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TRANSLATION INVARIANT FUNCTIONALS ON $L^p(G)$ WHEN G IS NOT AMENABLE

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Abstract

It is shown that if G is a non-amenable group, then there are no non-zero translation invariant functionals on $L^p(G)$ for $1 < p < \infty$. Furthermore, if G contains a closed, non-abelian free subgroup, then there are no non-zero translation invariant functionals on $C_0(G)$. The latter is proved by showing that a certain non-invertible convolution operator on $C_0(G)$ is surjective.

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Let G be a locally compact group. Throughout this paper m will denote left invariant Haar measure on G, and $L^p(G)$, $1 \leqslant p \leqslant \infty$, will denote the corresponding Lebesgue spaces. The space of continuous functions on G which converge to zero at infinity will be denoted by $C_0(G)$.

For each x in G, δ_x will be the point mass at x. Define the convolution product of δ_x and a function f in $L^p(G)$ or $C_0(G)$ by

$$(\delta_x * f)(y) = f(x^{-1}y), \qquad (y \in G).$$

Convolution by δ_x is an operator on $L^p(G)$ or $C_0(G)$ and is usually called *translation* by x. Translation by x is an isometry because m is left invariant. If $\sum_{i=1}^n c_i\delta_{x_i}$ is a linear combination of point masses, then the operator of convolution by this will be denoted by $(\sum_{i=1}^n c_i\delta_{x_i})*$ and the norm of this operator on $L^p(G)$ will be denoted by

$$\left|\left(\sum_{i=1}^{n} c_i\delta_{x_i}\right)*\right|_p.$$

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A linear functional λ on $L^p(G)$ or $C_0(G)$ is said to be *translation invariant* if

$$\lambda(\delta_x * f) = \lambda(f), \qquad (x \in G,\ f \in L^p(G) \text{ or } C_0(G)).$$

The purpose of this paper is to show that, for many groups G, there are no non-zero translation invariant functionals on $L^p(G)$, $1 < p \leqslant \infty$, or on $C_0(G)$. It follows that there will be no discontinuous translation invariant functionals on these spaces. The question of the existence of discontinuous translation invariant functionals on $L^p(G)$ and $C_0(G)$ has already been answered for some groups and p's, and a survey of the results obtained prior to 1981 is given in [6].

It is shown in [10] that there are discontinuous translation invariant functionals on $L^1(G)$ for every G (see also [4], [12]). Various results have been obtained for $L^p(G)$ when $p > 1$, and for $C_0(G)$. All of the examples of discontinuous translation invariant functionals which have been found (apart from those on $L^1(G)$) have been for amenable groups. Indeed, some of the constructions of discontinuous translation invariant functionals on $L^p(G)$ or $C_0(G)$ have explicitly used the amenability of G (see [10] and [12], for example). The only result concerning non-amenable groups which has been obtained previously is that there are no non-zero translation invariant functionals on $L^\infty(\mathbb{F}_2)$ [10].

In Section 1 it will be shown that there are no nonzero translation invariant functionals on $L^p(G)$ when $1 < p < \infty$ and G is a non-amenable group. This will be an easy consequence of a known characterization of amenable groups.

In Section 2 it will be shown that there are no nonzero translation invariant functionals on $C_0(G)$ when G has a closed subgroup isomorphic to a non-abelian free group. This will be achieved by showing that, if a and b belong to G and are such that the subgroup they generate is closed and free, then the operator $(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b)*$ is surjective on $C_0(G)$ (Theorem 2.2). Theorem 2.2 has some further consequences which will also be given in Section 2.

The final section contains an automatic continuity theorem for linear transformations which commute with translations. It is shown first of all that, if a and b generate a closed free subgroup of G, then the ranges of the operators $(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b)*$ and $(\delta_e + e^{-2\pi i/3}\delta_a + e^{2\pi i/3}\delta_b)*$ on $L^1(G)$ have only the zero vector in common (Theorem 3.1). This is used, together with Theorem 2.2, to show that, if X is a Banach space on which G acts, then every linear transformation T: $X \to L^1(G)$ which commutes with translations is continuous. It seems likely that Theorems 2.2 and 3.1 will have many more applications than those given here.

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1. $L^p(G)$ when G is not amenable

The fact that there are no non-zero translation invariant linear functionals on $L^p(G)$ when G is non-amenable and $1 < p < \infty$ will follow from the lemma below. This lemma is well known in the case when G is discrete (see [1, Theorem 1] and [7, 8.3.7]), but the non-discrete case seems not to be in the literature.

1.1. LEMMA. *Let G be a non-amenable locally compact group, and let $1 < p < \infty$. Then there are elements $x_1, x_2, \ldots, x_n$ in G and non-negative numbers $c_1, c_2, \ldots, c_n$ with $\sum_{i=1}^n c_n = 1$ such that $|(\sum_{i=1}^n c_i \delta_{x_i}) * |_p < 1$.*

PROOF. Suppose, for every $x_1, x_2, \ldots, x_n$ in G and every convex combination $\sum_{i=1}^n c_i \delta_{x_i}$, that $|(\sum_{i=1}^n c_i \delta_{x_i}) * |_p = 1$. We will show that G is amenable.

Let $y_1, y_2, \ldots, y_n$ belong to G. Then $|(n+1)^{-1} \sum_{i=0}^n \delta_{y_i}) * |_p = 1$, where $y_0 = e$, the group identity. Hence there is a sequence $(f_k)_{k=1}^\infty$ in $L^p(G)$ with $\|f_k\|_p = 1$ for each k such that

$$\lim_{k \to \infty} \left\| \frac{1}{n+1} \left(\sum_{i=0}^n \delta_{y_i} * f_k \right) \right\|_p = 1.$$

It follows, by the uniform convexity of the unit ball in $L^p(G)$, that

$$\left\| \delta_{y_0} * f_k - \delta_{y_i} * f_k \right\|_p = \left\| f_k - \delta_{y_i} * f_k \right\|_p \to 0$$

for each $i = 1, 2, \ldots, n$. Since $\|f - \delta_y * f\|_p \geqslant \| |f| - \delta_y * |f| \|_p$, we may suppose that f_k is a non-negative function for each k.

Now, by [7, Chapter 8, Section 3.2], there is a sequence $(f_k')_{k=1}^\infty$ of non-negative functions in $L^1(G)$ with $\|f_k'\|_1 = 1$ for each k such that $\lim_{k \to \infty} \|f_k' - \delta_{y_i} * f_k'\|_1 = 0$ for $i = 1, 2, \ldots, n$. Therefore, by [8, Section 3, Proposition 1], G is amenable.

1.2. COROLLARY. *Let G be a non-amenable locally compact group, and let $1 < p < \infty$. Then there are no non-zero translation invariant linear functionals on $L^p(G)$.*

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PROOF. Let $\sum_{i=1}^n c_i\delta_{x_i}$ be a convex combination of point masses such that $\|(\sum_{i=1}^n c_i\delta_{x_i})*\|_p < 1$. Then convolution by $\delta_e - (\sum_{i=1}^n c_i\delta_{x_i})$ is an invertible operator on $L^p(G)$. Hence, for each f in $L^p(G)$, there is a g in $L^p(G)$ such that

$$\begin{aligned} f &= g - \left(\sum_{i=1}^n c_i\delta_{x_i}\right) * g \\ &= \sum_{i=1}^n c_i\left(g - \delta_{x_i} * g\right). \end{aligned}$$

Now let λ be a translation invariant linear functional on $L^p(G)$, and let f belong to $L^p(G)$. Then

$$\lambda(f) = \sum_{i=1}^n c_i\lambda\left(g - \delta_{x_i} * g\right) = 0.$$

Therefore $\lambda = 0$.

2. $C_0(G)$ when G contains a free group

In this section we will show that, if G has a closed subgroup isomorphic to the free group on two generators, then there are no non-zero translation invariant linear functionals on $C_0(G)$.

To do this for non-discrete groups we must first construct a partial Borel transversal for the free group which has certain additional properties. This is done in the following lemma.

2.1. LEMMA. *Let G be a locally compact group, H a closed, discrete subgroup of G, and K a compact subset of G. Then there is a Borel subset E of G such that*

(i) *$\bar{E}$ is compact;*

(ii) *for each x, $y \in H$, $xE \cap yE = \varnothing$ unless $x = y$; and*

(iii) *there is a finite subset C of H such that $K \setminus (\bigcup_{x\in C} xE)$ has zero measure.*

PROOF. We may suppose that $e \in K$. Let $\{x_1, x_2, \ldots, x_n\} = H \cap K$. This set is nonempty, and it is finite since H is closed and discrete and K is compact. Put $L = \bigcup_{i=1}^n x_i^{-1}K$. Then L is compact and not empty.

Let $\mathscr{E}$ be the set of all Borel subsets of L which satisfy condition (ii) above. Then $\mathscr{E}$ is not empty because $\phi \in \mathscr{E}$. Define an equivalence relation $\sim$ on $\mathscr{E}$ by putting $E_1 \sim E_2$ if $m(E_1\Delta E_2) = 0$. Next, define a partial order $\leqslant$ on $\mathscr{E}/\sim$ by putting $[E_1] \leqslant [E_2]$ if there are $E_1' \in [E_1]$ and $E_2' \in [E_2]$ with $E_1' \subseteq E_2'$, where $[E_1]$ and $[E_2]$ are any two equivalence classes in $\mathscr{E}$.

Now let $([E_\alpha])$ be an increasing chain in $\mathscr{E}/\sim$. Then, since each $E_\alpha \subseteq L$, which is compact and so has finite measure. $([E_\alpha])$ has a countable cofinal subchain, $([E_n])_{n=0}^{\infty}$ say. Put $E_\infty = \bigcup_{n=0}^{\infty} \bigcap_{m=n}^{\infty} E_m$. Then E_∞ is a Borel subset of L. Also, $\bigcap_{m=n}^{\infty} E_m \subseteq E_n$ for each n and so satisfies condition (ii). Since E_∞ is an increasing union of sets which satisfy (ii), E_∞ satisfies condition (ii). Hence $E_\infty \in \mathscr{E}$. Since $([E_n])_{n=0}^{\infty}$ is an increasing chain, $E_n \setminus E_{n+1}$ has measure zero for each n. Hence, for each n, we have

$$E_n \setminus E_\infty \subseteq E_n \setminus \left(\bigcap_{m=n}^{\infty} E_m \right) \subseteq \bigcup_{m=n}^{\infty} (E_m \setminus E_{m+1}),$$

which has measure zero. Thus $[E_n] \leqslant [E_\infty]$ for each n, and so $[E_\infty]$ is an upper bound for the chain $([E_\alpha])$. It now follows, by Zorn's Lemma, that there is a maximal element in $(\mathscr{E}/\sim, \leqslant)$. Let E be such that $[E]$ is maximal. Then E satisfies (ii) and $E \subseteq L$, from which it follows that $\overline{E}$ is compact.

In order to show that E satisfies (iii), suppose that $m(K \setminus (\bigcup_{x \in H} xE)) > 0$. Choose a Borel subset D of $K \setminus (\bigcup_{x \in H} xE)$ such that $m(D) > 0$, but which is sufficiently small that $H \cap (DD^{-1}) = \{e\}$. Consider $E' = E \cup (x_1^{-1}D)$, which is a subset of L. If $xE' \cap yE' \neq \varnothing$ for some x, y in H, then either $xE \cap yE \neq \varnothing$, $xE \cap y(x_1^{-1}D) \neq \varnothing$ (or $x(x_1^{-1}D) \cap yE \neq \varnothing$), or $x(x_1^{-1}D) \cap y(x_1^{-1}D) \neq \varnothing$. The first alternative implies that $x = y$ because $E \in \mathscr{E}$; the second implies that $D \cap (x_1 y^{-1} xE) \neq \varnothing$ (or $D \cap (x_1 x^{-1} yE) \neq \varnothing$), which is not possible because $D \cap (\bigcup_{x \in H} xE) = \varnothing$; and the third alternative implies that $x_1 y^{-1} x x_1^{-1} \in DD^{-1}$ which, since $DD^{-1} \cap H = \{e\}$, implies that $x = y$. Hence $E' \in \mathscr{E}$. Since $m(D) > 0$, we have $[E'] > [E]$, which contradicts the maximality of $[E]$. Therefore $m(K \setminus (\bigcup_{x \in H} xE)) = 0$. Now let $C = \{x \in H \mid xE \cap K \neq \varnothing\}$. Then $m(K \setminus (\bigcup_{x \in C} xE)) = 0$ and $C \subseteq KL^{-1}$, which is compact. Hence C is finite, and so E satisfies (iii).

Now let $\mathbb{F}_2$ denote the free group on two generators a and b. We will be regarding $\mathbb{F}_2$ as a closed subgroup of the locally compact group G, and so a and b will be elements of G. If x belongs to $\mathbb{F}_2$, the expression "$x = a$__" will mean that the reduced word for x has the form $a^{\varepsilon} b^{\delta_1} a^{\varepsilon_1} \cdots b^{\delta_n} a^{\varepsilon_n}$, where $n \geqslant 0$, and where $\delta_1, \varepsilon_1, \ldots, \delta_n, \varepsilon_n$ are integers which are all non-zero except possibly for ε_n, and where ε is a positive integer (see [5, Chapter 1, Section 1] for terminology concerning free groups). The meanings of expressions such as "$x = b$__," "$x = a^{-1}$__" and "$x = b^{-1}$__" are defined in a corresponding way. In the following, χ will be a character on $\mathbb{F}_2$, so that $\chi(a)$ and $\chi(b)$ will be complex numbers with modulus one. Conversely, if θ and ψ are two complex numbers with modulus one, then there is a unique character χ on $\mathbb{F}_2$ with $\chi(a) = \theta$ and $\chi(b) = \psi$.

The fact that there are no non-zero translation-invariant linear functionals on $C_0(G)$ when G contains a non-abelian free group will follow from the next result.

2.2. THEOREM. *Let G be a locally compact group which contains $\mathbb{F}_2$ as a closed subgroup and let χ be a character on $\mathbb{F}_2$. Then the operator $(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b)*$ is surjective on $C_0(G)$ and also on $L^p(G)$ for $p > 2$. More precisely, let $p > 2$ and $\varepsilon > 0$. Then*

(a) *for each $f \in L^p(G)$, there is $g \in L^p(G)$ such that*

$$f = (\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g \quad \text{and} \quad \|g\|_p \leqslant \left(\tfrac{1}{2}\right)^{1/p} \frac{2^{2/p}}{2 - 2^{2/p}} \|f\|_p + \varepsilon;$$

and

(b) *for each $f \in C_0(G)$, there is a $g \in C_0(G)$ such that*

$$f = (\delta_e + \chi(x)\delta_a + \chi(b)\delta_b) * g \quad \text{and} \quad \|g\|_\infty \leqslant \|f\|_\infty + \varepsilon.$$

PROOF. We shall show that, for every bounded, measurable function f with compact support in G, there is a bounded, measurable function g such that $f(t) = [(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g](t)$ almost everywhere, such that

$$\|g\|_p \leqslant \left(\tfrac{1}{2}\right)^{1/p} \frac{2^{2/p}}{2 - 2^{2/p}} \|f\|_p \quad \text{and} \quad \|g\|_\infty \leqslant \|f\|_\infty,$$

and such that g converges to zero at infinity. This will suffice to prove the theorem.

To deduce the $C_0(G)$ case for example, let f' belong to $C_0(G)$. Then there are bounded measurable functions $f_0, f_1, f_2, \ldots$ which have compact support and which satisfy $\|f_0\|_\infty \leqslant \|f'\|_\infty$ and $\|f_n\|_\infty \leqslant \varepsilon 2^{-n}$, $n = 1, 2, 3, \ldots$, and there is a symmetric function ϕ in $L^1(G)$ with $\|\phi\|_1 < 1$ such that $f' = \sum_{n=0}^{\infty} f_n * \phi$. By what we are about to show, there are functions $g_0, g_1, g_2, \ldots$ such that, for $n = 0, 1, 2, \ldots$, $f_n = (\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g_n$, $\|g\|_\infty \leqslant \|f_n\|_\infty$, and g_n converges to zero at infinity. Hence $f' = \sum_{n=0}^{\infty} f_n * \phi = (\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * (\sum_{n=0}^{\infty} g_n * \phi)$, where $\sum_{n=0}^{\infty} g_n * \phi$ belongs to $C_0(G)$ and $\|\sum_{n=0}^{\infty} g_n * \phi\|_\infty \leqslant \|f'\|_\infty + \varepsilon$. The result for $L^p(G)$ may be deduced in a similar way.

In order to prove the above claim it is necessary to introduce some set valued functions on $\mathbb{F}_2$. For each $x \in \mathbb{F}_2 \setminus \{e\}$, let $\eta(x)$ be the set defined by

$$\eta(x) = \begin{cases} \{a^{-1}x, b^{-1}x, ax, b^{-1}ax\}, & \text{if } x = b^{-1}\_\_, \\ \{a^{-1}x, b^{-1}x, bx, a^{-1}bx\}, & \text{if } x = a^{-1}\_\_, \\ \{ax, b^{-1}ax, bx, a^{-1}bx\}, & \text{if } x = a\_\_ \text{ or } b\_\_. \end{cases}$$

Then $\eta(x) \subseteq \mathbb{F}_2 \setminus \{e\}$ and $|\eta(x)| = 4$ for each x. By considering reduced words, it may be seen that $\eta(x_1) \cap \eta(x_2) = \varnothing$ unless $x_1 = x_2$.

Now for each $x \in \mathbb{F}_2$ define sets $\mathscr{M}(n,x)$, $n = 1,2,3,\ldots$, inductively as follows. Define

$$\mathscr{M}(1,x) = \begin{cases} \{a^{-1}x, b^{-1}x\}, & \text{if } x = a^{-1}\_\_,\ b^{-1}\_\_, \text{ or } x = e, \\ \{x, b^{-1}x\}, & \text{if } x = a\_\_, \\ \{a^{-1}x, x\}, & \text{if } x = b\_\_. \end{cases}$$

Then $\mathscr{M}(1,x) \subseteq \mathbb{F}_2 \setminus \{e\}$ for each x in $\mathbb{F}_2$. Once $\mathscr{M}(n,x)$ has been defined, put $\mathscr{M}(n+1,x) = \eta(\mathscr{M}(n,x))$. Then $|\mathscr{M}(n,x)| = 2^{2n-1}$ and, for each n, $\mathscr{M}(n,x_1) \cap \mathscr{M}(n,x_2) = \varnothing$ unless $x_1 = x_2$, as may be proved by induction using the properties of η stated above.

Now let f be a bounded measurable function on G and assume that $K = \operatorname{supp}(f)$ is compact. For this K, and with $H = \mathbb{F}_2$, choose E, a Borel subset of G, and C, a finite subset of $\mathbb{F}_2$, as in 2.1.

For each $n = 1,2,3,\ldots$, define a function h_n on G by

$$h_n(yw) = \begin{cases} (\tfrac{1}{2})^n \chi(yx^{-1}) f(xw), & \text{if } w \in E \text{ and } x, y \in \mathbb{F}_2 \textit{ with } y \in \mathscr{M}(n,x), \\ 0, & \text{otherwise.} \end{cases}$$

Then h_n is well-defined by 2.1(ii), and because $\mathscr{M}(n,x_1) \cap \mathscr{M}(n,x_2) = \varnothing$ if $x_1 \neq x_2$. By 2.1, parts (i) and (ii), and because $\mathscr{M}(n,x)$ is finite, the closure of the essential support of h_n is compact. It is clear that $\|h_n\|_\infty = (\tfrac{1}{2})^n \|f\|_\infty$ and that

$$\begin{aligned} \|h_n\|_p &= \left(\sum_{y \in \mathbb{F}_2} \int_E |h_n(yw)|^p \, dm(w) \right)^{1/p} \\ &= \left(\sum_{x \in \mathbb{F}_2} \left(\sum_{\mathscr{M}(n,x)} \int_E \left| \left(\frac{1}{2}\right)^n f(xw) \right|^p dm(w) \right) \right)^{1/p} \\ &= \left(\frac{1}{2}\right)^n (2^{2n-1})^{1/p} \left(\sum_{x \in \mathbb{F}_2} \int_E |f(xw)|^p \, dm(w) \right)^{1/p} \\ &= (2^{n(2-p)-1})^{1/p} \|f\|_p, \end{aligned}$$

by 2.1(iii). Now put $g = \sum_{n=1}^{\infty} (-1)^{n+1} h_n$. Then g is bounded and measurable, it converges to zero at infinity, $\|g\|_\infty \leqslant \|f\|_\infty$, and

$$\|g\|_p \leqslant \left(\sum_{n=1}^{\infty} (2^{n(2-p)-1})^{1/p} \right) \|f\|_p = 2^{-1/p} \frac{2^{2/p}}{2 - 2^{2/p}} \|f\|_p.$$

It remains to show that $(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g = f$. To prove this it will suffice to show that $(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g(xw) = f(xw)$ for every $x \in \mathbb{F}_2$ and every $w \in E$, because $g(t) = 0$ for every $t \in G \setminus (\mathbb{F}_2 E)$, and because $f(t) = 0$ for almost every $t \in G \setminus (\mathbb{F}_2 E)$.

Let $w \in E$ and $x \in \mathbb{F}_2 \setminus \{e\}$, and suppose that $x = a^{-1}\_\_$ or $b^{-1}\_\_$. Then $a^{-1}x$ and $b^{-1}x$ are in $\mathscr{M}(1, x)$ and also in $\eta(x)$, and so $\chi(a)h_1(a^{-1}xw) = f(xw)/2 = \chi(b)h_1(b^{-1}xw)$ and $\chi(a)h_{n+1}(a^{-1}xw) = h_n(xw)/2 = \chi(b)h_{n+1}(b^{-1}xw)$ for $n = 1, 2, 3, \ldots$. Hence,

$$\begin{aligned}
\chi(a)g(a^{-1}xw) &= \chi(a)\sum_{n=1}^{\infty}(-1)^{n+1}h_n(a^{-1}xw) \\
&= \chi(a)h_1(a^{-1}xw) - \chi(a)\sum_{n=1}^{\infty}(-1)^{n+1}h_{n+1}(a^{-1}xw) \\
&= \frac{1}{2}f(xw) - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}h_n(xw) \\
&= \frac{1}{2}(f(xw) - g(xw)) \\
&= \chi(b)g(b^{-1}xw).
\end{aligned}$$

Therefore, if $x = a^{-1}\_\_$ or $b^{-1}\_\_$, then

$$\begin{aligned}
&[(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g](xw) \\
&\qquad = g(xw) + \chi(a)g(a^{-1}xw) + \chi(b)g(b^{-1}xw) = f(xw)
\end{aligned}$$

for every $w \in E$.

Next let $x \in \mathbb{F}_2 \setminus \{e\}$ and suppose that $x = a\_\_$. Then either $x = a$ or $x = ay$, where $y = b^{-1}\_\_$, $a\_\_$, or $b\_\_$. If $x = a$, then x and $b^{-1}x$ are in $\mathscr{M}(1, x)$ and not in $\eta(z)$ for any z, and so $h_1(xw) = f(xw)/2 = \chi(b)h_1(b^{-1}xw)$ and $h_n(xw) = h_n(b^{-1}xw) = 0$ for $n = 2, 3, 4, \ldots$. Hence,

$$g(xw) = h_1(xw) = \tfrac{1}{2}f(xw) = \chi(b)g(b^{-1}xw).$$

Also, $g(a^{-1}xw) = g(ew) = 0$. Therefore, if $x = a$, then $[(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g](xw) = f(xw)$. If $x = ay$, then x and $b^{-1}x$ are in $\mathscr{M}(1, x)$ and also in $\eta(y)$, and so $h_1(xw) = f(xw)/2 = \chi(b)h_1(b^{-1}xw)$ and $h_{n+1}(xw) = \chi(a)h_n(yw)/2 = \chi(b)h_{n+1}(b^{-1}xw)$ for $n = 1, 2, 3, \ldots$. Hence

$$\begin{aligned}
g(xw) &= h_1(xw) - \sum_{n=1}^{\infty}(-1)^{n+1}h_{n+1}(xw) \\
&= \frac{1}{2}f(xw) - \frac{1}{2}\chi(a)\sum_{n=1}^{\infty}(-1)^{n+1}h_n(yw) \\
&= \frac{1}{2}(f(xw) - \chi(a)g(yw)) \\
&= \chi(b)g(b^{-1}xw).
\end{aligned}$$

Therefore, if $x = ay$, then

$$\begin{aligned}[(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g](xw) \\ = g(xw) + \chi(a)g(yw) + \chi(b)g(b^{-1}xw) = f(xw)\end{aligned}$$

for every $w \in E$. A similar argument shows that this also holds when $x = b\_\_$.

Finally, suppose that $x = e$. Then $g(xw) = 0$ and $\chi(a)g(a^{-1}xw) = f(xw)/2 = \chi(b)g(b^{-1}xw)$, and the required identity follows. Therefore, for every $x \in \mathbb{F}_2$ and $w \in E$, we have

$$[(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * g](xw) = f(xw).$$

2.3. Remark. When G is discrete, the proof of 2.2 in fact describes a construction for a right inverse for the operator $(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b)*$ on $C_0(G)$. It does not do so when G is not discrete because the smoothing function ϕ cannot be chosen independently of f' (see the second paragraph of the proof). However, if a full transversal for $\mathbb{F}_2$ is chosen, then a right inverse for $(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b)*$ on $L^p(G)$, $2 < p \leqslant \infty$, may be constructed in the way described in the proof even when G is not discrete, because the smoothing function is not required in the $L^p(G)$ cases of Theorem 2.2.

2.4. Corollary. *Let G be a locally compact group which contains $\mathbb{F}_2$ as a closed subgroup. Then there are no non-zero translation-invariant linear functionals on $C_0(G)$ or on $L^\infty(G)$.*

Proof. Let λ be a translation-invariant linear functional on $C_0(G)$ and let $f \in C_0(G)$. Then there is $g \in C_0(G)$ such that $f = (\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b) * g$. Hence,

$$\lambda(f) = \lambda(g) + e^{2\pi i/3}\lambda(\delta_a * g) + e^{-2\pi i/3}\lambda(\delta_b * g) = 0.$$

A similar fact holds for $L^\infty(G)$.

The operator $(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b)*$ is not surjective on $L^p(G)$ for $1 \leqslant p \leqslant 2$. However, it is bounded from below on $L^p(G)$ for $1 \leqslant p < 2$, as the next corollary shows. This result is clearly equivalent to the dual of Theorem 2.2.

2.5. Corollary. *Let G be a locally compact group which contains $\mathbb{F}_2$ as a closed subgroup, let χ be a character on $\mathbb{F}_2$, and let $1 \leqslant p < 2$. Then, for every $f \in L^p(G)$, we have*

$$\|(\delta_e + \chi(a)\delta_a + \chi(b)\delta_b) * f\|_p \geqslant \left(\tfrac{1}{2}\right)^{1/p}(2^{2/p} - 2)\|f\|_p.$$

For each locally compact group G, let $L_0^1(G) = \{f \in L^1(G) \mid \int_G f\,dm = 0\}$. Then $L_0^1(G)$ is an ideal in $L^1(G)$ and is called the *augmentation ideal.* A well-known characterization of amenability asserts that G is amenable if and only

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if $L_0^1(G)$ has bounded approximate units (see [9]). It is shown in [3] that not only does $L_0^1(\mathbb{F}_2)$ not have bounded approximate units, but it also does not even satisfy the weaker condition of having right approximate units. In particular, it is shown that if $h_1 = \delta_e - \delta_a + \delta_b - \delta_{ab}$, then $h_1 \in L_0^1(\mathbb{F}_2)$, but $h_1 \notin (h_1 * L_0^1(\mathbb{F}_2))^-$.

The next result states that $h_2 = \delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b$ also has the property that $h_2 \in L_0^1(\mathbb{F}_2)$, but $h_2 \notin (h_2 * L_0^1(\mathbb{F}_2))^-$. This is another corollary of Thoerem 2.2. We have already seen that it follows from 2.2 that there are no translation-invariant linear functionals on $L^\infty(\mathbb{F}_2)$. Thus 2.2 combines two different proofs that $\mathbb{F}_2$ is not amenable.

2.6. COROLLARY (B. E. Johnson [3]). *Let G be a locally compact group which contains $\mathbb{F}_2$ as a closed subgroup. Then $L_0^1(G)$ does not have right approximate units.*

PROOF. Choose $f \in L^1(G)$ with $\int_G f\,dm = 1$ and put $h = (\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b) * f$. Then $h \in L_0^1(G)$.

Now let $u \in L_0^1(G)$. Then

$$\begin{aligned}\|h - h * u\|_1 &= \left\|\left(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b\right) * (f - f * u)\right\|_1 \\ &\geqslant \|f - f * u\|_1 \qquad \text{by } 2.5 \\ &\geqslant \left|\int_G (f - f * u)\,dm\right| = 1.\end{aligned}$$

Therefore $h \notin (h * L_0^1(G))^-$, and so $L_0^1(G)$ does not have right approximate units.

3. Linear transformations into $L^1(G)$ which commute with translations

Let G be a group and X a Banach space on which G acts. In other words, let there be a group homomorphism $\Delta\colon G \to \mathscr{B}^{-1}(X)$, where $\mathscr{B}^{-1}(X)$ is the group of invertible linear operators on X. A linear transformation $S\colon X \to L^1(G)$ is said to *commute with translations* if

$$S(\Delta(g)x) = \delta_g * S(x), \qquad (g \in G, x \in X).$$

If $\Delta(G)$ is a bounded subgroup of $\mathscr{B}^{-1}(X)$ then X becomes a Banach G-module and S becomes a G-module homomorphism in the sense of [2]. In this case it follows from Corollary 4.2 of [2] that, if G has a closed subgroup isomorphic to $\mathbb{Z}$, then every G-module homomorphism $S\colon X \to L^1(G)$ is continuous. The remark after Example 4.4 in [2] indicates that the condition that $\Delta(G)$

be bounded may be weakened somewhat, but the question as to whether any such condition is necessary is open. In this section we shall show that it is not necessary if G has a closed subgroup isomorphic to $\mathbb{F}_2$. This will follow from the next theorem.

3.1. THEOREM. *Let G be a locally compact group which contains $\mathbb{F}_2$ as a closed subgroup. Then*

(i) $[(\delta_e - \delta_a) * L^1(G)] \cap [(\delta_e - \delta_b) * L^1(G)] = \{0\}$; *and*

(ii) $[(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b) * L^1(G)] \cap [(\delta_e + e^{-2\pi i/3}\delta_a + e^{2\pi i/3}\delta_b) * L^1(G)] = \{0\}$.

PROOF. To prove (i), let $f \in [(\delta_e - \delta_a) * L^1(G)] \cap [(\delta_e - \delta_b) * L^1(G)]$ and suppose that $f \neq 0$. Then there is an open set U such that $|\int_U f\,dm| > 0$, and we may suppose that U is sufficiently small that $xU \cap yU = \varnothing$ unless $x = y$, $(x, y \in \mathbb{F}_2)$.

We shall show by induction on k that

$$\sum_{\substack{(n_1, n_2, \ldots, n_k) \in \mathbf{Z}^k \\ n_1, n_2, \ldots, n_k \neq 0}} \left| \int_{U(n_1, n_2, \ldots, n_k)} f\,dm \right| \geqslant \left| \int_U f\,dm \right|,$$

where $U(n_1, n_2, \ldots, n_k) = \ldots b^{n_4}a^{n_3}b^{n_2}a^{n_1}U$. To prove this when $k = 1$, note that, since $f \in (\delta_e - \delta_a) * L^1(G)$, we have

$$\sum_{n \in \mathbf{Z}} \int_{a^n U} f\,dm = 0,$$

and so

$$\sum_{\substack{n_1 \in \mathbf{Z} \\ n_1 \neq 0}} \left| \int_{a^{n_1}U} f\,dm \right| \geqslant \left| \sum_{\substack{n_1 \in \mathbf{Z} \\ n_1 \neq 0}} \int_{a^{n_1}U} f\,dm \right| = \left| \int_U f\,dm \right|.$$

Now suppose that we have proved the inequality for some k. If k is odd, then $U(n_1, n_2, \ldots, n_k, n_{k+1}) = b^{n_{k+1}}a^{n_k} \ldots b^{n_2}a^{n_1}U = b^{n_{k+1}}U(n_1, n_2, \ldots, n_k)$. Thus, since $f \in (\delta_e - \delta_b) * L^1(G)$, we have

$$\sum_{n \in \mathbf{Z}} \int_{b^n U(n_1, n_2, \ldots, n_k)} f\,dm = 0,$$

and so

$$\sum_{\substack{n_{k+1} \in \mathbf{Z} \\ n_{k+1} \neq 0}} \left| \int_{U(n_1, n_2, \ldots, n_k, n_{k+1})} f\,dm \right| \geqslant \left| \int_{U(n_1, n_2, \ldots, n_k)} f\,dm \right|.$$

Therefore,

$$\sum_{\substack{(n_1,n_2,\ldots,n_k,n_{k+1})\in\mathbf{Z}^{k+1}\\ n_1,n_2,\ldots,n_k,n_{k+1}\neq 0}} \left|\int_{U(n_1,n_2,\ldots,n_k,n_{k+1})} f\,dm\right|$$

$$\geqslant \sum_{\substack{(n_1,n_2,\ldots,n_k)\in\mathbf{Z}^{k}\\ n_1,n_2,\ldots,n_k\neq 0}} \left|\int_{U(n_1,n_2,\ldots,n_k)} f\,dm\right| \geqslant \left|\int_U f\,dm\right|$$

by the induction hypothesis. If k is even, then a similar argument using the fact that $f \in (\delta_e - \delta_a) * L^1(G)$ gives the same result.

If $(n_1, n_2, \ldots, n_k) \in \mathbf{Z}^k$ with $n_1, n_2, \ldots, n_k \neq 0$, and if $(m_1, m_2, \ldots, m_l) \in \mathbf{Z}^l$ with $m_1, m_2, \ldots, m_l \neq 0$, then $\ldots b^{n_4}a^{n_3}b^{n_2}a^{n_1} \neq \ldots b^{m_4}a^{m_3}b^{m_2}a^{m_1}$ unless $k = l$ and $n_1 = m_1$, $n_2 = m_2, \ldots, n_k = m_k$. Hence, by the definition of $U(n_1, n_2, \ldots, n_k)$ and by the choice of U, $U(n_1, n_2, \ldots, n_k) \cap U(m_1, m_2, \ldots, m_l) = \varnothing$ unless $k = l$ and $n_1 = m_1$, $n_2 = m_2, \ldots, n_k = m_k$. Hence, for each $N \geqslant 1$, we have

$$\|f\|_1 \geqslant \sum_{k=1}^{N} \left(\sum_{\substack{(n_1,n_2,\ldots,n_k)\in\mathbf{Z}^{k}\\ n_1,n_2,\ldots,n_k\neq 0}} \int_{U(n_1,n_2,\ldots,n_k)} |f|\,dm \right)$$

$$\geqslant \sum_{k=1}^{N} \left(\sum_{\substack{(n_1,n_2,\ldots,n_k)\in\mathbf{Z}^{k}\\ n_1,n_2,\ldots,n_k\neq 0}} \left|\int_{U(n_1,n_2,\ldots,n_k)} f\,dm\right| \right)$$

$$\geqslant N\left|\int_U f\,dm\right|.$$

This is a contradiction. Therefore $f = 0$.

To prove (ii), now, let

$$f \in \left[\left(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b\right) * L^1(G)\right]$$

$$\cap \left[\left(\delta_e + e^{-2\pi i/3}\delta_a + e^{2\pi i/3}\delta_b\right) * L^1(G)\right].$$

Then $f = (\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b) * g_1 = (\delta_e + e^{-2\pi i/3}\delta_a + e^{2\pi i/3}\delta_b) * g_2$ for some g_1 and g_2 in $L^1(G)$. It follows that $(\delta_e - \delta_a)*(e^{2\pi i/3}g_1 - e^{-2\pi i/3}g_2) = (\delta_e - \delta_b)*(-e^{-2\pi i/3}g_1 + e^{2\pi i/3}g_2)$ and so, by (i), both sides equal zero. Since the

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operators $(\delta_e - \delta_a)*$ and $(\delta_e - \delta_b)*$ are injective on $L^1(G)$, we have $e^{2\pi i/3}g_1 - e^{-2\pi i/3}g_2 = 0 = -e^{-2\pi i/3}g_1 + e^{2\pi i/3}g_2$. Therefore, $g_1 = 0 = g_2$, and so $f = 0$.

3.2. THEOREM. *Let G be a locally compact group which contains $\mathbb{F}_2$ as a closed subgroup, let X be a Banach space on which G acts, and let S: $X \to L^1(G)$ be a linear transformation which commutes with translations. Then S is continuous.*

PROOF. Let $\mathfrak{S}$ be the separating space of S (the separating space is defined in [11], p. 7). Since S commutes with translations, $\mathfrak{S}$ is invariant under translations, and so $\mathfrak{S}$ is a closed, left ideal in $L^1(G)$.

Let T be the operator $\Delta(e) + e^{2\pi i/3}\Delta(a) + e^{-2\pi i/3}\Delta(b)$ on X, and let R be the operator $(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b)*$ on $L^1(G)$. Then T and R are continuous linear operators on X and $L^1(G)$, respectively, and $ST\text{-}RS = 0$ because S commutes with translations. Hence, by [11], Lemma 1.6, there is an integer N such that $(R^n\mathfrak{S})^- = (R^N\mathfrak{S})^-$ for all $n \geqslant N$.

Suppose that $\mathfrak{S} \neq \{0\}$. Then $(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b)*\mathfrak{S}$ and $(\delta_e + e^{-2\pi i/3}\delta_a + e^{2\pi i/3}\delta_b)*\mathfrak{S}$ are subspaces of $\mathfrak{S}$ whose intersection is $\{0\}$ by 3.1. Since R^N is injective, it follows from 2.5 that $R^N(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b)*\mathfrak{S}$ and $R^N(\delta_e + e^{-2\pi i/3}\delta_a + e^{2\pi i/3}\delta_b)*\mathfrak{S}$ are subspaces of $R^N\mathfrak{S}$ whose intersection is $\{0\}$. Now $R^N(\delta_e + e^{-2\pi i/3}\delta_a + e^{2\pi i/3}\delta_b)*\mathfrak{S} \neq \{0\}$ by 2.5, and so $R^N(\delta_e + e^{2\pi i/3}\delta_a + e^{-2\pi i/3}\delta_b)*\mathfrak{S} = R^{N+1}\mathfrak{S}$ is a proper subspace of $R^N\mathfrak{S}$. By 2.5 again, $R^N\mathfrak{S}$ and $R^{N+1}\mathfrak{S}$ are closed. Hence $(R^{N+1}\mathfrak{S})^- \neq (R^N\mathfrak{S})^-$, which is a contradiction. Therefore $\mathfrak{S} = \{0\}$ and S is continuous, by [11, Lemma 1.2].

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