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ON A RESULT OF LEVITZKI

BY
B. FELZENSZWALB[1]

A well known result of Levitzki [2, Lemma 1.1] is the following:

THEOREM. *Let R be a ring and U a non-zero one-sided ideal of R. Suppose that given $a \in U$, $a^n = 0$ for a fixed integer $n \geq 1$; then R has a non-zero nilpotent ideal.*

The purpose of this note is to observe some additional results which are related to the above.

We being with

THEOREM 1. *Let R be a ring with no non-zero nil ideals and U an ideal of R. Suppose that $a \in R$ is such that for every $x \in U$, $ax^{n(x)} = 0$ where $n(x) \geq 1$ depends on x; then $aU = Ua = 0$.*

Proof. Let $V = \{r \in R \mid rx^n = 0,\ n = n(r, x) \geq 1, \text{ all } x \in U\}$. Then V is a left ideal of R and $V \cap U$ is nil. We proceed now to show that $V^2U \subset V \cap U$.

Let $v_1 \in V$ and $u \in U$. Then there exists $k \geq 1$ such that $v_1(uv_1)^k = 0$. Hence, if $y = v_1u$, $y^{k+1} = 0$ and, in particular y is quasi-regular.

Let $v_2 \in V$. If $x \in U$, $z = (1+y)x(1+y)^{-1} \in U$; so, there exists $n \geq 1$ such that $v_2z^n = 0$. Since $x = (1+y)^{-1}z(1+y)$ we have that $(1+y)^{-1}v_2(1+y)x^n = 0$. That is, $(1+y)^{-1}v_2(1+y) \in V$.

Thus, if $x \in U$

$$v_2x^m = 0$$

$$(1+y)^{-1}v_2(1+y)x^m = 0$$

holds for a suitable $m \geq 1$. It follows that $(v_2v_1)ux^m = v_2yx^m = v_2(1+y)x^m = 0$; hence $(v_2v_1)u \in V$. In other words, we have shown that $V^2U \subset V \cap U$. Since $V \cap U$ is nil, by our assumptions on R, $V^2U = 0$. But this gives $(VU)^3 = (UV)^3 = 0$ and consequently, since R has no non-zero nilpotent ideals, $VU = UV = 0$. Since $a \in V$ the result follows.

It was shown in [1] that if R is a ring with no non-zero nil right ideals and $a \in R$ is such that for every $x \in R$, $ax^na = 0$ where $n = n(x) \geq 1$ depends on x, then $a = 0$. The idea used to prove this can also be used to prove the following

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THEOREM 2. *Let R be a ring with no non-zero nil right ideals and U a left (respectively right) ideal of R. Suppose that $a \in R$ is such that for every $x \in U$, $ax^{n(x)} = 0$ where $n(x) \geq 1$ depends on x; then $aU = Ua = 0$ (respectively $aU = 0$).*

Proof. The situation when U is a right ideal of R follows immediately because in this case, if $x \in U$, $a(xa)^k = 0$ for some $k \geq 1$; so $(ax)^{k+1} = 0$. That is, aU is a nil right ideal of R. By our assumptions on R we must have $aU = 0$.

Suppose now U is a left ideal of R. Let $r \in U$ with $r^2 = 0$. Then, by hypothesis, there exists $m \geq 1$ such that $a(ar)^m = 0$ and $a(r + ar)^m = 0$. Hence, on expansion of the last equation we get $(ar)^m = 0$. In other words, ar is nilpotent for every $r \in U$ with $r^2 = 0$.

Let $V = \{y \in R \mid yx^n = 0,\ n = n(y, x) \geq 1, \text{ all } x \in U\}$. Then V is a left ideal of R and, by what we deduced before, Vr is nil for every $r \in U$ with $r^2 = 0$. Since R has no non-zero nil right ideals, and hence no non-zero nil left ideals, we obtain

$$(1) \qquad Vr = 0 \text{ for every } r \in U \text{ with } r^2 = 0.$$

Let $x \in U$. We claim that if $y \in V$, yx is nilpotent. We go by induction on n such that $yx^n = 0$. If $n = 1$ there is nothing to show. Suppose $n > 1$; then $xyx^{n-1} \in U$ and $(xyx^{n-1})^2 = 0$, so, by (1), $yxyx^{n-1} = 0$. Since $yxy \in V$ our induction gives us $(yx)^2$ nilpotent; hence yx is nilpotent.

So, Uy is nil for every $y \in V$. Since R has no non-zero nil right ideals we get $UV = 0$ and, a fortiori, $VU = 0$. Since $a \in V$ this proves the theorem.

We note that if in Theorem 2 the integers $n(x)$ have a finite maximum as x ranges over U the conclusion remains valid if we replace the hypothesis "with no non-zero nil right ideals" by "with no non-zero nilpotent ideals" (the proof goes just the same with the use of Levitzki's result and with the induction hypothesis "$yx^n = 0$ implies $(yx)^{2^n} = 0$"). This yields a generalization of Levitzki's Theorem.

The question of whether or not Theorem 2 remains valid if we replace the hypothesis "with no non-zero nil right ideals" by its two-sided version "with no non-zero nil ideals" is now readily seen to be equivalent to the Koethe conjecture: a nil one-sided ideal in a ring R generates a two-sided nil ideal.

REFERENCES

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INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
C.P. 1835 ZC-00
20.000 RIO DE JANEIRO, R. J., BRAZIL

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