ARCS, SEMIGROUPS, AND HYPERSPACES

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Introduction. Several years ago Kelley (**2**) showed that if X is a metric continuum then $S(X)$, the space of non-null, closed subsets of X, and $C(X)$, the space of non-null, closed, connected subsets of X, with the Vietoris topology, are arcwise connected continua. He further showed that $S(X)$ is acyclic. In this note we extend these results to non-metric continua.

The methods employed in this paper will, in general, be order-theoretic and similar to those appearing in (**1**; **3**; **4**; **7**). We do, in fact, generalize some of the theorems of (**4**). In particular, we exhibit conditions sufficient to ensure the arcwise connectedness and decomposability of a semigroup.

In the final section we state a theorem which, in some sense, describes the topological position of a zero element in certain semigroups. In view of the fact that any Hausdorff space, S, becomes a semigroup with zero under the multiplication $xy = p$, where p is an arbitrary, but fixed, point in S, it is clear that some restriction must be placed on S if anything at all is to be concluded about the position of a zero element. We are able to show, for example, that a compact connected semilattice is locally arcwise connected at its zero-element.

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1. We state some preliminary definitions and the reader is referred to (**3**) for those terms not defined here.

If $(S, \leqq)$ is a partially ordered space, then for $x \in S$ and $A \subset S$, we let $L(x) = \{y | \, y \leqq x\}$ and $U(A) = \{x | \, L(x) \subset A\}$. We use $F(A)$ to denote the boundary of A, and A^* the closure of A. If S is a semigroup we set $E = \{x: x = x^2\}$. We need the following results from (**3**). An *arc* is a Hausdorff continuum with exactly two non-cut points.

THEOREM. *Let $(X, \leqq)$ be a compact partially ordered space and let W be an open set in X. If*

(1) *$\leqq$ is a closed subset of $X \times X$ and*

(2) *for any $x \in W$, each open set about x contains an element y with $y < x$, then any element x of W belongs to a compact connected chain C with $C \cap F(W) \neq \square$ and $x = \sup C$.*

COROLLARY. *Let $(X, \leqq)$ be a compact partially ordered space with zero. If*

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(1) $\leqq$ *is a closed subset of* $X \times X$ *and*
(2) $L(x)$ *is connected for each* x *in* X,
then X *is arcwise connected.*

2. We begin with the following theorem.

THEOREM. *Let* S *be a compact semigroup with minimal ideal* K *such that* $S = SE$ *with* E *connected and commutative. Then* S *is*
(1) *arcwise connected if* K *is arcwise connected and*
(2) *decomposable if* $K \neq S$, *and* K *is connected.*

Proof. If x and y are in S, define $x \leqq y$ if and only if $x \in yE$. Then for any x in S we have $x \in SE$, hence $x = ze$ for some z in S and e in E, and thus $xe = zee = ze = x$. Therefore $\leqq$ is reflexive. If $x \in yE$ and $y \in xE$, then $y = xe$ and $x = yf$ for some e, f in E. Hence

$$y = xe = yfe = xefe = xfee = xfe = xe = x$$

and R is anti-symmetric. Finally, if $x \in yE$ and $y \in zE$, then $x \in zEE = zE$ and R is transitive. Thus $\leqq$ is a partial order and it is easily checked that the graph of $\leqq$ is closed. Now $L(x) = xE$ is connected for each $x \in S$ and we note that the set of minimal elements is contained in K. For if a is minimal, then $ae \leqq a$ for any $e \in E \cap K$, hence $a = ae \in K$. Furthermore, if a is minimal in $L(x)$, then a is minimal in S, hence $L(x) \cap K \neq \square$ for each $x \in S$. We now verify that $S \backslash K$ is an open set satisfying the hypotheses of the theorem of the previous section. If $x \in S \backslash K$ and V is an open subset of $S \backslash K$ containing x, then there exists an open set W containing x such that $W^* \subset V$. But $L(x)$ meets W and $S \backslash W$ (since $L(x)$ meets $K \subset S \backslash W$), hence

$$F(W) \cap L(x) \neq \square.$$

Then $y \in L(x) \cap F(W) \subset V$ is less than x. Now S is connected, since $S = U\{L(x): x \in S\}$ and $L(x)$ meets K for each x in S. Thus for the purpose of showing (1) we may assume that $S \backslash K$ is a proper open non-null subset of a continuum, and consequently, $F(S \backslash K)$ is also non-null. Thus by the theorem cited above, each element of $S \backslash K$ can be joined to $F(S \backslash K) \subset K$ by a compact connected chain and (1) follows immediately.

For the second part of the conclusion, let V be an open set such that $K \subset V \subset V^* \neq S$, then $U(V)$ is open since S is compact. Now if $x \in U(V)$, then $L(x) \subset V$, hence $x \in L(x)$ implies $U(V) \subset V$. Also, let $x \in U(V)$, then if $z \in L(x)$ we have that $L(z) \subset L(x) \subset V$, thus $z \in U(V)$, and hence $L(x) \subset U(V)$. Since K is connected and $L(x)$ meets K for each $x \in S$ we now have that $U(V) = \bigcup\{L(x): x \in U(V)\}$ is connected. Thus $U(V)^* \subset V^*$ is a proper subcontinuum of S containing interior points and (2) follows readily.

COROLLARY 1. *A compact connected inverse semigroup* S *is arcwise connected if its minimal ideal is arcwise connected, and is a group if* S *is indecomposable.*

Proof. If $x \in S$, then $x = x(x^{-1}x)$ and $x^{-1}x \in E$. E is commutative and, as shown in (**4**), a retract of S, hence connected.

COROLLARY 2. *A compact connected semilattice is arcwise connected and decomposable.*

3. In (**6**) Wallace showed that a compact connected semigroup S, with zero, $S = SE = ES$ and E commutative, is acyclic. Hence the semigroups of Corollary 1 (when K is trivial), and Corollary 2, are, in addition, known to be acyclic. Now, if X is a continuum, then the space $S(X)$ of all non-empty closed subsets of X with the Vietoris topology [i.e., if U and V are open subsets of X, $N(U, V) = \{C | \, C \in S(X),\ C \subset U \text{ and } C \cap V \neq \square\}$, then $\{N(U, V) | \, U, V \text{ open}\}$ is a sub-basis for the open sets of $S(X)$] is also a continuum (**5**).

THEOREM. *If X is a continuum, then $S(X)$ is decomposable, arcwise connected, and acyclic.*

Proof. If X is any continuum, one easily verifies that $S(X)$ under the operation of set-theoretic union is a compact connected semilattice. Thus Corollary 2 and the above-stated theorem apply.

THEOREM. *If X is a continuum, then $C(X)$ is arcwise connected.*

Proof. We use the corollary of § 1. Define $A \leqq B$ if and only if $B \subset A$. A routine check shows that $\leqq$ is a partial order with closed graph. Clearly, X is a zero for $(C(X), \leqq)$. Now let $A \in C(X)$, then $L(A) = \{C | \, A \subset C\}$ is connected, otherwise $L(A)$ is the union of two non-null, disjoint, closed sets D and E, and we may suppose that $A \in D$. Since $E \neq \square$ there exists a maximal element M in E. Let U_i and V_i be open sets in X such that

$$M \in \cap\{N(U_i, V_i) | \, 1 \leqq i \leqq m\} \cap L(A) \subset E.$$

If $m = 1$, then $M \in N(U, V) \cap L(A) \subset E$. Since $A \notin N(U, V)$, we have that $A \cap V = \square$. Let W be an open set in X such that $W^* \subset V$ and $W \cap M \neq \square$. Then $M \backslash W^*$ is an open subset of M containing A, hence, if C is the component of $M \backslash W^*$ containing A, then $C^* \cap F(M \backslash W^*) \neq \square$. Thus, $C^* \cap V \neq \square$ and we have that $C^* \in N(U, V) \cap L(A) \subset E$, but $C^* \cap V$ is a proper subset of $M \cap V$, hence $C^* > M$, a contradiction. It follows from an easy induction argument that such a maximal element cannot exist and hence, $L(A)$ is connected for each $A \in C(X)$.

4. We now offer a mild extension of the corollary in §1, and an application of the revised version.

KOCH'S ARC THEOREM. *Let $(S, \leqq)$ be a compact partially ordered space, with zero, such that $L(x)$ is connected for each $x \in S$ and the graph of $\leqq$ is closed.*

Then S is arcwise connected (**3**) *and*

(1) *$L(x)$ is arcwise connected for each $x \in S$,*

(2) *zero has an arcwise connected base.*

Proof. For (1) we note that $(L(x), \leqq)$ satisfies the hypotheses of the theorem, hence by Koch's result is itself arcwise connected. For (2) we show that $\{U(V): V$ is open and contains zero$\}$ is the required base. If V is an open set containing zero, then the proof of the second part of the theorem of §2 yields $U(V) = \bigcup\{L(x): x \in U(V)\}$ and $U(V) \subset V$. By (1), $L(x)$ is arcwise connected but $L(x)$ contains x and zero and is a subset of $U(V)$ for each $x \in U(V)$.

COROLLARY. *If S is a compact connected semigroup with zero, $S = SE$, with E connected and commutative, then S is locally arcwise connected at zero.*

Proof. We have seen that such a semigroup admits a closed partial order with $L(x)$ connected for each x, and such that an algebraic zero becomes an order zero.

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