Ergod. Th. & Dynam. Sys., page 1 of 7
doi:10.1017/etds.2025.10214

CORRIGENDUM

An uncountable Furstenberg–Zimmer structure theory – CORRIGENDUM

ASGAR JAMNESHAN

Mathematical Institute, University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
(e-mail: ajamnesh@math.uni-bonn.de)

(*Received* 22 *April* 2025 *and accepted in revised form* 28 *July* 2025)

doi: https://doi.org/10.1017/etds.2022.43, Published by Cambridge University Press, 21 June 2022

Abstract. This is a corrigendum to 'An uncountable Furstenberg–Zimmer structure theory' [*Ergod. Th. & Dynam. Sys.* **43**(7) (2023), 2404–2436]. We report two issues in that paper. First, Lemma A.5 and Proposition A.6 in the Appendix, which supported a spectral analysis of conditional Hilbert–Schmidt operators, are incorrect. These results were used in the proof of Lemma 4.4, which establishes part of the equivalences in Theorem 4.1. We provide a correction for this issue here. While the proof strategy of Lemma 4.4 remains valid, the details have been revised using known auxiliary results in the non-commutative setting of tracial von Neumann algebras, replacing the faulty arguments from the Appendix. Second, the proof of the implication (iii) $\Rightarrow$ (iii)$'$ in Lemma 4.10 is incorrect. We supply a new argument to address this. We also take this opportunity to correct several minor issues that have come to our attention since the paper's publication. A fully revised version, including these corrections, as well as updated references and some fixed typos, is now available on arXiv.

1. *Introduction*

We quote from [3] and adopt its notation without further comment. Both issues addressed in this corrigendum arise in the proof of Theorem 4.1, which is divided into several lemmas in [3, §4].

The main problematic part lies in Lemma A.5 and Proposition A.6 of the Appendix. These results are modifications of [5, Lemma C.14 and Proposition C.15], respectively. The issue arises in replacing $L^\infty(Y)$-modules in the conditional L^2-space from [5] with $L^\infty(Y)$-modules in the classical L^2-space, and using the L^2-norm instead of the L^∞-norm of the conditional norm when computing the operator norm of a conditional

Hilbert–Schmidt operator in Lemma A.5. More precisely, the first inequality in the final displayed equation at the end of the proof of Lemma A.5 is incorrect. We thank Henrik Kreidler for bringing this oversight to our attention.

This error propagates to Proposition A.6, which was used in Lemma 4.4 to argue that the range of a conditional Hilbert–Schmidt operator, viewed as an operator on $L^2(X)$, is the closure of finitely generated $L^\infty(Y)$-modules in $L^2(X)$. This conclusion can, however, be obtained through a different route: using a criterion for when an $L^\infty(Y)$-module in $L^2(X)$ is finitely generated, which we found in the non-commutative setting of tracial von Neumann algebras in [1]. (In [4], Spaas and the author establish a non-commutative version of Theorem 4.1 for inclusions of tracial von Neumann algebras. A similar issue as in the proof of Lemma 4.4 was identified in that paper [4, Lemma 2.11 and Proposition 2.12] and fixed using the non-commutative version of the same argument as here.)

This correction also necessitates a modification of assertion (ii)′ in Theorem 4.1, specifically by removing the L^2-closedness condition on the $L^\infty(Y)$-modules in $L^2(X)$. Details are provided in §2.

The second issue concerns the proof of the implication (iii) ⇒ (iii)′ in Lemma 4.10, where it is claimed that if $f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ has a conditionally totally bounded orbit, then so does $f \cdot 1_{\|f\|_{L^2(X|Y)} \le M}$ for any $M > 0$, due to Proposition 3.4(ii). This claim is flawed on multiple levels: first, Proposition 3.4(ii) does not justify such a conclusion; second, the conclusion itself is incorrect; and third, even if these issues were absent, the argument would still fall short, as the conditional total boundedness of the orbit of $f \cdot 1_{\|f\|_{L^2(X|Y)} \le M}$ would only hold in the larger space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, not in $L^2(X)$. However, a more careful truncation argument does yield a correct proof of the implication (iii) ⇒ (iii)′. The details are provided in §2.

Several minor issues have also come to our attention and are corrected below.

- The proof of the conditional triangle inequality in Proposition 3.4 relies on the conditional Cauchy–Schwarz inequality, which was not explicitly mentioned in the original paper.
- In the first step of the proof of the conditional Cauchy–Schwarz inequality (Proposition 3.4(iii)), there are minor errors: a missing square in the first displayed equation and an incorrect definition of a in the second one. A corrected version of the full first step is as follows.

  First, suppose that $f, g \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ satisfy $\|f\|_{L^2(X|Y)}, \|g\|_{L^2(X|Y)}, \|f - ag\|_{L^2(X|Y)} > 0$ for all $a \in L^0(Y)$. Then, we have

  $$0 \le \|f + ag\|^2_{L^2(X|Y)} = \|f\|^2_{L^2(X|Y)} + a\langle g, f\rangle_{L^2(X|Y)} + \bar{a}\langle f, g\rangle_{L^2(X|Y)} + |a|^2\|g\|^2_{L^2(X|Y)}$$

  for all $a \in L^0(Y)$. Setting $a = c(\langle f, g\rangle_{L^2(X|Y)}/\|g\|_{L^2(X|Y)})$, where $c \in L^0(Y)$ satisfies $|c| = 1$ and $c\langle f, g\rangle_{L^2(X|Y)} = |\langle f, g\rangle_{L^2(X|Y)}|$, and after some elementary algebraic manipulations, we obtain the conditional Cauchy–Schwarz inequality in this case.
- At the end of the proof of Proposition 3.7, the definition of the atoms of the partition $\mathcal{P}_0$ is incorrect. A corrected construction is as follows.

Let $F_i = \{\|g_i\|_{L^2(X|Y)} > 0\}$ for $i = 1, \dots, n$. Form all finite intersections $E_1 \cap E_2 \cap \cdots \cap E_n$, where each E_i is either F_i or F_i^c.

Accordingly, the definitions of $\mathcal{P}$ and the sets $\mathcal{F}_E$ must be adjusted.

- In the statements of properties (i) and (i)′ in Theorem 4.1, one must consider the $L^0(Y)$- or $\mathbb{C}$-linear span (respectively) of the sets appearing therein. A corrected statement of Theorem 4.1 is provided in §2.
- In Remark 3.5 on topos-theoretic aspects of Proposition 3.4, Proposition 3.4(iv) and (v), as originally stated, do *not* describe the $\mathtt{Sh}(Y)$-spectral theorem of $\mathtt{Sh}(Y)$-Hilbert–Schmidt operators.
- Due to the corrections in the proof of Lemma 4.4 and the adjustments in the statement of Theorem 4.1, the beginning of the proof of Theorem 5.3 requires the following modification: the family $(\mathcal{M}_\alpha)$ must be chosen in the larger $L^0(Y)$-module $\mathrm{L}^2(X|Y)$. Indeed, since we removed the L^2-closedness condition on the $L^\infty(Y)$-modules in $L^2(X)$ in Theorem 4.1, we cannot guarantee the existence of a conditional orthonormal basis in such modules (cf. [3, Proposition 3.7] and Proposition 2.2 below). This change does not affect the remainder of the proof.

2. *Fixing the proofs of Lemmas 4.4 and 4.10 in the original paper*

The following preliminary results are needed to achieve a conditional spectral decomposition for conditional Hilbert–Schmidt operators. Due to our use of the Borel functional calculus, we treat conditional Hilbert–Schmidt operators on the classical Hilbert space $L^2(X)$ which we however view as an $L^\infty(Y)$-module by multiplication.

Definition 2.1. (Conditional orthonormal basis) Let $\mathcal{M}$ be an $L^2(X)$-closed $L^\infty(Y)$-submodule of $L^2(X)$. A subset M of $\mathcal{M}$ is said to be a *conditional orthonormal basis* if the following properties are satisfied:

(i) $\langle f, g\rangle_{L^2(X|Y)} = 0$ for all $f, g \in M$;

(ii) $\langle f, f\rangle_{L^2(X|Y)} = 1_E$ for some $E \in Y$ for all $f \in M$ (where E may depend on f);

(iii) $\mathcal{M} = \bigoplus_{f\in M} \overline{L^\infty(Y) f}$.

PROPOSITION 2.2. *Any $L^2(X)$-closed $L^\infty(Y)$-submodule of $L^2(X)$ has a conditional orthonormal basis.*

Proof. This is a commutative special case of the existence of Pimsner–Popa orthonormal basis in right modules over tracial von Neumann algebras, see [1, Proposition 8.4.11]. □

PROPOSITION 2.3. *Let $\mathcal{M}$ be an $L^2(X)$-closed $L^\infty(Y)$-submodule of $L^2(X)$, and let $K \in L^\infty(X \times_Y X)$. Then, $K *_Y : \mathcal{M} \to L^2(X)$ is a well-defined $L^\infty(Y)$-linear classical bounded operator. Moreover, it is conditionally Hilbert–Schmidt in the sense that for any conditional orthonormal basis M of $\mathcal{M}$,*

$$\sup_{F\subset M \text{ finite}} \sum_{f\in F} \|K *_Y f\|^2_{L^2(X|Y)} < \infty,$$

where the (essential) supremum of the measurable functions on the left-hand side exists by completeness of Y (see [3, Remark 2.2]*), and a priori is measurable with values in* $[0, \infty]$.

Proof. Suppose $\|K\|_{L^\infty(X\times_Y X)} = C$ for some constant $C > 0$. By inequality (8) in property (v) of [3, Proposition 3.4],

$$\begin{aligned}\|K *_Y f\|^2_{L^2(X)} &= \int_Y \|K *_Y f\|^2_{L^2(X|Y)}\, d\nu \\ &\leq \int_Y C^2\|f\|^2_{L^2(X|Y)}\, d\nu \\ &= C^2\|f\|^2_{L^2(X)}.\end{aligned}$$

This proves the first claim.

As for the second claim, let M be a conditional orthonormal basis of $\mathcal{M}$ and let $F \subset M$ be a finite subset of M. Applying Bessel's inequality pointwise almost everywhere, we have

$$\sum_{f\in F} |(K *_Y f)(x)|^2 \leq \|K(x,\cdot)\|^2_{L^2(X|Y)}(\tilde{\pi}(x)) < C^2.$$

The claim follows from the monotone convergence theorem for conditional expectations since the essential supremum is attained by a countable subfamily due to the countable chain condition (see [3, Remark 2.2]) and since the family $\sum_{f\in M}\|K *_Y f\|^2_{L^2(X|Y)}$ parameterized by finite subsets of M is directed upwards. □

The following criterion helps to decide when an $L^2(X)$-closed $L^\infty(Y)$-submodule of $L^2(X)$ is finitely generated.

PROPOSITION 2.4. *Let $\mathcal{M}$ be an $L^2(X)$-closed $L^\infty(Y)$-submodule of $L^2(X)$. Then, $\mathcal{M}$ is finitely generated if and only if there is a constant $C > 0$ such that for every conditional orthonormal basis M in $\mathcal{M}$, it holds that $\sup_{F\subset M \text{finite}} \sum_{f\in F}\|f\|^2_{L^2(X|Y)} < C$.*

Proof. This is the special case of [1, Proposition 9.3.2 (i)] in the setting of commutative tracial von Neumann algebras, once one observes that $\hat{E}_Z(1)$ as defined in that proposition equals $\sup_{F\subset M \text{ finite}} \sum_{f\in F}\|f\|^2_{L^2(X|Y)}$ by [1, Lemma 8.4.8] and the observations at the beginning of [1, §9.3]. □

We state the corrected version of Theorem 4.1 in the original paper.

THEOREM 2.5. *Let $\mathcal{X} = (X, \mu, T)$ and $\mathcal{Y} = (Y, \nu, S)$ be $\mathbf{PrbAlg}_\Gamma$-systems and $\pi : \mathcal{X} \to \mathcal{Y}$ be a $\mathbf{PrbAlg}_\Gamma$-extension. Then, the following are equivalent.*

(i) *The conditional Hilbert space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is the $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$-closure of the $L^0(Y)$-linear span of*

$$\{K *_Y f : K \in \mathtt{HS}(\mathtt{X}|\mathtt{Y})\ \Gamma\text{-invariant},\ f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})\}.$$

(ii) *The conditional Hilbert space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is the $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$-closure of the union of all its finitely generated and Γ-invariant $L^0(Y)$-submodules.*

(iii) *There exists a dense set $\mathcal{G}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ with respect to the metric $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$ such that for all $f \in \mathcal{G}$ and every $\varepsilon > 0$, there is a finite set $\mathcal{F}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ such that for all $\gamma \in \Gamma$,*

$$\min_{g\in\mathcal{F}} \|(T^\gamma)^* f - g\|_{L^2(X|Y)} < \varepsilon.$$

(i)′ *The classical Hilbert space $L^2(X)$ is the L^2-closure of the $\mathbb{C}$-linear span of*

$$\{K *_Y f : K \in L^\infty(X \times_Y X)\ \Gamma\text{-invariant},\ f \in L^2(X)\}.$$

(ii)′ *The classical Hilbert space $L^2(X)$ is the L^2-closure of the union of all its Γ-invariant and finitely generated $L^\infty(Y)$-submodules.*

(iii)′ *There exists a dense set $\mathcal{H}$ in $L^2(X)$ such that for all $f \in \mathcal{H}$ and every $\varepsilon > 0$, there is a finite set $\mathcal{F}$ in $L^2(X)$ such that for all $\gamma \in \Gamma$,*

$$\min_{g\in\mathcal{F}} \|(T^\gamma)^* f - g\|_{L^2(X|Y)} < \varepsilon.$$

A $\mathbf{PrbAlg}_\Gamma$*-morphism π fulfilling one (and therefore all) of the above six properties is called a* relatively compact $\mathbf{PrbAlg}_\Gamma$-extension.

The following assertion is [3, Lemma 4.4].

LEMMA 2.6. *Assertion (i)′ implies assertion (ii)′ in Theorem 2.5.*

Proof. Since the finite sum of Γ-invariant and finitely generated $L^\infty(Y)$-submodules is a Γ-invariant and finitely generated $L^\infty(Y)$-submodule, it suffices to show that the ranges of $K*_Y$, where $K \in L^\infty(X \times_Y X)$ is Γ-invariant, are contained in the closure of the union of all Γ-invariant and finitely generated $L^\infty(Y)$-submodules of $L^2(X)$.

Let $K \in L^\infty(X \times_Y X)$ be Γ-invariant. By decomposing

$$K(x, y) = \frac{K(x, y) + \overline{K(y, x)}}{2} + \mathrm{i}\frac{K(x, y) - \overline{K(y, x)}}{2\mathrm{i}},$$

we may reduce to the case that $K(x, y) = \overline{K(y, x)}$, and then by Proposition 2.3, $K *_Y : L^2(X) \to L^2(X)$ is a bounded self-adjoint operator. Additionally, by treating the positive and negative spectrum separately, we can assume that K is a positive operator. For $\varepsilon > 0$, consider the spectral projection

$$P_\varepsilon := 1_{[\varepsilon, \|K*_Y\|]}(K*_Y).$$

By standard properties of spectral projections, we have

$$P_\varepsilon \circ (K*_Y) = (K*_Y) \circ P_\varepsilon \geq \varepsilon P_\varepsilon, \tag{1}$$

where the inequality means that

$$\langle K *_Y P_\varepsilon f \mid P_\varepsilon f\rangle_{L^2(X)} \geq \varepsilon \langle P_\varepsilon f \mid P_\varepsilon f\rangle_{L^2(X)}$$

for all $f \in L^2(X)$.

Since P_ε arises as a limit of polynomials in $K *_Y$ in the strong operator topology, P_ε is Γ-equivariant. Additionally, P_ε is $L^\infty(Y)$-linear since $L^\infty(Y)$-linearity is preserved when passing to strong operator limits. It follows that $\mathcal{H}_\varepsilon := P_\varepsilon(L^2(X))$ is a Γ-invariant $L^\infty(Y)$-submodule of $L^2(X)$. Since P_ε is an orthogonal projection, $\mathcal{H}_\varepsilon$ is also $L^2(X)$-closed.

We now show that $\mathcal{H}_\varepsilon$ is finitely generated. By equation (1),

$$\langle K *_Y f, f\rangle_{L^2(X)} \geq \varepsilon \langle f, f\rangle_{L^2(X)}$$

for all $f \in \mathcal{H}_\varepsilon$. We claim that

$$\langle K *_Y f, f\rangle_{L^2(X|Y)} \geq \varepsilon\langle f, f\rangle_{L^2(X|Y)} \tag{2}$$

for each $f \in \mathcal{H}_\varepsilon$. Indeed, let $E = \{\langle K *_Y f, f\rangle_{L^2(X|Y)} < \varepsilon\langle f, f\rangle_{L^2(X|Y)}\}$. Since $\mathcal{H}_\varepsilon$ is an $L^\infty(Y)$-module, we have $g := 1_E f \in \mathcal{H}_\varepsilon$. By equation (1),

$$\begin{aligned} 0 \leq \langle K *_Y g, g\rangle_{L^2(X)} - \varepsilon\langle f, f\rangle_{L^2(X)} &= \int_Y \langle K *_Y g, g\rangle_{L^2(X|Y)} - \varepsilon\langle g, g\rangle_{L^2(X|Y)}\, d\nu \\ &= \int_E \langle K *_Y f, f\rangle_{L^2(X|Y)} - \varepsilon\langle f, f\rangle_{L^2(X|Y)}\, d\nu \leq 0. \end{aligned}$$

Thus, $\nu(E) = 0$, proving the claim.

By the conditional Cauchy–Schwarz inequality applied to equation (2), we have

$$\|K *_Y f\|_{X|Y} \cdot \|f\|_{L^2(X|Y)} \geq \varepsilon\|f\|^2_{L^2(X|Y)}$$

and this implies $\|f\|_{L^2(X|Y)} \leq (1/\varepsilon)\|K *_Y f\|_{X|Y}$ for $f \in \mathcal{H}_\varepsilon$. Using that $K *_Y : L^2(X) \to L^2(X)$ is a conditional Hilbert–Schmidt operator (see Proposition 2.3), we find $C > 0$ such that

$$\sum_{f \in M} \|K *_Y f\|^2_{X|Y} \leq C$$

for every conditionally orthonormal basis $M \subseteq L^2(X)$. In particular, if $M \subseteq \mathcal{H}_\varepsilon$ is a conditionally orthonormal basis of $\mathcal{H}_\varepsilon$, then

$$\sum_{f \in M} \|f\|^2_{L^2(X|Y)} \leq \frac{1}{\varepsilon}\sum_{f \in M} \|K *_Y f\|^2_{X|Y} \leq \frac{C}{\varepsilon}.$$

Using Proposition 2.4, we obtain that the $L^\infty(Y)$-submodules $\mathcal{H}_\varepsilon$ are finitely generated.

If $f \in L^2(X)$, we obtain by the properties of spectral projections that

$$K *_Y f = \lim_{n\to\infty} P_{1/n}(K *_Y f).$$

Therefore, the image of $K *_Y$ is contained in the $L^2(X)$-closure of the union of all Γ-invariant and finitely generated $L^\infty(Y)$-submodules of $L^2(X)$, this concludes the proof. □

The following assertion is [**3**, Lemma 4.10].

LEMMA 2.7. *Assertion (iii) is equivalent to assertion (iii)′ in Theorem 2.5.*

Proof. We show that assertion (iii) implies assertion (iii)′. By assumption, there is a dense set $\mathcal{G}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ such that the orbit of every $g \in \mathcal{G}$ has the conditional total boundedness property in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$. We will construct a dense set $\mathcal{H}$ in $L^2(X)$ such that the orbit of every $f \in \mathcal{H}$ has the same conditional total boundedness property but within $L^2(X)$.

Let $\varepsilon > 0$ and let $0 < \delta < 1$. Let $f \in L^2(X)$ and let $g \in \mathcal{G}$ be such that $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}(f, g) < \delta\varepsilon/2$. Then, the measure of $B = \{\|f - g\|_{L^2(X|Y)} < \delta/2\}$ is greater than $1 - \varepsilon/2$. There is a finite subset $\mathcal{F}$ of $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ such that for all $\gamma \in \Gamma$,

$$\min_{h\in\mathcal{F}} \|(T^\gamma)^* g - h\|_{L^2(X|Y)} < \frac{\delta}{2}.$$

By the conditional triangle inequality, for all $\gamma \in \Gamma$,

$$\min_{h\in\mathcal{F}\cup\{0\}} \|(T^\gamma)^*(f1_B) - h\|_{L^2(X|Y)} < \delta.$$

Suppose $\mathcal{F}$ has n elements. By monotone convergence, for each $h \in \mathcal{F}$, pick a measurable subset A_h of Y such that $\nu(A_h) \geq 1 - \varepsilon/2n$ and $h1_{A_h} \in L^2(X)$. Let $A = \bigcap_{i=1}^n A_h$. Then, $\nu(A) \geq 1 - \varepsilon/2$. By the conditional triangle inequality,

$$\min_{h\in\mathcal{F}\cup\{0\}} \|(T^\gamma)^*(f1_B) - h1_{A_h}\|_{L^2(X|Y)} < \delta \quad \text{on } A.$$

By the completeness of the probability algebra Y and the countable chain condition (cf. [3, Remark 2.2]), we have

$$A^* := \bigcup_{\gamma\in\Gamma} T^\gamma(A) = \bigcup_{n\in\mathbb{N}} T^{\gamma_n}(A)$$

for some countable set $\{\gamma_n\} \subset \Gamma$. Using a trick of Furstenberg (cf. proof of [2, Theorem 6.13, $C_1 \Rightarrow C_2$]), by modifying the $h1_{A_h}$ (while keeping them in $L^2(X)$), we reach that

$$\min_{\tilde{h}\in\tilde{\mathcal{F}}\cup\{0\}} \|(T^\gamma)^*(f1_B) - \tilde{h}\|_{L^2(X|Y)} < \delta \quad \text{on } A^*,$$

where $\tilde{\mathcal{F}}$ collects the modified $h1_{A_h}$ for all $h \in \mathcal{F}$. Since A^* is Γ-invariant, we obtain

$$\min_{\tilde{h}\in\tilde{\mathcal{F}}\cup\{0\}} \|(T^\gamma)^*(f1_{B\cap A^*}) - \tilde{h}1_{A^*}\|_{L^2(X|Y)} < \delta$$

globally and the measure of $B \cap A^*$ is at least $1 - \varepsilon$. By [3, Proposition 3.4(vii)], the collection $\mathcal{H}$ of $f1_{B\cap A^*}$ as constructed above starting with any $f \in L^2(X)$ is dense in $L^2(X)$. □

Acknowledgment. The author was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 547294463.

REFERENCES

[1] C. Anantharaman-Delaroche and S. Popa. An introduction to II_1 factors, 2017. Book manuscript available at https://www.math.ucla.edu/~popa/Books/IIunV15.pdf.

[2] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory (Porter Lectures, Princeton Legacy Library)*. Princeton University Press, Princeton, NJ, 2014.

[3] A. Jamneshan. An uncountable Furstenberg–Zimmer structure theory. *Ergod. Th. & Dynam. Sys.* **43**(7) (2023), 2404–2436.

[4] A. Jamneshan and P. Spaas. On compact extensions of tracial W^*-dynamical systems. *Groups Geom. Dyn.*, published online first, 2024.

[5] D. Kerr and H. Li. *Ergodic Theory: Independence and Dichotomies (Springer Monographs in Mathematics)*. Springer, Cham, 2016.