent to $\delta_{l,m}$. We write $L^2(\mathbb{X})^{l,m}$ for the collection of functions of K-type (l,m). The projection $\mathrm{P}^{l,m}$ of $L^2(\mathbb{X})$ onto $L^2(\mathbb{X})^{l,m}$ is given by:

$$\mathrm{P}^{l,m} f(x) = d_{l,m} \int_K \chi_{l,m}(k^{-1}) f(k \cdot x)\, dk, \qquad f \in L^2(\mathbb{X}),$$

for $x \in \mathbb{X}$, see for example, [**9**, Chapter V, Section 3] and [**10**, Chapter III, Section 5]. There are similar definitions and results for functions in $L^2(\mathbb{Y})$ and also for functions in $C^\infty(\mathbb{X})$ and $C^\infty(\mathbb{Y})$.

The algebra of left-G-invariant differential operators on $\mathbb{X}$ is generated by the Laplace–Beltrami operator $\Delta_{\mathbf{X}}$, see for example, [**8**, Part II, Example 4.1], which in spherical coordinates is given by:

$$\Delta_{\mathbf{X}} f = \frac{1}{J(t)}\frac{\partial}{\partial t}\Big(J(t)\frac{\partial f}{\partial t}\Big) - \frac{1}{\cosh^2 t}\Delta_p f + \frac{1}{\sinh^2 t}\Delta_q f, \qquad f \in C^\infty(\mathbb{X}),$$

see for example, [**14**, p. 455]. It reduces to a differential operator $\Delta_{\mathbf{X}}^{l,m}$ in the t-variable when acting on functions of K-type (l,m):

$$\Delta_{\mathbf{X}}^{l,m} f = \Delta_{\mathbf{X}} f = \frac{1}{J(t)}\frac{\partial}{\partial t}\Big(J(t)\frac{\partial f}{\partial t}\Big) + \frac{l(l+p-2)}{\cosh^2 t} f - \frac{m(m+q-2)}{\sinh^2 t} f, \qquad f \in C^\infty(\mathbb{X})^{l,m}.$$

Consider the differential equation:

$$\Delta_{\mathbf{X}} f = \Delta_{\mathbf{X}}^{l,m} f = (\lambda^2 - \rho^2) f, \qquad f \in C^\infty(\mathbb{X})^{l,m}, \tag{3}$$

where $\rho = (p+q-2)/2$. Altering the proof of [**10**, Chapter I, Proposition 2.7] to fit our setup, we see that we can write any function $f \in C^\infty(\mathbb{X})^{l,m}$ in spherical coordinates as:

$$f\big(x(t,y)\big) = \sum_i f_i(t)\phi_i^{l,m}(y), \tag{4}$$

where $\{\phi_i^{l,m}\} = \{\phi^l \otimes \phi^m\}_i$ is a (finite) basis for $\mathcal{H}^{l,m}$, and f_i is a function of the form $f_i(t) = t^{|m|} f_{i,o}(t)$, with $f_{i,o}$ even. Let $x = -\sinh^2 t$ and $g = (1-x)^{-|l|/2}(-x)^{-|m|/2} f_i$. Then g is a solution to the hypergeometric differential equation with parameters $1/2\big(\lambda + \rho + |l| + |m|\big)$, $1/2\big(-\lambda + \rho + |l| + |m|\big)$ and $q/2 + |m|$. Let $\Phi_\lambda^{l,m}$ denote the regular (for generic λ) solution to this hypergeometric differential equation satisfying the asymptotic condition $\Phi_\lambda^{l,m}(t) \sim e^{(\lambda-\rho)t}$ for $t \to \infty$ (for $\Re\lambda > 0$ and when defined), then

$$\begin{aligned}
&\Phi_\lambda^{l,m}(t)\\
&= 2^{\lambda-\rho-|l|-|m|}\cosh^{|l|} t \sinh^{|m|} t \frac{\Gamma\big((\lambda+\rho+|l|+|m|)/2\big)\Gamma\big((\lambda-\rho+q-|l|+|m|)/2\big)}{\Gamma(\lambda)\Gamma\big((1/2)q+|m|\big)}\\
&\qquad \times {}_2F_1\Big(\frac{1}{2}(\lambda+\rho+|l|+|m|), \frac{1}{2}(-\lambda+\rho+|l|+|m|); \frac{q}{2}+|m|; -\sinh^2 t\Big),
\end{aligned}$$

for $\Re\lambda > 0$, see [**1**, pp. 72 and 76]. We also note that the function $x(t,y) \mapsto \Phi_\lambda^{l,m}(t)\phi(y)$ extends to a solution of (3) on $\mathbb{X}$ for any $\phi \in \mathcal{H}^{l,m}$.

Let $\varepsilon \in \{0,1\}$ and define $C_\varepsilon^\infty(\mathbb{Y}) := \big\{\phi \in C^\infty(\mathbb{Y}) \,|\, \phi(-y) = (-1)^\varepsilon \phi(y)\big\}$. The Poisson transform, $F_{\varepsilon,\lambda} : C_\varepsilon^\infty(\mathbb{Y}) \to C^\infty(\mathbb{X})$, is defined as:

$$F_{\varepsilon,\lambda}\phi(x) = \int_{\mathbb{Y}} \big|\langle x,y\rangle\big|^{(-\lambda-\rho)} \operatorname{sign}^\varepsilon\langle x,y\rangle \phi(y)\,dy, \qquad \phi \in C_\varepsilon^\infty(\mathbb{Y}),$$

when $-\Re\lambda \geqslant \rho$.

LEMMA 3.1. *Let $\phi \in C_\varepsilon^\infty(\mathbb{Y})$. The (meromorphic extension of the) function $F_{\varepsilon,\lambda}\phi$ is an eigenfunction of the Laplace–Beltrami operator $\Delta_{\mathbf{X}}$ with eigenvalue $\lambda^2 - \rho^2$ (when defined), that is,:*

$$\Delta_{\mathbf{X}} F_{\varepsilon,\lambda}\phi = (\lambda^2 - \rho^2) F_{\varepsilon,\lambda}\phi\,.$$

The asymptotic behaviour of $F_{\varepsilon,\lambda}\phi$ for $t \to \infty$ is given by (when defined):

$$F_{\varepsilon,\lambda}\phi\big(x(t,y)\big) \sim e^{(\lambda-\rho)t} c(\varepsilon,\lambda)\phi(y)\,.$$

for $\Re\lambda > 0$, where $c(\varepsilon,\lambda)$ is the so-called c-function for $\mathbb{X}$ given by:

$$c(\varepsilon,\lambda) = \frac{2^{2\rho-1}\Gamma(p/2)\Gamma(q/2)}{\pi}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\rho)}\begin{cases} \tan\big(\pi/2\,(\lambda+\rho+\varepsilon)\big) & \text{if } \; p \; \text{ is even} \\ 1 & \text{if } \; p \; \text{ is odd.}\end{cases}$$

PROOF: The lemma follows from [**14**, Lemma 4, Lemma 5 and Lemma 7]. ∎

We define the (normalised) Fourier transform $\mathcal{F}f$ of any function $f \in C_c^\infty(\mathbb{X})$ as:

$$\mathcal{F}f(\varepsilon,\lambda,y) := c(\varepsilon,-\lambda)^{-1}\int_{\mathbf{X}} \big|\langle x,y\rangle\big|^{(\lambda-\rho)} \operatorname{sign}^\varepsilon\langle x,y\rangle f(x)\,dx,$$

for $\varepsilon \in \{0,1\}$, $\Re\lambda \geqslant \rho$ and $y \in \mathbb{Y}$. Let $f \in C_c^\infty(\mathbb{X})^{l,m}$ for some fixed K-type (l,m). We can (re)write the Fourier transform of f as (with $\varepsilon \equiv l+m \mod 2$):

$$\mathcal{F}f(\varepsilon,\lambda,y) = \int_{\mathbb{R}_+} \Phi_{-\lambda}^{l,m}(t) f\big(x(t,y)\big)J(t)\,dt,$$

using spherical coordinates, Schur's Lemma and properties of the Poisson transform, see [**1**, pp. 74–76] for details. We see that $\mathcal{F}f(\varepsilon,\lambda,y)$ extends to a meromorphic function in the λ-variable, with zeros and poles completely determined by the above expression for $\Phi_\lambda^{l,m}$. Due to the factor $\Gamma(\lambda)$ in the denominator there are no poles for purely imaginary λ.

THEOREM 3.2. [Hardy's uncertainty principle on $SO_o(p,q)/SO_o(p-1,q)$.] *Let f be a measurable function on $\mathbb{X}$ satisfying the following growth estimates:*

$$\big|f(x)\big| \leqslant Ae^{-\alpha|x|^2}, \quad x \in \mathbb{X} \;\text{ and }\; \big|\mathcal{F}f(\varepsilon,\lambda,y)\big| \leqslant Be^{-\beta|\lambda|^2}, \quad (\varepsilon,\lambda,y) \in \{0,1\}\times i\mathbb{R}\times\mathbb{Y},$$

for positive constants A, B, α, β. If $\alpha\beta > 1/4$, then $f = 0$ almost everywhere.

PROOF: Let f be a measurable function satisfying the above growth conditions. The very rapid decay again implies that $f \in L^1(\mathbb{X}) \cap L^2(\mathbb{X})$ and that the Fourier transform $\mathcal{F}f$ is well-defined.

Define $\widetilde{\rho} = \rho + |l| + |m|$, $a = |m| + (q/2) - 1$ and $b = |l| + (p/2) - 1$, then:

$$\Phi_\lambda^{l,m}(t) = 2^{\lambda-\widetilde{\rho}}\cosh^{|l|} t \sinh^{|m|} t \frac{\Gamma\big((\lambda+\widetilde{\rho})/2\big)\Gamma\Big(\big(\lambda-\widetilde{\rho}+q+2|m|\big)/2\Big)}{\Gamma(\lambda)\Gamma\Big(|m|+q/2\Big)}\varphi_{-i\lambda}^{a,b}(t).$$

Let $f_{l,m}(t,y) := \mathbf{P}^{l,m} f\big(x(t,y)\big)/\cosh^{|l|} t \sinh^{|m|} t$. By (4) we see that $f_{l,m}$ is a measurable function on $\mathbb{R} \times \mathbb{Y}$, even in the t-variable. Let also

$$Q_{l,m}(\lambda) := 2^{\lambda - 3\widetilde{\rho}} \frac{\Gamma\Big((\lambda + \widetilde{\rho})/2\Big)\Gamma\Big((\lambda - \widetilde{\rho} + q + 2|m|)/2\Big)}{\Gamma(\lambda)}.$$

Then:

$$\widehat{f}^{a,b}_{l,m}(i\lambda, y) := \Gamma(a+1)^{-1} \int_{\mathbb{R}_+} f_{l,m}(t,y)\varphi^{a,b}_{i\lambda}(t)\Delta^{a,b}(t)\, dt = Q_{l,m}(\lambda)^{-1} \mathcal{F}\mathbf{P}^{l,m} f(\varepsilon, \lambda, y).$$

We note that $\left|Q_{l,m}(i\lambda)^{-1}\right| \sim \text{const.}\, |\lambda|^{1/2-q-2|m|}$ for $|\lambda| \to \infty$, see [**6**, 1.18(6)], whence compactness of $[-1,1] \times \mathbb{Y}$ gives us the following estimates of $f_{l,m}$ and $\widehat{f}^{a,b}_{l,m}$:

$$\left|f_{l,m}(t,y)\right| \leqslant A' e^{-\alpha|t|^2}, \quad (t,y) \in \mathbb{R} \times \mathbb{Y} \quad \text{and} \quad \left|\widehat{f}^{a,b}_{l,m}(\lambda, y)\right| \leqslant B' e^{-\beta'|\lambda|^2}, \quad (\lambda, y) \in \mathbb{R} \times \mathbb{Y},$$

for positive constants A', B'. Hardy's uncertainty principle for the Jacobi transform, Theorem 2.3, implies that $\mathbf{P}^{l,m} f$ is zero almost everywhere. We conclude the theorem since $f = \sum_{l,m} \mathbf{P}^{l,m} f$. ∎

4. Remarks and further results

The space $\mathbb{X} = SO_o(p,q)/SO_o(p-1,q)$ is a semisimple Riemannian symmetric space of the non-compact type when $p = 1$ and of the non-Riemannian type when $p > 1$. Hardy's uncertainty principle for the Riemannian case is due to A. Sitaram and M. Sundari, see [**16**, Theorem 4.1]. Our proof generalises to all rank 1 Riemannian symmetric spaces of the non-compact type, using that the Fourier transform of K-finite functions can be expressed by Jacobi functions.

Let $\mathbb{F}$ be one of the two classical fields $\mathbb{C}$ or $\mathbb{H}$ and let $x \mapsto \overline{x}$ be the standard (anti)-involution of $\mathbb{F}$. Let p and q be two positive integers and let $[\,,\,]$ be the Hermitian form on $\mathbb{F}^{p+q}$ given by

$$[x,y] = x_1\overline{y}_1 + \cdots + x_p\overline{y}_p - x_{p+1}\overline{y}_{p+1} - \cdots - x_{p+q}\overline{y}_{p+q},$$

for $x, y \in \mathbb{F}^{p+q}$. Let $G = U(p,q;\mathbb{F})$ denote the group of all $(p+q) \times (p+q)$ matrices over $\mathbb{F}$ preserving $[\,,\,]$. Thus $U(p,q;\mathbb{C}) = U(p,q)$ and $U(p,q;\mathbb{H}) = Sp(p,q)$ in standard notation. Let H be the subgroup of G stabilising the line $\mathbb{F}(1,0,\dots,0)$ in $\mathbb{F}^{p+q}$. We can identify H with $U(1,0;\mathbb{F}) \times U(p-1,q;\mathbb{F})$ and the homogeneous space G/H (which is a reductive symmetric space) with the projective image of the space $\big\{z \in \mathbb{F}^{p+q} \mid [z,z]\big\} = 1$. The statement and proof of Hardy's uncertainty principle for the Fourier transform on G/H follows from the above, either embedding G/H into $SO_o(dp,dq)/SO_o(dp-1,dq)$, with $d = \dim_{\mathbb{R}} \mathbb{F}$, or again expressing the Fourier transform of K-finite functions using modified Jacobi functions. See [**1**, p. 117] for more details.

We finally note that we prove L^p versions of Hardy's uncertainty principle for the Jacobi transform and for the Fourier transform on real hyperbolic spaces in [**2**]. Also see [**13**] for a similar result in the Lie group case.

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