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Some remarks on the effect of risk sharing and diversification for infinite mean risks

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Abstract

The basic principle of any version of insurance is the paradigm that exchanging risk by sharing it in a pool is beneficial for the participants. In case of independent risks with a finite mean, this is the case for risk-averse decision-makers. The situation may be very different in case of infinite mean models. In that case it is known that risk sharing may have a negative effect, which is sometimes called the nondiversification trap. This phenomenon is well known for infinite mean stable distributions. In a series of recent papers, similar results for infinite mean Pareto and Fréchet distributions have been obtained. We further investigate this property by showing that many of these results can be obtained as special cases of a simple result demonstrating that this holds for any distribution that is more skewed than a Cauchy distribution. We also relate this to the situation of deadly catastrophic risks, where we assume a positive probability for an infinite value. That case gives a very simple intuition why this phenomenon can occur for such catastrophic risks. We also mention several open problems and conjectures in this context.

1. Introduction

The basic reason for the existence of any kind of insurance or other versions of risk sharing is the fact that risk sharing typically leads to a reduction of risk. Under the assumption of independent and identically distributed risks with a finite mean, it is a simple consequence of the law of large numbers that sharing risks will typically be beneficial for risk-averse decision-makers. The problem of an adverse effect of diversification in case of infinite mean distributions has already been described in a seminal papers by Fama (1965) and Samuelson (1967) for the case of stable distributions. Further studies of this problem with infinite mean stable distributions have followed, for example, by Ibragimov *et al*. (2009), where the term *nondiversification trap* has been introduced for the situation, where an individual does not have an incentive to share his risk with others, as it may have a detrimental effect to do so. Real-world situations where this may happen have been described by several authors. Hofert and Wüthrich (2012) consider nuclear power accidents and estimate the parameters of a Pareto tail to have infinite mean. Eling and Wirfs (2019) consider cyber risk and also find parameter values that lead to infinite mean models. Similar observations have already been found for operational risks in Nešlehová *et al*. (2006).

In recent years, several new papers have appeared that study this phenomenon under the assumption of Pareto distributions and more general. Chen *et al*. (2024b) have shown that i.i.d. Pareto distributions with infinite mean fulfill the inequality

$$X_1 \leq_{st} \sum_{i=1}^{n} \theta_i X_i, \tag{1.1}$$

for all $\theta_i \geq 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^n \theta_i = 1$. Notice that we can assume without loss of generality that the weights fulfill $\theta_i > 0$, as n is arbitrary. We will sometimes use this fact in proofs. This condition means that no rational decision-maker will agree to any version of linear risk sharing, see Chen *et al.* (2024a) for a detailed study of this case. We want to mention that linear peer-to-peer risk-sharing rules of this type have recently been studied also in the context of flood risk pooling by Feng *et al.* (2023). As flood risks can also have very heavy tails this problem may also arise there. Chen and Shneer (2024) have shown that the inequality (1.1) also holds more general for distributions that they call super-Fréchet distributions. Another recent study of linear risk-sharing rules can be found in Yang and Wei (2025).

In this paper, we want to further study the class of distributions fulfilling the condition (1.1). Our main results in Section 2 will demonstrate that this condition also holds for all distributions that we will call super Cauchy distributions and we will clarify that this property is strongly related to skewness properties of the distributions relative to a Cauchy distribution. In Section 3, we will consider the case of risks that may have a positive probability of an infinite value with the interpretation of deadly risks. In that case these results about a nondiversification trap are very intuitive, and this may help to understand why the situation may be similar in case of very heavy tailed infinite mean risks. Section 4 relates our results to other recent contributions and states some open problems.

Notation: Throughout the paper, we will assume the existence of a probability space $(\Omega, \mathcal{A}, P)$, on which we can define random variables $X : \Omega \to \mathbb{R}$ with arbitrary distributions. We will denote by $F_X(t) := P(X \leq t)$, $t \in \mathbb{R}$, the corresponding cumulative distribution function of X, and we will also call F_X just the distribution of X. For any distribution function F, we define the generalized inverse

$$F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \geq u\}, \ 0 < u < 1.$$

We will denote by $X_n \xrightarrow{d} X$ convergence in distribution, that is pointwise convergence of F_{X_n} to F_X in continuity points of F_X. We write $X =_{st} Y$ if $F_X = F_Y$ and we define the usual stochastic order by

$$X \leq_{st} Y, \quad \text{if } F_X(t) \geq F_Y(t) \text{ for all } t \in \mathbb{R}.$$

This usual stochastic order $\leq_{st}$ can also be found under the name *first-order stochastic dominance* in an economic context, and it is generally agreed that any rational decision will prefer a risk X to a risk Y, if $X \leq_{st} Y$ holds. It is the strongest of the well-known stochastic dominance rules used for decisions under risk and in contrast to weaker notions like the second-order stochastic dominance it is defined for any distributions, including infinite mean models. For properties of this and related stochastic orders, we refer to the books Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

2. Main results

Throughout the paper, we will consider risks X, where we assume that these are random variables on some probability space $(\Omega, \mathcal{A}, P)$ with a known distribution. We allow for arbitrary real values of X even though in many applications, in particular in an insurance context, we may have $X \geq 0$. Even if one is only interested in the case of nonnegative risks $X \geq 0$, it is helpful to consider the more general case of arbitrary random variables X as this will give additional insight and more elegant proofs. In particular, it is helpful to relate the problem to stable distributions like the Cauchy distribution, which plays a fundamental role in this context. We will now give the main definition of the class of distributions that we want to consider.

Definition 2.1. *Let $\mathcal{D}^-$ denote the set of distribution functions F_X with the property that*

$$X_1 \leq_{st} \sum_{i=1}^n \theta_i X_i, \tag{2.1}$$

for all $\theta_i \geq 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^n \theta_i = 1$, where $X_1, \ldots, X_n$ are i.i.d. random variables with distribution functions F_X. We will also write $X \in \mathcal{D}^-$, if $F_X \in \mathcal{D}^-$.

We denote by $\mathcal{D}^+$ the set of distribution functions F_X where inequality (2.1) holds in reverse order, that is,

$$\sum_{i=1}^{n} \theta_i X_i \leq_{st} X_1. \tag{2.2}$$

Notice that $X \in \mathcal{D}^+$ if and only if $-X \in \mathcal{D}^-$. Therefore, we will mainly study properties of $\mathcal{D}^-$ in the following. The property $X \in \mathcal{D}^-$ means in terms of risks that diversification of a portfolio of independent such risks has a negative effect by stochastically increasing the risk. This property may be a bit surprising at first sight, as one typically assumes that diversification is good at least for risk averse decision-makers. A detailed discussion of this property in the context of risk exchange is given in Chen *et al*. (2024a).

The following simple result unifies and generalizes part (ii)–(iv) of Proposition 1 in Chen and Shneer (2024).

Theorem 2.2. *Assume that $X, Y \in \mathcal{D}^-$ are independent and that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary increasing convex function. Then $\phi(X, Y) \in \mathcal{D}^-$.*

Proof. This simply follows from the fact that for i.i.d. $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ we get

$$\phi(X_1, Y_1) \leq_{st} \phi\left(\sum_{i=1}^{n} \theta_i X_i, \sum_{i=1}^{n} \theta_i Y_i\right) \leq \sum_{i=1}^{n} \theta_i \phi\left(X_i, Y_i\right).$$

Here the first inequality follows from (2.1) for any increasing function as shown in Theorem 1.2.16 of Müller and Stoyan (2002), and it is an immediate consequence of the definition of convexity that the second inequality holds pointwise for all $\omega \in \Omega$. □

We will now collect a list of several elementary properties of the set $\mathcal{D}^-$. Many of them can be found already in similar forms in the references Chen *et al*. (2024a); Chen *et al*. (2024b) and Chen and Shneer (2024), but for completeness we add the simple proofs for all of them.

Theorem 2.3.

(a) if $X_n \in \mathcal{D}^-$, $n \in \mathbb{N}$, and $X_n \xrightarrow{d} X$, then $X \in \mathcal{D}^-$.
(b) if $X \in \mathcal{D}^-$ is non-degenerate, then $E|X| = \infty$.
(c) if $X \in \mathcal{D}^-$, then $aX + b \in \mathcal{D}^-$ for all $a > 0$ and $b \in \mathbb{R}$.
(d) if $X, Y \in \mathcal{D}^-$ and X,Y are independent, then $X + Y \in \mathcal{D}^-$.
(e) if $X, Y \in \mathcal{D}^-$ and X,Y are independent, then $\max\{X, Y\} \in \mathcal{D}^-$.
(f) if $X \in \mathcal{D}^-$ and ϕ:$\mathbb{R} \to \mathbb{R}$ is increasing convex, then $\phi(X) \in \mathcal{D}^-$.

Proof.

(a) This follows immediately from the well-known facts that convergence in distribution is preserved under continuous mappings and thus when building convex combinations, and that $\leq_{st}$ is preserved under convergence in distribution, see for example Theorem 1.2.14 in Müller and Stoyan (2002).

(b) It is well known that $X \leq_{st} Y$ and $EX = EY$ implies $X =_{st} Y$, which is an easy consequence of

$$EY - EX = \int_{-\infty}^{\infty} (F_X(t) - F_Y(t))dt.$$

Therefore, $X \in \mathcal{D}^-$ and $E|X| < \infty$ implies $X =_{st} \bar{X}_n := \sum_{i=1}^{n} X_i/n$ for all $n \in \mathbb{N}$ and due to the law of large number and part a) finally $X =_{st} \lim_{n\to\infty} \bar{X}_n = EX$ a.s. under this assumption. Part (c)–(f) immediately follow from Theorem 2.2.

□

We can use Theorem 2.3 to show that $X \in \mathcal{D}^-$ implies that this property also holds for scaled limits of sums or maxima of i.i.d. random variables from the corresponding distribution. Recall that a random

variable X is said to be in the *sum-domain of attraction* of a random variable Z, if there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that

$$a_n(X_1 + \ldots + X_n) - b_n \xrightarrow{d} Z, \tag{2.3}$$

where $X_1, X_2, \ldots$ are i.i.d. copies of X. We write $X \in DOA^+(Z)$ in this case. The possible limits are called sum-stable distributions. Similarly, one can define the *max-domain of attraction* of a random variable Z, if there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that

$$a_n \max\{X_1, \ldots, X_n\} - b_n \xrightarrow{d} Z, \tag{2.4}$$

where $X_1, X_2, \ldots$ are i.i.d. copies of X. We write $X \in DOA^\vee(Z)$ in the case of the max-domain of attraction of a max-stable random variable Z. For details on sum-stable distributions and their domains of attraction, we refer to Nolan (2020). The topic of max-stability and their domain of attraction is considered in the extreme value theory literature. We refer to Resnick (2008) for a good reference on the details that we will use here. The following result is now easy to show.

Theorem 2.4.

(a) If $X \in \mathcal{D}^-$ and $X \in DOA^+(Z)$, then $Z \in \mathcal{D}^-$.
(b) If $X \in \mathcal{D}^-$ and $X \in DOA^\vee(Z)$, then $Z \in \mathcal{D}^-$.

Proof. This follows immediately from the preservation of $\mathcal{D}^-$ under independent sums and maxima, scaling, and distributional limits as mentioned in Theorem 2.3. □

The possible limits in (2.3) are necessarily what is called a stable distribution. We use here the following notation for stable distributions, which corresponds to parametrization 1 in Nolan (2020). For $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$, we say that $Z \sim S(\alpha, \beta)$ has a stable distribution, if the characteristic function fulfills

$$E(\exp(iuZ)) = \exp(-|u|^\alpha[1 - i\beta \tan(\pi\alpha/2)(sign(u))]), \ u \in \mathbb{R},$$

if $\alpha \neq 1$, and

$$E(\exp(iuZ)) = \exp(-|u|[1 + 2i\beta/\pi(sign(u)\ln(|u|))]), \ u \in \mathbb{R},$$

if $\alpha = 1$. Any more general stable distribution with 4 parameters $(\alpha, \beta, \gamma, \delta)$ can be obtained as a linear transformation $\gamma Z + \delta$ for $Z \sim S(\alpha, \beta)$, but for the purpose of this paper, it is sufficient to consider stable distributions $Z \sim S(\alpha, \beta)$, as the set $\mathcal{D}^-$ is invariant under linear transformations. The parameter α is typically called the characteristic exponent, and the parameter β is called the skewness parameter. These stable distributions have an infinite mean if $\alpha \leq 1$. Any convex combination of i.i.d. stable distributions is again a stable distribution and the obtained parameters can be found in equation (1.7) in Nolan (2020). Using this we can derive the following result for stable distributions.

Theorem 2.5. *We have $Z \sim S(\alpha, \beta) \in \mathcal{D}^-$ if and only if one of the following conditions holds:*

(a) $\alpha = 1$ and $\beta \geq 0$;
(b) $\alpha < 1$ and $\beta = 1$.

Proof. For $\alpha > 1$, we have $E|Z| < \infty$ and therefore $Z \notin \mathcal{D}^-$. In case $\alpha = 1$ we get from (1.7) in Nolan (2020) for any convex combination $\sum w_i X_i$ of i.i.d. stable distributions that $\sum w_i X_i =_{st} X_1 + \delta$, where

$$\delta = -\frac{2\beta}{\pi} \sum_{i=1}^{n} w_i \ln(w_i)$$

and therefore $Z \in \mathcal{D}^-$ if and only if $\delta \geq 0$, which holds if and only if $\beta \geq 0$.

If $\alpha < 1$, then we get $\sum w_i X_i =_{st} \gamma X_1$ with

$$\gamma = \left(\sum_{i=1}^{n} w_i^{\alpha} \right)^{1/\alpha} \geq 1,$$

and thus we have $\gamma > 1$, if $w_i > 0$ for at least two indices i. Therefore, $\sum w_i X_i \geq_{st} X_1$ in this case, if and only if $P(X_1 \geq 0) = 1$. This holds if and only if $\beta = 1$. □

Notice that we get a very special case, if we choose $\alpha = 1$ and $\beta = 0$. Then we get a Cauchy distribution with distribution function

$$F(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}, \ x \in \mathbb{R}.$$

This has the special property that we get equality in (2.1).

Combining Theorems 2.4 and 2.5 yields a necessary condition for $X \in \mathcal{D}^-$, as $X \in DOA^+(Z)$ and $Z \notin \mathcal{D}^-$ implies $X \notin \mathcal{D}^-$. As an example, we can consider X^3 for X a Cauchy distribution. In this case, we get $X \in DOA^+(Z)$ for $Z \sim S(1/3, 0) \notin \mathcal{D}^-$ and hence $X \notin \mathcal{D}^-$.

Among the max-stable distributions, the Fréchet distribution with cdf

$$F(x) = e^{-1/x^{\alpha}}, \ x \geq 0,$$

has an infinite mean, if $\alpha \leq 1$. A random variable X in the sum-domain of attraction of a stable distribution Z with parameter $\alpha \leq 1$ fulfills $X \in DOA^{\vee}(Z)$ for a Fréchet distribution Z with the same parameter α. This follows from the fact that both domains of attraction are characterized by regular variation of order α for the survival function, see Theorem 3.14 in Nolan (2020) and Proposition 1.11 in Resnick (2008) for details. Therefore, we can derive the following result.

Theorem 2.6. *We have $X \in DOA^{\vee}(Z)$ with $Z \in \mathcal{D}^-$ if and only if Z is a Fréchet distribution with $\alpha \leq 1$.*

Proof. According to Theorem 2.5, we have for any $\alpha \leq 1$ a stable distribution $X \in \mathcal{D}^-$. If $X \in DOA^{\vee}(Z)$, then it follows from Theorem 2.4 that $Z \in \mathcal{D}^-$. But this holds for Z a Fréchet distribution with $\alpha \leq 1$. As these Fréchet distributions are the only max-stable distributions with infinite mean, the result follows. □

Chen and Shneer (2024) gave an alternative direct proof of the result that infinite mean Fréchet distributions are in $\mathcal{D}^-$. We will demonstrate now that this result and the related results for Pareto distributions given in Chen *et al*. (2024b) are all special cases of the simple fact that all these distributions are convex transformations of Cauchy distributions. Notice that for continuous distributions, $Y = \phi(X)$ can only hold for an increasing ϕ, if we choose $\phi(x) = F_Y^{-1}(F_X(x))$. This relative inverse distribution function ϕ is strongly related to the statistical concept of a quantile–quantile plot (or shortly Q-Q plot). The condition of the relative inverse being convex is a well-known order relation between distributions describing an increase in skewness. This concept is discussed in detail in Zwet (1964) and Oja (1981). We will use the following notation.

Definition 2.7. *A cdf G is said to be more skewed than F (written $F \leq_{skew} G$), if the function $\phi(x) = G^{-1}(F(x))$ is convex.*

If densities f and g exist, then taking the derivative of ϕ yields the condition that $F \leq_{skew} G$ holds if and only if

$$u \mapsto \frac{f(F^{-1}(u))}{g(G^{-1}(u))} \quad \text{is non-decreasing, } u \in (0, 1). \tag{2.5}$$

In the case of a Cauchy distribution, where we get equality in distribution in (2.1), we can even choose arbitrary convex transformations to remain in $\mathcal{D}^-$.

Theorem 2.8. *Assume that X has a Cauchy distribution and that $\phi : \mathbb{R} \to \mathbb{R}$ is an arbitrary convex function. Then $\phi(X) \in \mathcal{D}^-$.*

Proof. This simply follows from the fact that convexity of ϕ implies for any convex combination

$$\phi(X_1) =_{st} \phi\left(\sum_{i=1}^{n} \theta_i X_i\right) \leq \sum_{i=1}^{n} \theta_i \phi\,(X_i)$$

P.-a.s. □

For a distribution F with a heavy right tail, a convex transformation leads to a distribution, which has even more heavy right tails. Therefore, several authors introduced classes of *super heavy-tailed distributions* by requiring a convex transformation of a specific heavy-tailed distribution. We will consider now the relationship between some of these concepts and their relation to the class $\mathcal{D}^-$.

Definition 2.9. *(a) For any c.d.f. F, we can define the class of distributions*

$$\mathcal{S}(F) := \{G : F \leq_{skew} G\}.$$

(b) If

$$F(x) = 1 - \frac{1}{x},\ x \geq 1,$$

is the c.d.f. of a Pareto(1) distribution, then $\mathcal{S}_P := \mathcal{S}(F)$ is called the class of super Pareto distributions.

If

$$F(x) = e^{-1/x},\ x \geq 0,$$

is the c.d.f. of a standard Fréchet distribution, then $\mathcal{S}_F := \mathcal{S}(F)$ is called the class of super Fréchet distributions.

If

$$F(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2},\ x \in \mathbb{R},$$

is the c.d.f. of a Cauchy distribution, then $\mathcal{S}_C := \mathcal{S}(F)$ is called the class of super Cauchy distributions.

The class of super Pareto distributions has been introduced in Chen *et al*. (2024b), and the class of super Fréchet distributions was considered in Chen and Shneer (2024). It is natural to also consider the class of super Cauchy distributions, and we will show now the relationship between these classes and that they are all contained in $\mathcal{D}^-$.

Theorem 2.10. *The following relations hold: $\mathcal{S}_P \subset \mathcal{S}_F \subset \mathcal{S}_C \subset \mathcal{D}^-$.*

Proof. The relationship $\mathcal{S}_P \subset \mathcal{S}_F$ follows from Example 11 in Chen and Shneer (2024), where it is shown that a Pareto(1) distribution is super Fréchet. Therefore, we mainly have to show $\mathcal{S}_F \subset \mathcal{S}_C$. Consider a Fréchet distribution with $G(x) = e^{-1/x}$ and a Cauchy distribution $F(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}$. We have to show that $G^{-1}(F(x))$ is convex, or equivalently that $h(x) = F^{-1}(G(x))$ is concave. We get

$$h(x) = \tan(\pi e^{-1/x} - \pi/2),\ x > 0.$$

Taking the second derivative yields

$$h''(x) = -\frac{\pi e^{-2/x}(e^{1/x}(2x-1) + 2\pi \cot(\pi/e^{1/x}))}{\sin^2(\pi e^{-1/x})x^4}.$$

For $x \leq \pi/4$, the slope of $\tan(x)$ is smaller than 2 and thus we have $\tan(x) \leq 2x$ and hence $\cot(x) \geq 1/(2x)$. This implies $h''(x) \leq 0$ for $x \leq 1/2 \leq \pi/4$. For $x \geq 1/2$, it obviously holds that $h''(x) \leq 0$. Hence, h is concave and thus F is less skewed than G. The relation $\mathcal{S}_C \subset \mathcal{D}^-$ is an immediate consequence of Theorem 2.2. □

This result shows that there are additional distributions in $\mathcal{D}^-$ compared to the classes considered in Chen *et al*. (2024b) and Chen and Shneer (2024).

We will now show that even the restriction of $\mathcal{S}_C$ to nonnegative distributions is strictly larger than $\mathcal{S}_F$.

Theorem 2.11. *If X has a Cauchy distribution, then the distribution function $F(x) = 2\arctan(x)/\pi$, $x \geq 0$, of $|X|$ fulfills $F \in \mathcal{S}_C$, but $F \notin \mathcal{S}_F$.*

Proof. To show that this distribution with $F(x) = 2\arctan(x)/\pi$ is not a super Fréchet distribution, we have to consider the relative inverse with respect to $G(x) = e^{-1/x}$. We get

$$\phi(x) = F^{-1}(G(x)) = \tan(\pi e^{-1/x}/2).$$

A straightforward calculation similar to the one in Theorem 2.10 shows that this function is concave for large x and thus $F \notin \mathcal{S}_F$. □

Theorem 2.12. *If $X \geq 0$ is non-degenerate, then $X \notin \mathcal{D}^+$.*

Proof. Assume that $X \geq 0$ is non-degenerate and $X \in \mathcal{D}^+$ so that in particular

$$X \geq_{st} \frac{1}{n}\sum_{i=1}^{n} X_i =: Y_n \tag{2.6}$$

for $X_1, X_2, \ldots$ i.i.d. with the same distribution as X and for all $n \in \mathbb{N}$. As the Cauchy distribution is the only non-degenerate distribution with stochastic equality in (2.6) for all n, we must have $F_X(t_0) < F_{Y_n}(t_0)$ for some $t_0 > 0$ and for some $n \in \mathbb{N}$, if $X \geq 0$ fulfills (2.6). For $t > t_0$, we get

$$E(t - X)_+ = \int_0^t F_X(z)dz < \int_0^t F_{Y_n}(z)dz = E(t - Y_n)_+.$$

However, if X has a finite mean, then we have $E[X_1|Y_n] = Y_n$ and thus $X \geq_{cx} Y_n$. Therefore, we get for the convex function $f(x) = (t - x)_+$ that

$$Ef(X) = E(t - X)_+ \geq E(t - Y_n)_+,$$

a contradiction. If X does not have a finite mean, this still applies, as the proof is only based on values of X below t, and therefore, we can apply the argument to the bounded random variable $\tilde{X} := \min\{X, t\}$. □

We want to mention that there are of course also random variables with $X \geq 0$, $EX = \infty$, and $X \notin \mathcal{D}^-$. The following example demonstrates that $X \in \mathcal{D}^-$ leads to conditions also for the values of the c.d.f. for small x and thus also demonstrates that there cannot be any sufficient conditions for $X \in \mathcal{D}^-$ based only on the tail behavior of the distribution of X.

Example 2.13. *Assume that X_1, X_2 are i.i.d. with $X_1 \geq 0$, $EX_1 = \infty$ and $P(0 \leq X_1 \leq 2) = P(2 < X_1 \leq 4) = 0.4$. Then*

$$P((X_1 + X_2)/2 \leq 3) \geq P(0 \leq X_1 \leq 4, 0 \leq X_2 \leq 2) + P(0 \leq X_1 \leq 2, 2 < X_2 \leq 4) = 0.48.$$

Hence, $F \notin \mathcal{D}^-$, if $F(0) = 0, F(2) = 0.4, F(3) < 0.48$, and $F(4) = 0.8$.

Notice that we can derive similar and sharper bounds for Super-Pareto and related distributions for $F(3)$, if we assume $F(0) = 0, F(2) = 0.4$, and $F(4) = 0.8$. If we denote by G the distribution function of a Pareto(1) distribution, then F is Super-Pareto, if and only if $G^{-1}(F(x))$ is concave. This implies in case $F(0) = 0, F(2) = 0.4$, and $F(4) = 0.8$ that

$$G^{-1}(F(3)) \geq \frac{1}{2}(G^{-1}(F(2)) + G^{-1}(F(4))) = \frac{10}{3}$$

and thus $F(3) \geq 0.7$. A similar calculation yields that in the case of a super-Fréchet distribution we get the condition $F(3) \geq 0.698$ and in the case of a super-Cauchy distribution we get the necessary condition $F(3) \geq 0.654$.

3. Deadly risks with $P(X = \infty) > 0$

Let us now consider the case that the random variables have values in $\mathbb{R} \cup \{\infty\}$. In particular, we consider the case of $P(X_1 = \infty) = p = 1 - P(X_1 = 0)$ for some $p > 0$. Then it obviously holds that for a convex combination

$$Y = \sum_{i=1}^{n} \theta_i X_i \text{ we get } P(Y = 0) = \prod_{i=1}^{n} (1-p) 1_{[\theta_i > 0]} = 1 - P(Y = \infty),$$

and hence this distribution is in $\mathcal{D}^-$. Notice that we can write $X_i = \phi(Z_i)$ for any continuous real valued random variable Z by choosing

$$\phi(x) = \begin{cases} \infty \ , \text{ if } x \geq F_X^{-1}(p), \\ 0 \ , \text{ else.} \end{cases}$$

This function ϕ is convex, so that we can consider this distribution as the extreme case of the most skewed infinity mean distribution that exists. We can give a nice intuitive interpretation of this result by assuming that $X = \infty$ means a deadly risk. Assume that a pool of people stranded on a deserted island. Every individual collects some food. You have to eat a certain amount of food to survive, but all available items have independent of the others a probability p of being poisoned and leading to death even if you only eat a small amount of it. So if you must eat something to survive, it is intuitively clear that it is optimal to just eat from one item leading to a probability of death of p, whereas you can only increase this probability if you eat from several potentially deadly items. In this case, we have the property that X_1 and cX_1 have the same distribution for every $c > 0$, and this makes it immediately obvious that such a result holds. So in this case of deadly risks it is intuitively clear that risk sharing is not a good idea. Knowing this it may appear a bit less surprising that similar things can happen in the case of infinite mean models with extremely heavy tails.

4. Relation to other recent contributions and open problems.

Several other authors recently considered this question of finding large classes of distributions in $\mathcal{D}^-$ independently of our approach. Chen and Shneer (2024) define the class $\mathcal{H}$ of distributions with the property that $x \mapsto h_F(x) := -\log(F(1/x))$ is subadditive. Arab *et al.* (2024) consider the class of distributions with the property that $x \mapsto g_F(x) := 1 - F(1/x)$ is subadditive. Let us call this class of distributions by $\mathcal{G}$. It is easy to see that $\mathcal{H} \subset \mathcal{G}$. Both papers contain examples of distributions F in their respective class with $F \notin \mathcal{S}_C$. A simple example with $F \in \mathcal{H}$, but $F \notin \mathcal{S}_C$ can be constructed by using a subadditive function h_F with a kink with increasing slope. As a concrete example we can choose

$$h_F(x) = \begin{cases} 2x, & x < 1, \\ 2, & 1 \leq x < 2, \\ x, & x \geq 2. \end{cases}$$

As $x \mapsto h_F(x)/x$ is obviously decreasing, this is a subadditive function. The corresponding distribution function F fulfills $F(x) = \exp(-h_F(1/x))$ and therefore also has a kink with increasing slope. But this implies that $F \notin \mathcal{S}_C$.

We give now an example of a nonnegative distribution with $F \in \mathcal{S}_C$ with $F \notin \mathcal{G}$. We consider the following convex function $\phi : \mathbb{R} \to (0, \infty)$:

$$\phi(x) := \begin{cases} \frac{1}{10-x}, & x < 5, \\ \frac{x}{25}, & x \geq 5. \end{cases}$$

If X has a Cauchy distribution, then we get for the distribution F of $\phi(X)$ that

$$g_F(x) = \frac{1}{2} - \frac{1}{\pi} \cdot \begin{cases} \arctan(25/x), & x < 5, \\ \arctan(10 - x), & x \geq 5. \end{cases}$$

Thus, we have $g_F(10) = 0.5 > 2 \cdot g_F(5) \approx 0.12$ and thus g_F is not subadditive and hence $F \notin \mathcal{G}$.

Chen and Shneer (2024) claim in their Example 3 that there is an example of a discrete distribution $F \in \mathcal{H}$. Unfortunately, this example is wrong, as the corresponding function h_F is not subadditive. Indeed it is easy to see that any increasing subadditive function h_F with $\lim_{x\to 0} h_F(x) = 0$ must be continuous. This can be shown along the same lines as the proof of Theorem 1 in Matkowski and Światkowski (1993). For increasing functions, the proof is indeed much simpler. Therefore, any distribution in $\mathcal{H}$ also must be continuous. This also holds for distributions in $\mathcal{G}$. The example of a discrete distribution in $\mathcal{G}$ mentioned in Example 2.7 in Arab *et al*. (2024) has positive probability $P(X = \infty) > 0$, and this is similar to the example that we consider in the previous section. It seems to remain an open problem, whether there is a discrete distribution in $\mathcal{D}^-$ that assumes only finite values.

It also seems to be an open problem whether all stable distributions in $\mathcal{D}^-$ are super Cauchy distributions.

For stable distributions with parameter $\alpha < 1$ and $\beta = 1$, some numerical experiments seem to support this conjecture, but we have no idea how to prove it, as an explicit formula for the cdf is only known for the case $\alpha = 1/2$, which is known as Levy distribution. So the question arises whether one can prove that these positive infinite-mean stable distributions are convex transformations of a Cauchy distribution for any $\alpha < 1$ without having a closed form for the cdf?

A similar problem arises for stable distributions with parameter $\alpha = 1$ and $\beta > 0$. We know from Theorem 2.5 that these distributions are also in $\mathcal{D}^-$, but eventhough the parameter β is commonly denoted as skewness parameter, we could not find a formal proof that an increase in β yields an increase with respect to the stochastic order $\leq_{skew}$ for stable distributions. We also conjecture that this is true in general.

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