Combinatorics, Probability and Computing (2024), 1–49
doi:10.1017/S0963548324000336

ARTICLE

Noisy group testing via spatial coupling

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(Received 9 February 2024; revised 4 September 2024; accepted 20 September 2024)

Abstract
We study the problem of identifying a small number $k \sim n^{\theta}$, $0 < \theta < 1$, of infected individuals within a large population of size n by testing groups of individuals simultaneously. All tests are conducted concurrently. The goal is to minimise the total number of tests required. In this paper, we make the (realistic) assumption that tests are noisy, that is, that a group that contains an infected individual may return a negative test result or one that does not contain an infected individual may return a positive test result with a certain probability. The noise need not be symmetric. We develop an algorithm called SPARC that correctly identifies the set of infected individuals up to $o(k)$ errors with high probability with the asymptotically minimum number of tests. Additionally, we develop an algorithm called SPEX that exactly identifies the set of infected individuals w.h.p. with a number of tests that match the information-theoretic lower bound for the constant column design, a powerful and well-studied test design.

Keywords: Group testing; random graphs; inference; efficient algorithms

2020 MSC Codes: Primary: 62B10; Secondary: 05C80, 68P30, 68R05

1. Introduction

1.1. Background and motivation

Few mathematical disciplines offer as abundant a supply of easy-to-state but hard-to-crack problems as combinatorics does. Group testing is a prime example. The problem goes back to the 1940s [15]. Within a population of size n, we are to identify a subset of k infected individuals. To this end, we test groups of individuals simultaneously. In an idealised scenario called 'noiseless group testing', each test returns a positive result if and only if at least one member of the group is infected. All tests are conducted in parallel. The problem is to devise a (possibly randomised) test design that minimises the total number of tests required.

Noiseless group testing has inspired a long line of research, which has led to optimal or near-optimal results for several parameter regimes [4, 9]. But the assumption of perfectly accurate tests is unrealistic. Real tests are noisy [29]. More precisely, in medical terms, the *sensitivity* of a test is defined as the probability that a test detects an actual infection, namely, that a group that contains an infected individual displays a positive test result. Moreover, the *specificity* of a test refers to the

Amin Coja-Oghlan and Lena Krieg are supported by DFG CO 646/3 and DFG CO 646/5. Max Hahn-Klimroth and Olga Scheftelowitsch are supported by DFG CO 646/5.

probability that a healthy group returns a negative test result. If these accuracies are reasonably high (say 99%), one might be tempted to think that noiseless testing provides a good enough approximation. Yet remarkably, we will discover that this is far from correct. Even a seemingly tiny amount of noise has a discernible impact not only on the number of tests required but also on the choice of a good test design; we will revisit this point in Section 1.5. Hence, in group testing, like in several other inference problems, the presence of noise adds substantial mathematical depth. As a rough analogy, think of error-correcting linear codes. In the absence of noise, the decoding problem just boils down to solving linear equations. By contrast, the noisy version, the closest vector problem, is NP-hard [13].

In the present paper, we consider a very general noise model that allows for arbitrary sensitivities and specificities. To be precise, we assume that if a test group contains an infected individual, then the test displays a positive result with probability p_{11} and a negative result with probability $p_{10} = 1 - p_{11}$. Similarly, if the group consists of healthy individuals only, then the test result will display a negative outcome with probability p_{00} and a positive result with probability $p_{01} = 1 - p_{00}$. Every test is subjected to noise independently.

Under the common assumption that the number k of infected individuals scales as a power $k \sim n^\theta$ of the population size n with an exponent $0 < \theta < 1$, we contribute new *approximate* and *exact recovery algorithms* SPARC and SPEX. These new algorithms come with randomised test designs. We will identify a threshold $m_{\mathtt{SPARC}} = m_{\mathtt{SPARC}}(n, k, \boldsymbol{p})$ such that SPARC correctly identifies the set of infected individuals up to $o(k)$ errors with high probability over the choice of the test design, provided that at least $(1+\varepsilon)m_{\mathtt{SPARC}}$ tests are deployed. SPARC is efficient, that is, has polynomial running time in terms of n. By contrast, we will prove that with $(1-\varepsilon)m_{\mathtt{SPARC}}$ tests it is *impossible* to identify the set of infected individuals up to $o(k)$ errors, regardless of the choice of test design or the running time allotted. In other words, SPARC solves the approximate recovery problem optimally.

In addition, we develop a polynomial time algorithm SPEX that correctly identifies the status of *all* individuals w.h.p., provided that at least $(1+\varepsilon)m_{\mathtt{SPEX}}$ tests are available, for a certain $m_{\mathtt{SPEX}}(n, k, \boldsymbol{p})$. Exact recovery has been the focus of much of the previous literature on group testing [4]. In particular, for noisy group testing, the best previous exact recovery algorithm is the DD algorithm from [19]. DD comes with a simple randomised test design called the *constant column design*. Complementing the positive result on SPEX, we show that on the constant column design, exact recovery is information-theoretically impossible with $(1-\varepsilon)m_{\mathtt{SPEX}}$ tests. As a consequence, the number $m_{\mathtt{SPEX}}$ of tests required by SPEX is an asymptotic lower bound on the number of tests required by *any* algorithm on the constant column design, including DD. Indeed, as we will see in Section 1.5, for most choices of the specificity/sensitivity and of the infection density θ, SPEX outperforms DD dramatically.

Throughout the paper, we write $\boldsymbol{p} = (p_{00}, p_{01}, p_{10}, p_{11})$ for the noisy channel to which the test results are subjected. We may assume without loss of generality that

$$p_{11} > p_{01}, \tag{1.1}$$

that is, that a test is more likely to display a positive result if the test group actually contains an infected individual; otherwise, we could just invert the test results. A test design can be described succinctly by a (possibly random) bipartite graph $G = (V \cup F, E)$, where V is a set of n individuals and F is a set of m tests. Write $\boldsymbol{\sigma} \in \{0, 1\}^V$ for the (unknown) vector of Hamming weight k whose 1-entries mark the infected individuals. Further, let $\boldsymbol{\sigma}' \in \{0, 1\}^F$ be the vector of *actual test results*, that is, $\boldsymbol{\sigma}'_a = 1$ if and only if $\boldsymbol{\sigma}_v = 1$ for at least one individual v in test a. Finally, let $\boldsymbol{\sigma}'' \in \{0, 1\}^F$ be the vector of *displayed test results*, where noise has been applied to $\boldsymbol{\sigma}'$, that is,

$$\mathbb{P}\left[\boldsymbol{\sigma}''_a = \sigma'' \mid G,\ \boldsymbol{\sigma}'_a = \sigma'\right] = p_{\sigma' \sigma''} \qquad \text{independently for every } a \in F. \tag{1.2}$$

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The objective is to infer $\boldsymbol{\sigma}$ from $\boldsymbol{\sigma}''$ given G. As per common practice in the group testing literature, we assume throughout that $k = \lceil n^{\theta} \rceil$ and that k and the channel $\boldsymbol{p}$ are known to the algorithm [4].[1]

1.2. Approximate recovery

The first main result provides an algorithm that identifies the status of all but $o(k)$ individuals correctly with the optimal number of tests. For a number $z \in [0, 1]$, let $h(z) = -z \ln(z) - (1 - z) \ln(1 - z)$. Further, for $y, z \in [0, 1]$, let $D_{\mathrm{KL}}\left(y \| z\right) = y \ln(y/z) + (1 - y) \ln((1 - y)/(1 - z))$ signify the Kullback–Leibler divergence. We use the convention that $0 \ln 0 = 0 \ln \frac{0}{0} = 0$. Given the channel $\boldsymbol{p}$ define

$$\phi = \phi(\boldsymbol{p}) = \frac{h(p_{00}) - h(p_{10})}{p_{00} - p_{10}}, \qquad c_{\mathrm{Sh}} = c_{\mathrm{Sh}}(\boldsymbol{p}) = \frac{1}{D_{\mathrm{KL}}\left(p_{10} \| (1 - \tanh(\phi/2))/2\right)}. \tag{1.3}$$

The value $1/c_{\mathrm{Sh}} = D_{\mathrm{KL}}\left(p_{10} \| (1 - \tanh(\phi/2))/2\right)$ equals the capacity of the $\boldsymbol{p}$-channel [19, Lemma F.1]. Let

$$m_{\mathtt{SPARC}}(n, k, p) = c_{\mathrm{Sh}} k \ln(n/k).$$

Theorem 1.1. *For any $\boldsymbol{p}$, $0 < \theta < 1$ and $\varepsilon > 0$, there exists $n_0 = n_0(\boldsymbol{p}, \theta, \varepsilon)$ such that for every $n > n_0$, there exists a randomised test design $\mathbf{G}_{\mathrm{sc}}$ with $m \leq (1 + \varepsilon) m_{\mathtt{SPARC}}(n, k, \boldsymbol{p})$ tests and a deterministic polynomial time inference algorithm* `SPARC` *such that*

$$\mathbb{P}\left[\|\mathtt{SPARC}(\mathbf{G}_{\mathrm{sc}}, \boldsymbol{\sigma}''_{\mathbf{G}_{\mathrm{sc}}}) - \boldsymbol{\sigma}\|_1 < \varepsilon k\right] > 1 - \varepsilon. \tag{1.4}$$

In other words, once the number of tests exceeds $m_{\mathtt{SPARC}} = c_{\mathrm{Sh}} k \ln(n/k)$, SPARC applied to the test design $\mathbf{G}_{\mathrm{sc}}$ identifies the status of all but $o(k)$ individuals correctly w.h.p. The test design $\mathbf{G}_{\mathrm{sc}}$ employs an idea from coding theory called 'spatial coupling' [37, 41]. As we will elaborate in Section 2, spatial coupling blends a randomised and a topological construction. A closely related design has been used in noiseless group testing [9].

The following theorem shows that Theorem 1.1 is optimal in the strong sense that it is information-theoretically impossible to approximately recover the set of infected individuals with fewer than $(1 - \varepsilon) m_{\mathtt{SPARC}}$ tests on *any* test design. In fact, approximate recovery w.h.p. is impossible even if we allow *adaptive* group testing, where tests are conducted one by one and the choice of the next group to be tested may depend on all previous results.

Theorem 1.2. *For any $\boldsymbol{p}$, $0 < \theta < 1$ and $\varepsilon > 0$, there exist $\delta = \delta(\boldsymbol{p}, \theta, \varepsilon) > 0$ and $n_0 = n_0(\boldsymbol{p}, \theta, \varepsilon)$ such that for all $n > n_0$, all adaptive test designs with $m \leq (1 - \varepsilon) m_{\mathtt{SPARC}}(n, k, \boldsymbol{p})$ tests in total and any function $\mathscr{A}\colon \{0, 1\}^m \to \{0, 1\}^n$, we have*

$$\mathbb{P}\left[\|\mathscr{A}(\boldsymbol{\sigma}'') - \boldsymbol{\sigma}\|_1 < \delta k\right] < 1 - \delta. \tag{1.5}$$

Theorem 1.2 and its proof are a relatively simple adaptation of [19, Corollary 2.3], where $c_{\mathrm{Sh}} k \ln(n/k)$ was established as a lower bound on the number of tests required for *exact* recovery.

1.3. Exact recovery

How many tests are required in order to infer the set of infected individuals precisely, not just up to $o(k)$ mistakes? Intuitively, apart from an information-theoretic condition such as (1.3), exact recovery requires a kind of local stability condition. More precisely, imagine that we managed to correctly diagnose all individuals $y \neq x$ that share a test with individual x. Then towards ascertaining the status of x itself, only those tests are relevant that contain x but no other infected individual

[1]These assumptions could be relaxed at the expense of increasing the required number of tests (details omitted).

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y, for the outcome of these tests hinges on the status of x. Hence, to achieve exact recovery, we need to make certain that it is possible to tell the status of x itself from these tests w.h.p.

The required number of tests to guarantee local stability on the test design $\mathbf{G}_{\mathrm{sc}}$ from Theorem 1.1 can be expressed in terms of a mildly involved optimisation problem. For $c, d > 0$ and $\theta \in (0, 1)$, let

$$\mathscr{Y}(c, d, \theta) = \left\{y \in [0, 1] : cd(1-\theta)D_{\mathrm{KL}}\left(y \| \exp(-d)\right) < \theta\right\}. \tag{1.6}$$

This set is a nonempty interval because $y \mapsto D_{\mathrm{KL}}\left(y\|\exp(-d)\right)$ is convex and $y = \exp(-d) \in \mathscr{Y}(c, d, \theta)$. Let

$$c_{\mathrm{ex},0}(d,\theta) = \begin{cases} \inf\left\{c > 0 : \inf_{y \in \mathscr{Y}(c,d,\theta)} cd(1-\theta)\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right)\right.\right. & \text{if } p_{11} < 1, \\ \left.\left.+yD_{\mathrm{KL}}\left(p_{01}\|p_{11}\right)\right) \geq \theta\right\} & \\ \inf\{c > 0 : 0 \notin \mathscr{Y}(c, d, \theta)\} & \text{otherwise.} \end{cases} \tag{1.7}$$

If $p_{11} = 1$, let $\mathfrak{z}(y) = 1$ for all $y \in \mathscr{Y}(c, d, \theta)$. Further, if $p_{11} < 1$, then the function $z \mapsto D_{\mathrm{KL}}\left(z\|p_{11}\right)$ is strictly decreasing on $[p_{01}, p_{11}]$; therefore, for any $c > c_{\mathrm{ex},0}(d, \theta)$ and $y \in \mathscr{Y}(c, d, \theta)$, there exists a unique $\mathfrak{z}(y) = \mathfrak{z}_{c,d,\theta}(y) \in [p_{01}, p_{11}]$ such that

$$cd(1-\theta)\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(\mathfrak{z}(y)\|p_{11}\right)\right) = \theta. \tag{1.8}$$

In either case set

$$c_{\mathrm{ex},1}(d,\theta) = \begin{cases} \inf\left\{c > c_{\mathrm{ex},0}(d,\theta) : \inf_{y \in \mathscr{Y}(c,d,\theta)} cd(1-\theta)\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right)\right.\right. & \text{if } p_{01} > 0, \\ \left.\left.+yD_{\mathrm{KL}}\left(\mathfrak{z}(y)\|p_{01}\right)\right) \geq 1\right\} & \\ c_{\mathrm{ex},0}(d,\theta) & \text{otherwise.} \end{cases} \tag{1.9}$$

Finally, define

$$c_{\mathrm{ex},2}(d) = 1\Big/\left(h(p_{00}\exp(-d) + p_{10}(1-\exp(-d))) - \exp(-d)h(p_{00})\right.$$
$$\left. -(1-\exp(-d))h(p_{10})\right), \tag{1.10}$$

$$c_{\mathrm{ex}}(\theta) = \inf_{d>0} \max\{c_{\mathrm{ex},1}(d,\theta), c_{\mathrm{ex},2}(d)\}, \tag{1.11}$$
$$m_{\mathtt{SPEX}}(n, k, \boldsymbol{p}) = c_{\mathrm{ex}}(\theta)k\ln(n/k).$$

Theorem 1.3. *For any $\boldsymbol{p}$, $0 < \theta < 1$ and $\varepsilon > 0$, there exists $n_0 = n_0(\boldsymbol{p}, \theta, \varepsilon)$ such that for every $n > n_0$, there exists a randomised test design $\mathbf{G}_{\mathrm{sc}}$ with $m \leq (1+\varepsilon)m_{\mathtt{SPEX}}(n, k, \boldsymbol{p})$ tests and a deterministic polynomial time inference algorithm* SPEX *such that*

$$\mathbb{P}\left[\mathtt{SPEX}(\mathbf{G}_{\mathrm{sc}}, \boldsymbol{\sigma}''_{\mathbf{G}_{\mathrm{sc}}}) = \boldsymbol{\sigma}\right] > 1 - \varepsilon. \tag{1.12}$$

Like SPARC from Theorem 1.1, SPEX uses the spatially coupled test design $\mathbf{G}_{\mathrm{sc}}$. Crucially, apart from the numbers n and m of individuals and tests, the value of d at which the infimum (1.11) is attained also enters into the construction of that test design. Specifically, the average size of a test group equals dn/k. Remarkably, while the optimal value of d for approximate recovery turns out to depend on the channel $\boldsymbol{p}$ only, a different value of d that also depends on k can be the right choice to facilitate exact recovery. We will revisit this point in Section 1.5.

1.4. Lower bound on the constant column design

Unlike in the case of approximate recovery, we do not have a proof that the positive result on exact recovery from Theorem 1.3 is optimal for *any* choice of test design. However, we can show that exact recovery with $(1-\varepsilon)c_{\mathrm{ex}}k\ln(n/k)$ tests is impossible on the *constant column design* $\mathbf{G}_{\mathrm{cc}}$. Under $\mathbf{G}_{\mathrm{cc}}$, each of the n individuals independently joins an equal number Δ of tests, drawn uniformly without replacement from the set of all m available tests. Let $\mathbf{G}_{\mathrm{cc}} = \mathbf{G}_{\mathrm{cc}}(n, m, \Delta)$ signify the

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outcome. The following theorem shows that exact recovery on $\mathbf{G}_{cc}$ is information-theoretically impossible with fewer than m_{SPEX} tests.

Theorem 1.4. *For any* $\boldsymbol{p}$, $0 < \theta < 1$ *and* $\varepsilon > 0$, *there exists* $n_0 = n_0(\boldsymbol{p}, \theta, \varepsilon)$ *such that for every* $n > n_0$ *and all* $m \leq (1-\varepsilon)m_{SPEX}(n, k, \boldsymbol{p})$, $\Delta > 0$ *and any* $\mathscr{A}_{\mathbf{G}_{cc}} : \{0,1\}^m \to \{0,1\}^n$, *we have*

$$\mathbb{P}\left[\mathscr{A}_{\mathbf{G}_{cc}}(\boldsymbol{\sigma}''_{\boldsymbol{G}}) = \boldsymbol{\sigma}\right] < \varepsilon. \tag{1.13}$$

An immediate implication of Theorem 1.4 is that the positive result from Theorem 1.3 is at least as good as the best prior results on exact noisy recovery from [19], which are based on running a simple combinatorial algorithm called DD on $\mathbf{G}_{cc}$. In fact, in Section 1.5, we will see that the new bound from Theorem 1.3 improves over the bounds from [19] rather significantly for most $\theta, \boldsymbol{p}$.

The proof of Theorem 1.4 confirms the combinatorial meaning of the threshold $c_{ex}(\theta)$. Specifically, for $c = m/(k \ln(n/k)) < c_{ex,2}(d)$ from (1.10), a moment calculation reveals that w.h.p., $\mathbf{G}_{cc}$ contains 'solutions' σ of at least the same posterior likelihood as the true $\boldsymbol{\sigma}$ such that σ and $\boldsymbol{\sigma}$ differ significantly, that is, $\|\boldsymbol{\sigma} - \sigma\|_1 = \Omega(k)$. By contrast, the threshold $c_{ex,1}(d, \theta)$ marks the onset of local stability. This means that for $c < c_{ex,1}(d, \theta)$, there will be numerous σ close to but not identical to $\boldsymbol{\sigma}$ (i.e. $0 < \|\boldsymbol{\sigma} - \sigma\|_1 = o(k)$) of the at least same posterior likelihood. In either case, any inference algorithm, efficient or not, is at a loss identifying the actual $\boldsymbol{\sigma}$.

In recent independent work, Chen and Scarlett [7] obtained Theorem 1.4 in the special case of symmetric noise (i.e. $p_{00} = p_{11}$). While syntactically their expression for the threshold $c_{ex}(\theta)$ differs from (1.6)–(1.11), it can be checked that both formulas yield identical results (see Appendix F). Apart from the information-theoretic lower bound (which is the part most relevant to the present work), Chen and Scarlett also proved that it is information-theoretically possible (by means of an exponential-time algorithm) to infer $\boldsymbol{\sigma}$ w.h.p. on the constant column design with $m \geq (1+\varepsilon)c_{ex}(\theta)k \ln(n/k)$ tests if $p_{00} = p_{11}$. Hence, the bound $c_{ex}(\theta)$ is tight in the case of symmetric noise.

1.5. Examples

We illustrate the improvements that Theorems 1.1 and 1.3 contribute by way of concrete examples of channels $\boldsymbol{p}$. Specifically, for the binary symmetric channel and the Z-channel, it is possible to obtain partial analytic results on the optimisation behind $c_{ex}(\theta)$ from (1.11) (see Appendices D–E). As we will see, even a tiny amount of noise has a dramatic impact on both the number of tests required and the parameters that make a good test design.

1.5.1. The binary symmetric channel

Although symmetry is an unrealistic assumption from the viewpoint of applications [29], the expression (1.3) and the optimisations (1.9)–(1.11) get much simpler on the binary symmetric channel, that is, in the case $p_{00} = p_{11}$. For instance, the value of d that minimises $c_{ex,2}(d)$ from (1.10) turns out to be $d = \ln 2$. The parameter d enters into the constructions of the randomised test designs $\mathbf{G}_{sc}$ and $\mathbf{G}_{cc}$ in a crucial role. Specifically, the average size of a test group equals dn/k. In effect, any test is actually negative with probability $\exp(-d + o(1))$ (see Proposition 2.3 and Lemma 5.1 below). Hence, if $d = \ln 2$, then w.h.p. about half the tests contain an infected individual. In effect, since $p_{11} = p_{00}$, after the application of noise about half the tests display a positive result w.h.p.

In particular, in the noiseless case $p_{11} = p_{00} = 1$, a bit of calculus reveals that the value $d = \ln 2$ also minimises the other parameter $c_{ex,1}(d, \theta)$ from (1.9) that enters into $c_{ex}(\theta)$ from (1.11). Therefore, in noiseless group testing, $d = \ln 2$ unequivocally is the optimal choice, the optimisation on d (1.11) effectively disappears, and we obtain

$$c_{ex}(\theta) = \max\left\{\frac{\theta}{(1-\theta)\ln^2 2}, \frac{1}{\ln 2}\right\}, \tag{1.14}$$

thereby reproducing the optimal result on noiseless group testing from [9].

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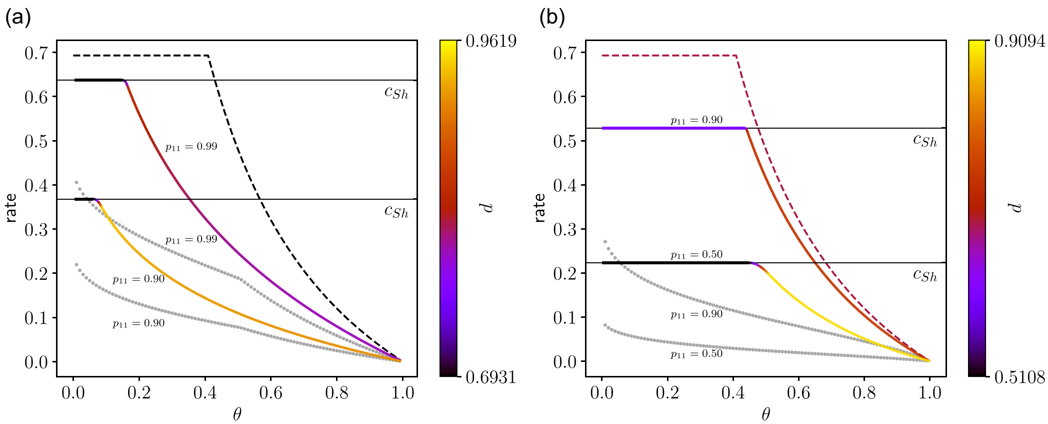

Rates on the binary symmetric channel for $p_{00} = p_{11} = 0.99$ and $p_{00} = p_{11} = 0.9$.

Rates on the Z-channel ($p_{00} = 1$) with $p_{11} = 0.9$ and $p_{11} = 0.5$.

Figure 1. Information rates on different channels in nats. The horizontal axis displays the infection density parameter $0 < \theta < 1$. The colour indicates the optimal value of d for a given θ.

But remarkably, as we verify analytically in Appendix D at positive noise $\frac{1}{2} < p_{11} = p_{00} < 1$, the value of d that minimises $c_{\mathrm{ex},1}(d, \theta)$ does *not* generally equal ln 2. Hence, if we aim for exact recovery, then at positive noise, it is no longer optimal for all $0 < \theta < 1$ to aim for about half the tests being positive/negative. The reason is the occurrence of a phase transition in terms of θ where the 'local stability' term $c_{\mathrm{ex},1}(d, \theta)$ takes over as the overall maximiser in (1.10). Consequently, the objective of minimising $c_{\mathrm{ex},1}(d, \theta)$ and the optimal choice $d = \ln 2$ for $c_{\mathrm{ex},2}(d)$ clash. In effect, the overall minimiser d for $c_{\mathrm{ex}}(d, \theta)$ depends on both $\boldsymbol{p}$ and the infection density parameter θ in a nontrivial way. Thus, the presence and level of noise have a discernible impact on the choice of a good test design.[2]

Figure 1 displays the performance of the algorithms SPARC and SPEX from Theorems 1.1 and 1.3 on the binary symmetric channel. For the purpose of graphical representation, the figure does not display the values of $c_{\mathrm{ex}}(\theta)$, which diverge as $\theta \to 1$, but the value $1/c_{\mathrm{ex}}(\theta)$. This value has a natural information-theoretic interpretation: it is the average amount of information that a test reveals about the set of infected individuals, measured in 'nats'. In other words, the plots display the information rate of a single test (higher is better). The optimal values of d are colour-coded into the curves. While in the noiseless case, $d = \ln 2$ remains constant, in the noisy cases, d varies substantially with both θ and $p_{00} = p_{11}$.

For comparison, the figure also displays the rate in the noiseless case (dashed line on top) and the best previous rates realised by the DD algorithm on $\mathbf{G}_{\mathrm{cc}}$ from [19] (dotted lines). As is evident from Figure 1, even noise as small as 1% already reduces the rate markedly: the upper coloured curve remains significantly below the noiseless black line. That said, Figure 1 also illustrates how the rate achieved by SPEX improves over the best previous algorithm DD from [19]. Somewhat remarkably, the 10%-line for $c_{\mathrm{ex}}(\theta)$ intersects the 1%-line for DD for an interval of θ. Hence, for these θ, the algorithm from Theorem 1.3 at 10% noise requires fewer tests than DD at 1%.

Figure 1 also illustrates how approximate and exact recovery compare. Both coloured curves start out as black horizontal lines. These bits of the curves coincide with the rate of the SPARC algorithm from Theorem 1.1. The rate achieved by SPARC, which does not depend on θ, is therefore just the extension of this horizontal line to the entire interval $0 < \theta < 1$ at the height c_{Sh} from

[2]This observation confirms a hypothesis stated in (19, Appendix F). As mentioned in Section 1.4, independent work of Chen and Scarlett [7] on the case of symmetric noise reaches the same conclusion.

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(1.3). Hence, particularly for large θ, approximate recovery achieves *much* better rates than exact recovery.

1.5.2. The Z-channel

In the case $p_{00} = 1$ of perfect specificity, known as the Z-channel, it is possible to derive simple expressions for the optimisation problems (1.6)–(1.11) (see Appendix E):

$$c_{ex,1}(d, \theta) = -\frac{\theta}{d(1-\theta)\ln(1-\exp(-d)p_{11})}, \tag{1.15}$$

$$c_{ex,2}(d) = \left(h(p_{10} + (1-p_{10})\exp(-d)) - (1-\exp(-d))h(p_{10})\right)^{-1}. \tag{1.16}$$

As in the symmetric case, there is a tension between the value of d that minimises (1.15) and the objective of minimising (1.16). Recall that since the size of the test groups is proportional to d, the optimiser d has a direct impact on the construction of the test design.

Figure 1b displays the rates achieved by SPEX (solid line) and, for comparison, the DD algorithm from [19] (dotted grey) on the Z-channel with sensitivities $p_{11} = 0.9$ and $p_{11} = 0.5$. Additionally, the dashed red line indicates the noiseless rate. Once again, the optimal value of d is colour-coded into the solid SPEX line. Remarkably, the SPEX rate at $p_{11} = 0.5$ (high noise) exceeds the DD rate at $p_{11} = 0.9$ for a wide range of θ. As in the symmetric case, the horizontal c_{Sh}-lines indicate the performance of the SPARC approximate recovery algorithm.

1.5.3. General (asymmetric) noise

While in the symmetric case, the term $c_{ex,2}(d)$ from (1.10) attains its minimum simply at $d = \ln 2$, with ϕ from (1.3), the minimum for general $\boldsymbol{p}$ is attained at

$$d = d_{Sh} = \ln(p_{11} - p_{01}) - \ln\left((1-\tanh(\phi/2))/2 - p_{10}\right) \qquad [19, \text{Lemma F.1}]. \tag{1.17}$$

Once again, the design of $\mathbf{G}_{sc}$ (as well as $\mathbf{G}_{cc}$) ensures that w.h.p., a $\exp(-d)$-fraction of tests is actually negative w.h.p. The choice (1.17) ensures that under the $\boldsymbol{p}$-channel, the mutual information between the channel input and the channel output is maximised (19, Lemma F.1):

$$\frac{1}{c_{Sh}} = \frac{1}{c_{ex,2}(d_{Sh})} = D_{KL}\left(p_{10} \| (1-\tanh(\phi/2))/2\right), \tag{1.18}$$

As can be checked numerically, the second contributor $c_{ex,1}(d, \theta)$ to $c_{ex}(\theta)$ may take its minimum at another d. However, we are not aware of simple explicit expressions for $c_{ex,2}(\theta)$ from (1.10) for general noise.

1.6. Related work

The monograph of Aldridge, Johnson, and Scarlett [4] provides an excellent overview of the group testing literature. The group testing problem comes in various different flavours: non-adaptive (where all tests are conducted concurrently) or adaptive (where tests are conducted in subsequent stages such that the tests at later stages may depend on the outcomes of earlier stages), as well as noiseless or noisy. An important result of Aldridge shows that noiseless non-adaptive group testing does not perform better than plain individual testing if $k = \Omega(n)$, that is, if the number of infected individuals is linear in the size of the population [3]. Therefore, research on non-adaptive group testing focuses on the case $k \sim n^\theta$ with $0 < \theta < 1$. For non-adaptive noiseless group testing with this scaling of k, two different test designs (Bernoulli and constant column) and various elementary algorithms have been proposed [6]. Among these elementary designs and algorithms, the best performance to date is achieved by the DD greedy algorithm on the constant column

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design [24]. However, the DD algorithm does not match the information-theoretic bound on the constant column design for all θ [8].

Coja-Oghlan, Gebhard, Hahn-Klimroth, and Loick proposed a more sophisticated test design for noiseless group testing based on spatial coupling [9], along with an efficient inference algorithm called SPIV. Additionally, they improved the information-theoretic lower bound for non-adaptive noiseless group testing. The number of tests required by the SPIV algorithm matches this new lower bound. In effect, the combination of SPIV with the spatially coupled test design solves the noiseless non-adaptive group testing problem optimally both for exact and approximate recovery.

The present article deals with the *noisy* non-adaptive variant of group testing. A noisy version of the efficient DD algorithm was previously studied on both the Bernoulli and the constant column design [19, 35]. The best previous exact recovery results for general noise were obtained by Johnson, Gebhard, Loick, and Rolvien [19] by means of DD on the constant column design (see Theorem 2.1 below). Theorem 1.4 shows in combination with Theorem 1.3 that the new SPEX algorithm performs at least as well as *any* algorithm on the constant column design, including and particularly DD.

Apart from the articles [19, 35] that dealt with the same general noise model as we consider here, several contributions focused on special noise models, particularly symmetric noise ($p_{00} = p_{11}$). In this scenario, Chen and Scarlett [7] recently determined the information-theoretically optimal number of tests required for exact recovery on the Bernoulli and constant column designs.

For both designs, Chen and Scarlett considered the "low overlap" and the "high overlap" regime separately. This corresponds to decodings that align weakly/strongly with the ground truth. For the low overlap regime, they bounded the mutual information by using results of [33]. For the high overlap regime, they worked out a local stability analysis for the symmetric case while taking the degree fluctuations for the Bernoulli design into account. The local stability analysis of Chen and Scarlett for the constant column design is equivalent to the special case of symmetric noise of the corresponding analysis that we perform towards the proof of Theorem 1.3. Indeed, the information-theoretic threshold identified by Chen and Scarlett matches the threshold $c_{\mathrm{ex}}(\theta)$ from (1.10) in the special case of symmetric noise; see Appendix F for details. In contrast to the present work, Chen and Scarlett do not investigate the issue of *efficient* inference algorithms. Instead, they pose the existence of an efficient inference algorithm that matches the information-theoretic threshold as an open problem. Theorem 1.3 applied to symmetric noise answers their question in the affirmative. While it might be interesting to investigate whether the present approach for exact recovery extends to (a spatially coupled variant of) the Bernoulli design, we expect that this would lead to weaker bounds than obtained in Theorem 1.3. In fact, there is no known case where the Bernoulli design outperformed the constant column design. Furthermore, for noiseless group testing, the constant column design outperforms the Bernoulli design [9, 1, 39], although it is not proven yet that this holds in general for the noisy case [4, 7]. In addition, the question of a tight lower bound for exact recovery for general noise and arbitrary test designs remains open.

A further contribution of Scarlett and Cevher [33] contains a result on approximate recovery under the assumption of symmetric noise. In this case, Scarlett and Cevher obtain matching information-theoretic upper and lower bounds, albeit without addressing the question of efficient inference algorithms. Theorem 1.1 applied to the special case of symmetric noise provides a polynomial time inference algorithm that matches their lower bound.

From a practical viewpoint, non-adaptive group testing (where all tests are conducted in parallel) is preferable because results are available more rapidly than in the adaptive setting, where several rounds of testing are required. That said, adaptive schemes may require a smaller total number of tests. The case of noiseless adaptive group testing has been studied since the seminal work of Dorfman [15] from the 1940s. For the case $k \sim n^{\theta}$, a technique known as generalised binary splitting gets by with the optimal number of tests [5, 22]. Aldridge [2] extended this approach to the case $k = \Theta(n)$, obtaining near-optimal rates. Recently, there has been significant progress on upper and lower bounds for noisy and adaptive group testing, although general optimal results remain elusive [32, 38].

Beyond group testing, in recent years, important progress has been made on several inference problems by means of a combination of spatial coupling and message passing ideas. Perhaps the most prominent case in point is the compressed sensing problem [14, 25]. Further applications include the pooled data problem [17–20] and CDMA [37], a signal processing problem. The basic idea of spatial coupling, which we are going to discuss in some detail in Section 2.3, goes back to work on capacity-achieving linear codes [37, 41, 26–28]. The SPIV algorithm from [9] combines a test design inspired by spatial coupling with a combinatorial inference algorithm. A novelty of the present work is that we replace this elementary algorithm with a novel variant of the Belief Propagation (BP) message passing algorithm [30, 31] that lends itself to a rigorous analysis.

The standard variant of BP has been studied for variants of group testing [17, 36, 43]. Additionally there have been empirical studies on BP for group testing [11, 40].

1.7. Organisation

After introducing a bit of notation and recalling some background in Section 1.8, we give an outline of the proofs of the main results in Section 2. Subsequently, Section 3 deals with the details of the proof of Theorem 1.1. Moreover, Section 4 deals with the proof of Theorem 1.3, while in Section 5, we prove Theorem 1.4. The proof of Theorem 1.2, which is quite short and uses arguments that are well established in the literature, follows in Appendix B. Appendix C contains the proof of a routine expansion property of the test design $\mathbf{G}_{\mathrm{sc}}$. Finally, in Appendices D and E, we investigate the optimisation problems (1.9)–(1.11) on the binary symmetric channel and the Z-channel, and in Appendix F, we compare our result to the recent result of Chen and Scarlett [7]. A table of used notations can be found in Appendix A.

1.8. Preliminaries

As explained in Section 1.1, a test design is a bipartite graph $G = (V \cup F, E)$ whose vertex set consists of a set V individuals and a set F of tests. The ground truth, that is, the set of infected individuals, is encoded as a vector $\boldsymbol{\sigma} \in \{0, 1\}^V$ of Hamming weight k. Since we will deal with randomised test designs, we may assume that $\boldsymbol{\sigma}$ is a *uniformly random* vector of Hamming weight k (by shuffling the set of individuals). Also recall that for a test a, we let $\boldsymbol{\sigma}'_a \in \{0, 1\}$ denote the *actual* result of test a (equal to one if and only if a contains an infected individual), while $\boldsymbol{\sigma}''_a \in \{0, 1\}$ signifies the *displayed* test result obtained via (1.2). It is convenient to introduce the shorthands

$$V_0 = \{x \in V : \boldsymbol{\sigma}_x = 0\}, \quad V_1 = \{x \in V : \boldsymbol{\sigma}_x = 1\}, \quad F_0 = \left\{a \in F : \boldsymbol{\sigma}'_a = 0\right\}, \quad F_1 = \left\{a \in F : \boldsymbol{\sigma}'_a = 1\right\},$$
$$F^- = \left\{a \in F : \boldsymbol{\sigma}''_a = 0\right\}, \quad F^+ = \left\{a \in F : \boldsymbol{\sigma}''_a = 1\right\}, \quad F_0^\pm = F^\pm \cap F_0, \qquad F_1^\pm = F^\pm \cap F_1$$

for the set of infected/healthy individuals, the set of actually negative/positive tests, the set of negatively/positively displayed tests, and the tests that are actually negative/positive and display a positive/negative result, respectively. For each node u of G, we denote $\partial u = \partial_G u$ be the set of neighbours of u. For an individual $x \in V$, we also let $\partial^\pm x = \partial_G^\pm x = \partial_G x \cap F^\pm$ be the set of positively/negatively displayed tests that contain x.

We need Chernoff bounds for the binomial and the hypergeometric distribution. Recall that the *hypergeometric distribution* $\mathrm{Hyp}(L, M, N)$ is defined by

$$\mathbb{P}\left[\mathrm{Hyp}(L, M, N) = k\right] = \binom{M}{k}\binom{L-M}{N-k}\binom{L}{N}^{-1} \qquad (k \in \{0, 1, \ldots, \min\{M, N\}\}). \tag{1.19}$$

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(Out of a total of L items of which M are special, we draw N items without replacement and count the number of special items in the draw.) The mean of the hypergeometric distribution equals MN/L.

Lemma 1.5 ([23, Equation 2.4]). *Let* $\boldsymbol{X}$ *be a binomial random variable with parameters* N, p. *Then*

$$\mathbb{P}\left[\boldsymbol{X} \geq qN\right] \leq \exp\left(-ND_{\mathrm{KL}}\left(q \| p\right)\right) \quad \textit{for } p < q < 1, \tag{1.20}$$

$$\mathbb{P}\left[\boldsymbol{X} \leq qN\right] \leq \exp\left(-ND_{\mathrm{KL}}\left(q \| p\right)\right) \quad \textit{for } 0 < q < p. \tag{1.21}$$

Lemma 1.6 ([21]). *For a hypergeometric variable* $\boldsymbol{X} \sim Hyp(L, M, N)$, *the bounds (1.20)–(1.21) hold with* $p = M/L$.

Throughout, we use asymptotic notation $o(\,\cdot\,), \omega(\,\cdot\,), O(\,\cdot\,), \Omega(\,\cdot\,), \Theta(\,\cdot\,)$ to refer to limit $n \to \infty$. It is understood that the constants hidden in, for example, a $O(\,\cdot\,)$-term may depend on $\theta, \boldsymbol{p}$ or other parameters and that a $O(\,\cdot\,)$-term may have a positive or a negative sign. To avoid case distinctions, we sometimes take the liberty of calculating with the values $\pm\infty$. The usual conventions $\infty + \infty = \infty \cdot \infty = \infty$ and $0 \cdot \infty = 0$ apply. Furthermore, we set $\tanh(\pm\infty) = \pm 1$. Also recall that $0 \ln 0 = 0 \ln \frac{0}{0} = 0$. Additionally, $\ln 0 = -\infty$ and $\frac{1}{0} = \ln \frac{1}{0} = \infty$.

Finally, for two random variables $\boldsymbol{X}, \boldsymbol{Y}$ defined on the same finite probability space $(\Omega, \mathbb{P}[\cdot])$, we write

$$I(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{\omega, \omega' \in \Omega} \mathbb{P}\left[\boldsymbol{X} = \omega,\ \boldsymbol{Y} = \omega'\right] \ln \frac{\mathbb{P}\left[\boldsymbol{X} = \omega,\ \boldsymbol{Y} = \omega'\right]}{\mathbb{P}\left[\boldsymbol{X} = \omega\right] \mathbb{P}\left[\boldsymbol{Y} = \omega'\right]}$$

for the *mutual information* of $\boldsymbol{X}, \boldsymbol{Y}$. We recall that $I(\boldsymbol{X}, \boldsymbol{Y}) \geq 0$.

2. Overview

We proceed to survey the functioning of the algorithms SPARC and SPEX. To get started, we briefly discuss the best previously known algorithm for noisy group testing, the DD algorithm from [19], which operates on the constant column design. We will discover that DD can be viewed as a truncated version of the BP message passing algorithm. BP is a generic heuristic for inference problems backed by physics intuition [30, 42]. Yet unfortunately, BP is notoriously difficult to analyse. Even worse, it seems unlikely that full BP will significantly outperform DD on the constant column design; evidence of this was provided in [10] in the noiseless case. The basic issue is the lack of a good initialisation of the BP messages.

To remedy this issue, we resort to the spatially coupled test design $\mathbf{G}_{\mathrm{sc}}$, which combines a randomised and a spatial construction. The basic idea resembles domino toppling. Starting from an easy-to-diagnose 'seed', the algorithm works its way forward in a well-defined direction until all individuals have been diagnosed. Spatial coupling has proved useful in related inference problems, including noiseless group testing [9]. Therefore, a natural stab at group testing would be to run BP on $\mathbf{G}_{\mathrm{sc}}$. Indeed, the BP intuition provides a key ingredient to the SPARC algorithm, namely, the update equations (see [2.30] below), of which the correct choice of the weights (Eq. [2.29] below) is the most important ingredient. But in light of the difficulty of analysing textbook BP, SPARC relies on a modified version of BP that better lends itself to a rigorous analysis. Furthermore, the SPEX algorithm for exact recovery combines SPARC with a cleanup step.

SPARC and SPEX can be viewed as generalised versions of the noiseless group testing algorithm called SPIV from [9]. However, [9] did not exploit the connection with BP. Instead, in the noiseless case, the correct weights were simply 'guessed' based on combinatorial intuition, an approach that it seems difficult to generalise. Hence, the present, systematic derivation of the weights (2.29) also casts new light on the noiseless case. In fact, we expect that the paradigm behind SPARC

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and SPEX, namely, to use BP heuristically to find the correct parameters for a simplified message passing algorithm, potentially generalises to other inference problems as well.

2.1. The DD algorithm

The noisy DD algorithm is the best previously known efficient algorithm for exact recovery [19]. SPEX uses DD as a subroutine for diagnosing a small fraction of individuals. Later, we compare the number of tests needed by SPEX with the one needed by DD. The DD algorithm from [19] utilises the constant column design $\mathbf{G}_{\mathrm{cc}}$.Thus, each individual independently joins an equal number Δ of random tests. Given the displayed test results, DD first declares certain individuals as uninfected by thresholding the number of negatively displayed tests. More precisely, DD declares as uninfected any individual that appears in at least $\alpha\Delta$ negatively displayed tests, with α a diligently chosen threshold. Having identified the respective individuals as uninfected, DD looks out for tests a that display a positive result and that only contain a single individual x that has not been identified as uninfected yet. Since such tests a hint at x being infected, in its second step, DD declares as infected any individual x that appears in at least $\beta\Delta$ positively displayed tests a where all other individuals $y \in \partial a \setminus x$ were declared uninfected by the first step. Once again, β is a carefully chosen threshold. Finally, DD declares as uninfected all remaining individuals.

The DD algorithm exactly recovers the infected set w.h.p. provided the total number m of tests is sufficiently large such that the aforementioned thresholds α, β exist. The required number of tests, which comes out in terms of a mildly delicate optimisation problem, was determined in [19]. Let

$$q_0^- = \exp(-d)p_{00} + (1 - \exp(-d))p_{10}, \qquad q_0^+ = \exp(-d)p_{01} + (1 - \exp(-d))p_{11}. \tag{2.1}$$

Theorem 2.1 ([19, Theorem 2.2]). *Let $\varepsilon > 0$ and with $\alpha \in (p_{10}, q_0^-)$ and $\beta \in (0, \exp(-d)p_{11})$, let*

$$c_{\mathrm{DD}} = \min_{\alpha,\beta,d} \max\left\{c_{\mathrm{DD},1}(\alpha, d), c_{\mathrm{DD},2}(\alpha, d), c_{\mathrm{DD},3}(\beta, d), c_{\mathrm{DD},4}(\alpha, \beta, d)\right\}, \qquad \text{where}$$

$$c_{\mathrm{DD},1}(\alpha, d) = \frac{\theta}{(1-\theta)D_{\mathrm{KL}}\left(\alpha\|p_{10}\right)}, \qquad c_{\mathrm{DD},2}(\alpha, d) = \frac{1}{dD_{\mathrm{KL}}\left(\alpha\|q_0^-\right)},$$

$$c_{\mathrm{DD},3}(\beta, d) = \frac{\theta}{d(1-\theta)D_{\mathrm{KL}}\left(\beta\|p_{11}\exp(-d)\right)},$$

$$c_{\mathrm{DD},4}(\alpha, \beta, d) = \max_{(1-\alpha)\vee\beta\le z\le 1}\left(d(1-\theta)\left(D_{\mathrm{KL}}\left(z\|q_0^+\right)\right.\right.$$
$$\left.\left. + \mathbb{1}\left\{\beta > \frac{z\exp(-d)p_{01}}{q_0^+}\right\} zD_{\mathrm{KL}}\left(\beta/z\|\exp(-d)p_{01}/q_0^+\right)\right)\right)^{-1}.$$

If $m \ge (1+\varepsilon)c_{\mathrm{DD}}k\ln(n/k)$, then there exists $\Delta > 0$ and $0 \le \alpha, \beta \le 1$ such that the DD algorithm outputs $\boldsymbol{\sigma}$ w.h.p.

The distinct feature of DD is its simplicity. However, the thresholding that DD applies does seem to leave something on the table. For a start, whether DD identifies a certain individual x as infected depends only on the results of tests that have a distance of at most three from x in the graph $\mathbf{G}_{\mathrm{cc}}$. Moreover, it seems wasteful that DD takes only those positively displayed tests into consideration where all but one individual were already identified as uninfected.

2.2. Belief propagation

BP is a message passing algorithm that is expected to overcome these deficiencies. In fact, heuristic arguments suggest that BP might be the ultimate recovery algorithm for a wide class of inference algorithms on random graphs [42]. That said, rigorous analyses of BP are few and far between.

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Following the general framework from [30], in order to apply BP to a group testing design $G=(V\cup F,E)$, we equip each test $a\in F$ with a weight function

$$\psi_a=\psi_{G,\boldsymbol{\sigma}'',a}:\{0,1\}^{\partial a}\to\mathbb{R}_{\geq 0},\quad \sigma_{\partial a_i}\mapsto\begin{cases}\mathbb{1}\{\|\sigma\|_1=0\}p_{00}+\mathbb{1}\{\|\sigma\|_1>0\}p_{10} & \text{if } a\in F^-\\ \mathbb{1}\{\|\sigma\|_1=0\}p_{01}+\mathbb{1}\{\|\sigma\|_1>0\}p_{11} & \text{if } a\in F^+.\end{cases} \tag{2.2}$$

Thus, ψ_a takes as argument a $\{0,1\}$-vector $\sigma=(\sigma_x)_{x\in\partial a}$ indexed by the individuals that take part in test a. The weight $\psi_a(\sigma)$ equals the probability of observing the result that a displays if the infection status were σ_x for every individual $x\in\partial a$. In other words, ψ_a encodes the posterior under the $\boldsymbol{p}$-channel. Given $G,\boldsymbol{\sigma}''$, the weight functions give rise to the total weight of $\sigma\in\{0,1\}^V$ by letting

$$\psi(\sigma)=\psi_{G,\boldsymbol{\sigma}''}(\sigma)=\mathbb{1}\left\{\|\sigma\|_1=k\right\}\prod_{a\in F}\psi_a(\sigma_{\partial a}). \tag{2.3}$$

Thus, we just multiply up the contributions (2.2) of the various tests and add in the prior assumption that precisely k individuals are infected. The total weight (2.3) induces a probability distribution

$$\mu_{G,\boldsymbol{\sigma}''}(\sigma)=\psi_{G,\boldsymbol{\sigma}''}(\sigma)/Z_{G,\boldsymbol{\sigma}''},\qquad \text{where}\qquad Z_{G,\boldsymbol{\sigma}''}=\sum_{\sigma\in\{0,1\}^V}\psi_{G,\boldsymbol{\sigma}''}(\sigma). \tag{2.4}$$

A simple application of Bayes' rule shows that μ_G matches the posterior of the ground truth $\boldsymbol{\sigma}$ given the test results.

Fact 2.2. *For any test design G, we have* $\mu_{G,\boldsymbol{\sigma}''}(\sigma)=\mathbb{P}\left[\boldsymbol{\sigma}=\sigma\mid\boldsymbol{\sigma}''\right]$. □

BP is a heuristic to calculate the marginals of $\mu_{G,\boldsymbol{\sigma}''}$ or, in light of Fact 2.2, the posterior probabilities $\mathbb{P}[\boldsymbol{\sigma}_{x_i}=1\mid\boldsymbol{\sigma}'']$. To this end, BP associates *messages* with the edges of G. Specifically, for any adjacent individual/test pair x,a, there is a message $\mu_{x\to a,t}(\,\cdot\,)$ from x to a and another one $\mu_{a\to x,t}(\,\cdot\,)$ in the reverse direction. The messages are updated in rounds and therefore come with a time parameter $t\in\mathbb{Z}_{\geq 0}$. Moreover, being probability distributions on $\{0,1\}$, the messages always satisfy

$$\mu_{x\to a,t}(0)+\mu_{x\to a,t}(1)=\mu_{a\to x,t}(0)+\mu_{a\to x,t}(1)=1. \tag{2.5}$$

The intended semantics is that $\mu_{x\to a,t}(1)$ estimates the probability that x is infected given all known information except the result of test a. Analogously, $\mu_{a\to x,t}(1)$ estimates the probability that x is infected if we disregard all other tests $b\in\partial x\setminus\{a\}$.

This slightly convoluted interpretation of the messages facilitates simple heuristic formulas for computing the messages iteratively. To elaborate, in the absence of a better a priori estimate, at time $t=0$, we simply initialise in accordance with the prior, that is,

$$\mu_{x\to a,0}(0)=1-k/n \qquad\qquad \mu_{x\to a,0}(1)=k/n. \tag{2.6}$$

Subsequently, we use the weight function (2.2) to update the messages: inductively for $t\geq 1$ and $r\in\{0,1\}$, let

$$\mu_{a\to x,t}(r)\propto\begin{cases}p_{11} & \text{if } r=1,\ a\in F^+,\\ p_{11}+(p_{01}-p_{11})\prod_{y\in\partial a\setminus\{x\}}\mu_{y\to a,t-1}(0) & \text{if } r=0,\ a\in F^+,\\ p_{10} & \text{if } r=1,\ a\in F^-,\\ p_{10}+(p_{00}-p_{10})\prod_{y\in\partial a\setminus\{x\}}\mu_{y\to a,t-1}(0) & \text{if } r=0,\ a\in F^-,\end{cases} \tag{2.7}$$

$$\mu_{x\to a,t}(r)\propto\left(\frac{k}{n}\right)^r\left(1-\frac{k}{n}\right)^{1-r}\prod_{b\in\partial x\setminus\{a\}}\mu_{b\to x,t-1}(r). \tag{2.8}$$

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The $\propto$-symbol hides the necessary normalisation to ensure that the messages satisfy (2.5). Furthermore, the $(k/n)^r$ and $(1-k/n)^{1-r}$-prefactors in (2.8) encode the prior that precisely k individuals are infected. The expressions (2.7)–(2.8) are motivated by the hunch that for most tests a, the values $(\boldsymbol{\sigma}_y)_{y\in\partial a}$ should be stochastically dependent primarily through their joint membership in test a. An excellent exposition of BP can be found in [30].

How do we utilise the BP messages to infer the actual infection status of each individual? The idea is to perform the update (2.7)–(2.8) for a 'sufficiently large' number of rounds, say, until an approximate fixed point is reached. The (heuristic) BP estimate of the posterior marginals after t rounds then reads

$$\mu_{x,t}(r) \propto \left(\frac{k}{n}\right)^r \left(1-\frac{k}{n}\right)^{1-r} \prod_{b\in\partial x} \mu_{b\to x,t-1}(r) \qquad (r\in\{0,1\}). \tag{2.9}$$

Thus, by comparison to (2.8), we just take the incoming messages from *all* tests $b\in\partial x$ into account. In summary, we 'hope' that after sufficiently many updates, we have $\mu_{x,t}(r)\approx\mathbb{P}[\boldsymbol{\sigma}_x = r\mid\boldsymbol{\sigma}'']$. We could then, for instance, declare the k individuals with the greatest $\mu_{x,t}(1)$ infected and everybody else uninfected.

For later reference, we point out that the BP updates (2.7)–(2.8) and (2.9) can be simplified slightly by passing to log-likelihood ratios. Thus, define

$$\eta_{x\to a,t} = \ln\frac{\mu_{G,\boldsymbol{\sigma}'',x\to a,t}(1)}{\mu_{G,\boldsymbol{\sigma}'',x\to a,t}(0)}, \qquad \eta_{G,\boldsymbol{\sigma}'',a\to x,t} = \ln\frac{\mu_{G,\boldsymbol{\sigma}'',a\to x,t}(1)}{\mu_{G,\boldsymbol{\sigma}'',a\to x,t}(0)}, \tag{2.10}$$

with the initialisation $\eta_{G,\boldsymbol{\sigma}'',x\to a,0} = \ln(k/(n-k)) \sim (\theta-1)\ln n$ from (2.6). Then (2.7)–(2.9) transform into

$$\eta_{a\to x,t} = \begin{cases} \ln p_{11} - \ln\left[p_{11} + (p_{01}-p_{11})\prod_{y\in\partial a\setminus\{x\}}\frac{1}{2}\left(1-\tanh\left(\frac{1}{2}\eta_{y\to a,t-1}\right)\right)\right] & \text{if } a\in F^+,\\ \ln p_{10} - \ln\left[p_{10} + (p_{00}-p_{10})\prod_{y\in\partial a\setminus\{x\}}\frac{1}{2}\left(1-\tanh\left(\frac{1}{2}\eta_{y\to a,t-1}\right)\right)\right] & \text{if } a\in F^-, \end{cases} \tag{2.11}$$

$$\eta_{x\to a,t} = (\theta-1)\ln n + \sum_{b\in\partial x\setminus\{a\}} \eta_{b\to x,t}, \qquad \eta_{x,t} = (\theta-1)\ln n + \sum_{b\in\partial x}\eta_{b\to x,t}. \tag{2.12}$$

In this formulation, BP ultimately diagnoses the k individuals with the largest $\eta_{x,t}$ as infected.

Under the assumptions of Theorem 2.1, the DD algorithm can be viewed as the special case of BP with $t=2$ applied to $\mathbf{G}_{cc}$. Indeed, the analysis of DD evinces that on $\mathbf{G}_{cc}$, the largest k BP estimates (2.9) with $t\geq 2$ correctly identify the infected individuals w.h.p.[3]

It is therefore an obvious question whether BP on the constant column design fits the bill of Theorem 1.1. Clearly, BP remedies the obvious deficiencies of DD by taking into account information from a larger radius around an individual (if we iterate beyond $t=2$). Also in contrast to DD's hard thresholding, the update rules (2.7)–(2.8) take information into account in a more subtle, soft manner. Nonetheless, we do not expect that BP applied to the constant column design meets the information-theoretically optimal bound from Theorem 1.1. In fact, there is strong evidence that BP does not suffice to meet the information-threshold for all θ even in the noiseless case [10]. The fundamental obstacle appears to be the 'cold' initialisation (2.6), which (depending on the parameters) can cause the BP messages to approach a meaningless fixed point. Yet for symmetry reasons on the constant column design, no better starting point than the prior (2.6) springs to mind; after all, $\mathbf{G}_{cc}$ is a nearly biregular random graph, and thus, all individuals look alike. To

[3]In the noiseless case, DD is actually a special case of a discrete message passing algorithm called Warning Propagation [30].

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overcome this issue, we will employ a different type of test design that enables a warm start for BP. This technique goes by the name of spatial coupling.

2.3. Spatial coupling

The thrust of spatial coupling is to blend a randomised construction, in our case, the constant column design, with a spatial arrangement so as to provide a propitious starting point for BP. Originally hailing from coding theory, spatial coupling has also been used in previous work on noiseless group testing [9]. In fact, the construction that we use is essentially identical to that from [9] (with suitably adapted parameters).

To set up the spatially coupled test design $\mathbf{G}_{\mathrm{sc}}$, we divide the set $V = V_n = \{x_1, \ldots, x_n\}$ of individuals into

$$\ell = \lceil \sqrt{\ln n} \rceil \tag{2.13}$$

pairwise disjoint *compartments* $V[1], \ldots, V[\ell] \subseteq V$ such that $\lfloor n/\ell \rfloor \leq |V[i]| \leq \lceil n/\ell \rceil$. We think of these compartments as being spatially arranged so that $V[i+1]$ comes 'to the right' of $V[i]$ for $1 \leq i < \ell$. More precisely, we arrange the compartments in a ring topology such that $V[\ell]$ is followed again by $V[1]$. Hence, for notational convenience, let $V[\ell + j] = V[j]$ and $V[1 - j] = V[\ell - j + 1]$ for $1 \leq j \leq \ell$. Additionally, we introduce ℓ compartments $F[1], \ldots, F[\ell]$ of tests arranged in the same way: think of $F[i]$ as sitting 'above' $V[i]$. We assume that the total number m of tests in $F[1] \cup \ldots \cup F[\ell]$ is divisible by ℓ and satisfies $m = \Theta(k \ln (n/k))$. Hence, let each compartment $F[i]$ contain precisely m/ℓ tests. As in the case of the individuals, we let $F[\ell + j] = F[j]$ for $0 < j \leq \ell$. Additionally, let

$$s = \lceil \ln \ln n \rceil \quad \text{and} \quad \Delta = \Theta(\ln n) \tag{2.14}$$

be integers such that Δ is divisible by s. Construct $\mathbf{G}_{\mathrm{sc}}$ by letting each $x \in V[i]$ join precisely Δ/s tests from $F[i + j - 1]$ for $j = 1, \ldots, s$. These tests are chosen uniformly without replacement and independently for different x and j. Additionally, $\mathbf{G}_{\mathrm{sc}}$ contains a compartment $F[0]$ of

$$m_0 = 2c_{\mathrm{DD}} \frac{ks}{\ell} \ln (n/k) = o(k \ln (n/k)) \tag{2.15}$$

tests. Every individual x from the first s compartments $V[1], \ldots, V[s]$ joins an equal number Δ_0 of tests from $F[0]$. These tests are drawn uniformly without replacement and independently. For future reference, we let

$$c = c_n = \frac{m}{k \ln (n/k)}, \qquad d = d_n = \frac{k\Delta}{m}; \tag{2.16}$$

the aforementioned assumptions on m, Δ ensure that $c, d = \Theta(1)$ and the total number of tests of $\mathbf{G}_{\mathrm{sc}}$ comes to

$$\sum_{i=0}^{\ell} |F[i]| = (c + o(1))k \ln (n/k). \tag{2.17}$$

In summary, $\mathbf{G}_{\mathrm{sc}}$ consists of ℓ equally sized compartments $V[i]$, $F[i]$ of tests plus one extra serving $F[0]$ of tests. Each individual $V[i]$ joins random tests in the s consecutive compartments $F[i + j - 1]$ with $1 \leq j \leq s$. Additionally, the individuals in the first s compartments $V[1] \cup \cdots \cup V[s]$, which we refer to as the *seed*, also join the tests in $F[0]$.

We will discover momentarily how $\mathbf{G}_{\mathrm{sc}}$ facilitates inference via BP. But first let us make a note of some basic properties of $\mathbf{G}_{\mathrm{sc}}$. Recall that $\boldsymbol{\sigma} \in \{0, 1\}^V$, which encodes the true infection status of each individual, is chosen uniformly and independently of $\mathbf{G}_{\mathrm{sc}}$ from all vectors of Hamming weight k. Let $V_1 = \{x \in V : \boldsymbol{\sigma}_x = 1\}$, $V_0 = V \setminus V_1$, and let $V_r[i] = V_r \cap V[i]$ be the set of individuals with infection status $r \in \{0, 1\}$ in compartment i. Furthermore, recall that $\boldsymbol{\sigma}'_a \in \{0, 1\}$ denotes the actual result of test $a \in F = F[0] \cup \cdots \cup F[\ell]$ and that $\boldsymbol{\sigma}''_a$ signifies the displayed result of a as per (1.2). For $r \in \{0, 1\}$ and $0 \le i \le \ell$, let

$$F_r[i] = F_r \cap F[i], \qquad F_r^+[i] = F_r[i] \cap F^+, \qquad F_r^-[i] = F_r[i] \cap F^-.$$

Thus, the subscript indicates the actual test result, while the superscript indicates the displayed result. In Section 3.1, we will prove the following.

Proposition 2.3. *The test design* $\mathbf{G}_{\mathrm{sc}}$ *enjoys the following properties with probability* $1 - o(n^{-2})$.

G1 The number of infected individuals in the various compartments satisfies

$$\frac{k}{\ell} - \sqrt{\frac{k}{\ell}} \ln n \le \min_{i \in [\ell]} |V_1[i]| \le \max_{i \in [\ell]} |V_1[i]| \le \frac{k}{\ell} + \sqrt{\frac{k}{\ell}} \ln n. \tag{2.18}$$

G2 For all $i \in [\ell]$, *the numbers of tests that are actually/displayed positive/negative satisfy*

$$\frac{m}{\ell} \exp(-d) p_{00} - \sqrt{m} \ln^3 n \le \left|F_0^-[i]\right| \le \frac{m}{\ell} \exp(-d) p_{00} + \sqrt{m} \ln^3 n, \tag{2.19}$$

$$\frac{m}{\ell} \exp(-d) p_{01} - \sqrt{m} \ln^3 n \le \left|F_0^+[i]\right| \le \frac{m}{\ell} \exp(-d) p_{01} + \sqrt{m} \ln^3 n, \qquad \text{comment} \tag{2.20}$$

$$\frac{m}{\ell}(1 - \exp(-d)) p_{10} - \sqrt{m} \ln^3 n \le \left|F_1^-[i]\right| \le \frac{m}{\ell}(1 - \exp(-d)) p_{10} + \sqrt{m} \ln^3 n, \tag{2.21}$$

$$\frac{m}{\ell}(1 - \exp(-d)) p_{11} - \sqrt{m} \ln^3 n \le \left|F_1^+[i]\right| \le \frac{m}{\ell}(1 - \exp(-d)) p_{11} + \sqrt{m} \ln^3 n. \tag{2.22}$$

2.4. Approximate recovery

We are going to exploit the spatial structure of $\mathbf{G}_{\mathrm{sc}}$ in a manner reminiscent of domino toppling. To get started, we will run DD on the seed $V[1] \cup \cdots \cup V[s]$ and the tests $F[0]$ only; this is our first domino. The choice (2.15) of m_0 ensures that DD diagnoses all individuals in $V[1] \cup \cdots \cup V[s]$ correctly w.h.p. The seed could then be used as an informed starting point from which we could iterate BP to infer the status of the individuals in $V[s + 1] \cup \ldots \cup V[\ell]$. However, this algorithm appears to be difficult to analyse. Instead, we will show under that the assumptions of Theorem 1.1 a modified, 'paced' version of BP that diagnoses one compartment (or 'domino') at a time and then re-initialises the messages ultimately classifies all but $o(k)$ individuals correctly. Let us flesh this strategy out in detail.

2.4.1. The seed

Recall that each individual $x \in V[1] \cup \cdots \cup V[s]$ independently joins Δ_0 random tests from $F[0]$. In the initial step, SPARC runs DD on the test design $\boldsymbol{G}_0$ comprising $V[1] \cup \cdots \cup V[s]$ and the tests $F[0]$ only. Throughout, SPARC maintains a vector $\tau \in \{0, 1, *\}^V$ that represents the algorithm's current estimate of the ground truth $\boldsymbol{\sigma}$, with $*$ indicating 'undetermined as yet'.

Since Proposition 2.3 shows that the seed contains $(1 + o(1))ks/\ell$ infected individuals w.h.p., the choice (2.15) of m_0 and Theorem 2.1 imply that DD will succeed to diagnose the seed correctly for a suitable Δ_0.

Proposition 2.4 ([19, Theorem 2.2]). *There exists* $\Delta_0 = \Theta(\ln n)$ *such that the output of DD satisfies* $\tau_x = \boldsymbol{\sigma}_x$ *for all* $x \in V[1] \cup \cdots \cup V[s]$ *w.h.p.*

2.4.2. A combinatorial condition

To simplify the analysis of the message passing step in Section 2.4.3, we observe that certain individuals can be identified as likely uninfected purely on combinatorial grounds. More precisely,

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Algorithm 1 SPARC, steps 1–2

Input: $\boldsymbol{G}, \boldsymbol{\sigma}$
Output: an estimate of $\boldsymbol{\sigma}$
1 Let $(\tau_x)_{x \in V[1]\cup\cdots\cup V[s]} \in \{0,1\}^{V[1]\cup\cdots\cup V[s]}$ be the result of applying DD to $V[1]\cup\cdots\cup V[s]$ and $F[0]$;
2 Set $\tau_x = *$ for all individuals $x \in V \setminus (V[1]\cup\cdots\cup V[s])$;

consider $x \in V[i]$ for $s < i \le \ell$. If x is infected, then any test $a \in \partial x$ is actually positive. Hence, we expect that x appears in about $p_{11}\Delta$ tests that display a positive result. In fact, the choice (2.14) of s ensures that w.h.p. even within each separate compartment $F[i+j-1]$, $1 \le j \le s$ the individual x appears in about $p_{11}\Delta/s$ positively displayed tests. Thus, let

$$V^+[i] = \left\{ x \in V[i] : \sum_{j=1}^{s} \left| |\partial x \cap F^+[i+j-1]| - \Delta p_{11}/s \right| \le \ln^{4/7} n \right\}. \tag{2.23}$$

The following proposition confirms that all but $o(k/s)$ infected individuals $x \in V[i]$ belong to $V^+[i]$. Additionally, the proposition determines the approximate size of $V^+[i]$. For notational convenience, we define

$$V_0^+[i] = V^+[i] \cap V_0, \qquad V_1^+[i] = V^+[i] \cap V_1, \qquad V^+ = \bigcup_{s<i\le\ell} V^+[i].$$

Recall q_0^+ from (2.1). The proof of the following proposition can be found in Section 3.2.

Proposition 2.5. *W.h.p., we have*

$$\sum_{i=s+1}^{\ell} |V_1[i] \setminus V^+[i]| \le k \exp\left(-\Omega(\ln^{1/7} n)\right) \quad \text{and} \tag{2.24}$$

$$\left|V^+[i] \setminus V_1[i]\right| \le \frac{n}{\ell} \exp\left(-\Delta D_{\mathrm{KL}}\left(p_{11} \| q_0^+\right) + O(\ln^{4/7} n)\right) \quad \text{for all } s+1 \le i \le \ell. \tag{2.25}$$

2.4.3. Belief propagation redux

The main phase of the SPARC algorithm employs a simplified version of the BP update rules (2.7)–(2.8) to diagnose one compartment $V[i]$, $s < i \le \ell$, after the other. The textbook way to employ BP would be to diagnose the seed, initialise the BP messages emanating from the seed accordingly, and then run BP updates until the messages converge. However, towards the proof of Theorem 1.1, this way of applying BP seems too complicated to analyse. Instead, SPARC relies on a 'paced' version of BP. Rather than updating the messages to convergence from the seed, we perform one round of message updates, then diagnose the next compartment, re-initialise the messages to coincide with the newly diagnosed compartment, and proceed to the next undiagnosed compartment.

We work with the log-likelihood versions of the BP messages from (2.10) to (2.12). Suppose we aim to process compartment $V[i]$, $s < i \le \ell$, having completed $V[1], \ldots, V[i-1]$ already. Then for a test $a \in F[i+j-1]$, $j \in [s]$, and an adjacent variable $x \in V[i+j-s] \cup V[i+j-1]$, we initialise

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$$\eta_{x\to a,0} = \begin{cases} -\infty & \text{if } \tau_x = 0, \\ +\infty & \text{if } \tau_x = 1, \\ \ln(k/(n-k)) & \text{if } \tau_x = *. \end{cases} \tag{2.26}$$

The third case above occurs if and only if $x \in V[i] \cup \cdots \cup V[i+j-1]$, that is, if x belongs to an as yet undiagnosed compartment. For the compartments that have been diagnosed already, we set $\eta_{x\to a,0}$ to $\pm\infty$, depending on whether x has been classified as infected or uninfected.

Let us now investigate the ensuing messages $\eta_{a\to x,1}$ for $x \in V[i]$ and tests $a \in \partial x \cap F[i+j-1]$. A glimpse at (2.11) reveals that for any test a that contains an individual $y \in V[i+j-s] \cup \cdots \cup V[i-1]$ with $\tau_y = 1$, we have $\eta_{a\to x,1} = 0$. This is because (2.26) ensures that $\eta_{y\to a,0} = \infty$ and $\tanh(\infty) = 1$. Hence, the test a contains no further information as to the status of x. Therefore, we call a test $a \in F[i+j-1]$ *informative* towards $V[i]$ if $\tau_y = 0$ for all $y \in \partial a \cap (V[i+j-s] \cup \cdots \cup V[i-1])$.

Let $\boldsymbol{W}_{i,j}(\tau)$ be the set of all informative $a \in F[i+j-1]$. Then any $a \in \boldsymbol{W}_{i,j}(\tau)$ receives $\eta_{y\to a,0} = -\infty$ from all individuals y that have been diagnosed already, that is, all $y \in V[h]$ with $i+j-s \leq h < i$. Another glance at the update rule shows that the corresponding terms simply disappear from the product on the r.h.s. of (2.11) because $\tanh(-\infty) = -1$. Consequently, only the factors corresponding to undiagnosed individuals $y \in V[i] \cup \cdots \cup V[i+j-1]$ remain. Hence, with $r = \mathbb{1}\{a \in F^+\}$, the update rule (2.11) simplifies to

$$\eta_{a\to x,1} = \ln p_{1r} - \ln\left[p_{1r} + (p_{0r} - p_{1r})\left(1 - k/n\right)^{|\partial a \cap (V[i]\cup\cdots\cup V[i+j-1])|-1}\right]. \tag{2.27}$$

The only random element in the expression (2.27) is the number $|\partial a \cap (V[i] \cup \cdots \cup V[i+j-1])|$ of members of test a from compartments $V[i] \cup \cdots \cup V[i+j-1]$. But by the construction of $\mathbf{G}_{\mathrm{sc}}$, this number has a binomial distribution with mean

$$\mathbb{E}\left|\partial a \cap (V[i] \cup \cdots \cup V[i+j-1])\right| = \frac{j\Delta n}{ms} + o(1) = \frac{djn}{ks} + o(1) \qquad \text{[using (2.16)]}.$$

Since the fluctuations of $|\partial a \cap (V[i] \cup \cdots \cup V[i+j-1])|$ are of smaller order than the mean, we conclude that w.h.p., (2.27) can be well approximated by a deterministic quantity:

$$\eta_{a\to x,1} = \begin{cases} w_j^+ + o(1), & \text{if } a \in F^+, \\ -w_j^- + o(1), & \text{if } a \in F^- \end{cases}, \qquad \text{where} \tag{2.28}$$

$$w_j^+ = \ln \frac{p_{11}}{p_{11} + (p_{01} - p_{11})\exp(-dj/s)} \geq 0, \qquad w_j^- = -\ln \frac{p_{10}}{p_{10} + (p_{00} - p_{10})\exp(-dj/s)} \geq 0. \tag{2.29}$$

Note that in the case $p_{10} = 0$, the negative test weight W_j^- evaluates to $w_j^- = \infty$, indicating that individual contained in negative test definitely are uninfected. Finally, the messages (2.28) lead to the BP estimate of the posterior marginal of x via (2.12), that is, by summing on all informative tests $a \in \partial x$. To be precise, letting

$$\boldsymbol{W}_{x,j}^{\pm}(\tau) = \partial x \cap \boldsymbol{W}_{i,j}(\tau) \cap F^{\pm}$$

be the positively/negatively displayed informative tests adjacent to x and setting

$$\boldsymbol{W}_x^+(\tau) = \sum_{j=1}^{s} w_j^+ \left|\boldsymbol{W}_{x,j}^+(\tau)\right|, \qquad \boldsymbol{W}_x^-(\tau) = \sum_{j=1}^{s} w_j^- \left|\boldsymbol{W}_{x,j}^-(\tau)\right|, \tag{2.30}$$

we obtain

$$\eta_{x,1} = \boldsymbol{W}_x^+(\tau) - \boldsymbol{W}_x^-(\tau) + \text{'lower order fluctuations'}. \tag{2.31}$$

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Algorithm 2 SPARC, steps 3–9.

for $i = s+1, \ldots, \ell$ **do**
  for $x \in V[i]$ **do**
    if $x \notin V^+[i]$ *or* $\boldsymbol{W}_x^+(\tau) < (1-\zeta)W^+$ *or* $\boldsymbol{W}_x^-(\tau) > (1+\zeta)W^-$ **then**
      $\tau_x = 0$ `// classify as uninfected`
    else
      $\tau_x = 1$ `// classify as infected`
return τ

One issue with the formula (2.28) is the analysis of the 'lower order fluctuations', which come from the random variables $|\partial a \cap (V^+[i] \cup \cdots \cup V^+[i+s-1])|$. Of course, one could try to analyse these deviations carefully by resorting to some kind of a normal approximation. But for our purposes, this is unnecessary. It turns out that we may simply use the sum on the r.h.s. of (2.31) to identify which individuals are infected. Specifically, instead of computing the actual BP approximation $\eta_{x,1}$ after one round of updating, we just compare $\boldsymbol{W}_x^+(\tau)$ and $\boldsymbol{W}_x^-(\tau)$ with the values that we would expect these random variables to take if x were infected. These conditional expectations work out to be

$$W^+ = p_{11}\Delta \sum_{j=1}^{s} \exp\left(d(j-s)/s\right) w_j^+, \quad W^- = \begin{cases} p_{10}\Delta \sum_{j=1}^{s} \exp\left(d(j-s)/s\right) w_j^- & \text{if } p_{10} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

Thus, SPARC will diagnose $V[i]$ by comparing $\boldsymbol{W}_x^{\pm}(\tau)$ with $W^{\pm}$. Additionally, SPARC takes into account that infected individuals likely belong to $V^+[i]$, as we learned from Proposition 2.5.

Let

$$\zeta = (\ln\ln\ln n)^{-1} \tag{2.33}$$

be a term that tends to zero slowly enough to absorb error terms. The following proposition, which we prove in Section 3.3, summarises the analysis of phase 3. Recall from (2.16) that $c = m/(k \ln(n/k))$.

Proposition 2.6. *Assume that for a fixed* $\varepsilon > 0$, *we have*

$$c > c_{\mathrm{ex},2}(d) + \varepsilon. \tag{2.34}$$

Then w.h.p., the output τ *of SPARC satisfies*

$$\sum_{x \in V^+} \mathbb{1}\{\tau_x \neq \boldsymbol{\sigma}_x\} \leq k \exp\left(-\Omega\left(\frac{\ln n}{(\ln\ln n)^5}\right)\right).$$

The proof of Proposition 2.6, which can be found in Section 3.3, is the centrepiece of the analysis of SPARC. The proof is based on a large deviations analysis that bounds the number of individuals $x \in V^+[i]$ whose corresponding sums $\boldsymbol{W}_x^{\pm}(\tau)$ deviate from their conditional expectations given $\boldsymbol{\sigma}_x$. We have all the pieces in place to complete the proof of the first theorem.

Proof of Theorem 1.1 (upper bound on approximate recovery by SPARC). Setting $d = d_{\mathrm{Sh}}$ from (1.17) and invoking (1.18), we see that the theorem is an immediate consequence of Propositions 2.4, 2.5, and 2.6. □

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2.5. Exact recovery

As we saw in Section 1.3, the threshold $c_{\mathrm{ex},1}(d, \theta)$ encodes a local stability condition. This condition is intended to ensure that w.h.p., no other 'solution' $\sigma \neq \boldsymbol{\sigma}$ with $\|\sigma - \boldsymbol{\sigma}\|_1 = o(k)$ of a similarly large posterior likelihood exists. In fact, because $\mathbf{G}_{\mathrm{sc}}$ enjoys fairly good expansion properties, the test results provide sufficient clues for us to home in on $\boldsymbol{\sigma}$ once we get close, provided the number of tests is as large as prescribed by Theorem 1.3. Thus, the idea behind exact recovery is to run SPARC first and then apply local corrections to fully recover $\boldsymbol{\sigma}$; a similar strategy was employed in the noiseless case in [9].

Though this may sound easy and a simple greedy strategy does indeed do the trick in the noiseless case [9], in the presence of noise, it takes a good bit of care to get the local search step right. Hence, as per (1.11), suppose that c, d from (2.16) satisfy $c > \max\{c_{\mathrm{ex},2}(d), c_{\mathrm{ex},1}(d, \theta)\} + \varepsilon$. Also suppose that we already ran SPARC to obtain $\tau \in \{0, 1\}^V$ with $\|\tau - \boldsymbol{\sigma}\|_1 = o(k)$ (as provided by Proposition 2.6). How can we set about learning the status of an individual x with perfect confidence?

Assume for the sake of argument that $\tau_y = \boldsymbol{\sigma}_y$ for all y that share a test $a \in \partial x$ with x. If a contains another infected individual $y \neq x$, then unfortunately, nothing can be learned from a about the status of x. In this case, we call the test a *tainted*. By contrast, if $\tau_y = 0$ for all $y \in \partial a \setminus \{x\}$, that is, if a is *untainted*, then the displayed result $\boldsymbol{\sigma}''_a$ hinges on the infection status of x itself. Hence, the larger the number of untainted positively displayed $a \in \partial x$, the likelier x is infected. Consequently, to accomplish exact recovery, we are going to threshold the number of untainted positively displayed $a \in \partial x$. But crucially, to obtain an optimal algorithm, we cannot just use a scalar, one-size-fits-all threshold. Instead, we need to carefully choose a threshold function that takes into account the total number of untainted tests $a \in \partial x$.

To elaborate, let

$$\boldsymbol{Y}_x = \left|\left\{a \in \partial x \setminus F[0] : \forall y \in \partial a \setminus \{x\} : \boldsymbol{\sigma}_y = 0\right\}\right| \tag{2.35}$$

be the total number of untainted tests $a \in \partial x$; to avoid case distinctions, we omit seed tests $a \in F[0]$. Routine calculations reveal that $\boldsymbol{Y}_x$ is well approximated by a binomial variable with mean $\exp(-d)\Delta$. Therefore, the fluctuations of $\boldsymbol{Y}_x$ can be estimated via the Chernoff bound. Specifically, the numbers of infected/uninfected individuals with $\boldsymbol{Y}_x = \alpha\Delta$ can be approximated as

$$\mathbb{E}\left|\{x \in V_1 : \boldsymbol{Y}_x = \alpha\Delta\}\right| = k \exp\left(-\Delta D_{\mathrm{KL}}\left(\alpha \| \exp(-d)\right) + o(\Delta)\right), \tag{2.36}$$

$$\mathbb{E}\left|\{x \in V_0 : \boldsymbol{Y}_x = \alpha\Delta\}\right| = n \exp\left(-\Delta D_{\mathrm{KL}}\left(\alpha \| \exp(-d)\right) + o(\Delta)\right). \tag{2.37}$$

Consequently, since $k = \lceil n^\theta \rceil = o(n)$ 'atypical' values of $\boldsymbol{Y}_x$ occur more frequently in healthy than in infected individuals. In fact, recalling (1.6), we learn from a brief calculation that for $\alpha \notin \mathscr{Y}(c, d, \theta)$ not a single $x \in V_1$ with $\boldsymbol{Y}_x = \alpha\Delta$ exists w.h.p. Hence, if $\boldsymbol{Y}_x/\Delta \notin \mathscr{Y}(c, d, \theta)$, we deduce that x is uninfected.

For x such that $\boldsymbol{Y}_x/\Delta \in \mathscr{Y}(c, d, \theta)$, more care is required. In this case, we need to compare the number

$$\boldsymbol{Z}_x = \left|\left\{a \in \partial^+ x \setminus F[0] : \forall y \in \partial a \setminus \{x\} : \boldsymbol{\sigma}_y = 0\right\}\right| \tag{2.38}$$

of *positively displayed* untainted tests to the total number $\boldsymbol{Y}_x$ of untainted tests. Since the test results are put through the $\boldsymbol{p}$-channel independently, $\boldsymbol{Z}_x$ is a binomial variable given $\boldsymbol{Y}_x$. The conditional mean of $\boldsymbol{Z}_x$ equals $p_{11}\boldsymbol{Y}_x$ if x is infected and $p_{01}\boldsymbol{Y}_x$ otherwise. Therefore, the Chernoff bound shows that

$$\mathbb{P}\left[\boldsymbol{Z}_x = \alpha\beta\Delta \mid \boldsymbol{Y}_x = \alpha\Delta\right] = \exp\left(-\alpha\Delta D_{\mathrm{KL}}\left(\beta \| p_{\boldsymbol{\sigma}_x 1}\right) + o(\Delta)\right). \tag{2.39}$$

In light of (2.39), we set up the definition (1.9) of $c_{\mathrm{ex},1}(d, \theta)$ so that $\mathfrak{z}(\,\cdot\,)$ can be used as a threshold function to tell infected from uninfected individuals. Indeed, given $\boldsymbol{Y}_x, \boldsymbol{Z}_x$, we should declare

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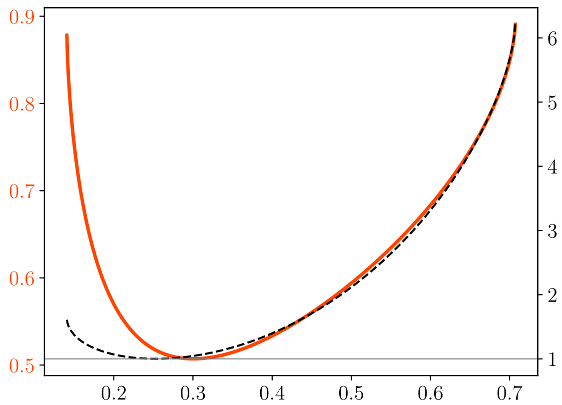

Figure 2. The threshold function $\mathfrak{z}(\,\cdot\,)$ (red) on the interval $\mathscr{Y}(c_{\mathrm{ex},1}(d,\theta),d,\theta)$ and the resulting large deviations rate $c_{\mathrm{ex},1}(d,\theta)d(1-\theta)(D_{\mathrm{KL}}(\alpha\|\exp(-d))+\alpha D_{\mathrm{KL}}(\mathfrak{z}(\alpha)\|p_{01}))$ (black) with $\theta=1/2$, $p_{00}=0.972$, $p_{11}=0.9$ at the optimal choice of d.

x uninfected if either $\boldsymbol{Y}_x/\Delta\notin\mathscr{Y}(c,d,\theta)$ or $\boldsymbol{Z}_x/\boldsymbol{Y}_x<\mathfrak{z}(\boldsymbol{Z}_x/\Delta)$ and infected otherwise; then the choice of $\mathfrak{z}(\,\cdot\,)$ and $c_{\mathrm{ex}}(\theta)$ would ensure that all individuals get diagnosed correctly w.h.p. Figure 2 displays a characteristic specimen of the function $\mathfrak{z}(\,\cdot\,)$ and the corresponding rate function from (1.9).

Yet trying to distil an algorithm from these considerations, we run into two obvious obstacles. First, the threshold $\mathfrak{z}(\,\cdot\,)$ may be hard to compute precisely. Similarly, the limits of the interval $\mathscr{Y}(c,d,\theta)$ may be irrational (or worse). The following proposition, which we prove in Section 4.1, remedies these issues.

Proposition 2.7. *Let $\varepsilon>0$ and assume that $c>c_{\mathrm{ex}}(d,\theta)+\varepsilon$. Then there exist $\delta>0$ and an open interval $\emptyset\neq\mathscr{I}=(l,r)\subseteq[\delta,1-\delta]$ with endpoints $l,r\in\mathbb{Q}$ such that for any $\varepsilon'>0$, there exist $\delta'>0$ and a step function $\mathscr{Z}:\mathscr{I}\to(p_{01},p_{11})\cap\mathbb{Q}$ such that the following conditions are satisfied.*

Z1: $cd(1-\theta)D_{\mathrm{KL}}\left(y\|\exp(-d)\right)>\theta+\delta$ *for all* $y\in(0,1)\setminus(l+\delta,r-\delta)$.

Z2: $cd(1-\theta)\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right)+yD_{\mathrm{KL}}\left(\mathscr{Z}(y)\|p_{11}\right)\right)>\theta+\delta$ *for all* $y\in\mathscr{I}$.

Z3: $cd(1-\theta)\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right)+yD_{\mathrm{KL}}\left(\mathscr{Z}(y)\|p_{01}\right)\right)>1+\delta$ *for all* $y\in\mathscr{I}$.

Z4: *If* $y,y'\in\mathscr{I}$ *satisfy* $|y-y'|<\delta'$, *then* $|\mathscr{Z}(y)-\mathscr{Z}(y')|<\varepsilon'$.

The second obstacle is that in the above discussion, we assumed that $\tau_y=\boldsymbol{\sigma}_y$ for all $y\in\partial a\setminus\{x\}$. But all that the analysis of SPARC provides is that w.h.p. $\tau_y\neq\boldsymbol{\sigma}_y$ for at most $o(k)$ individuals y. To cope with this issue, we resort to the expansion properties of $\mathbf{G}_{\mathrm{sc}}$. Roughly speaking, we show that w.h.p. for any small set S of individuals (such as $\{y:\tau_y\neq\boldsymbol{\sigma}_y\}$), the set of individuals x that occur in 'many' tests that also contain a second individual from S is significantly smaller than S. As a consequence, as we apply the thresholding procedure repeatedly, the number of misclassified individuals decays geometrically. We thus arrive at the following algorithm.

Proposition 2.8. *Suppose that $c>c_{\mathrm{ex}}(d,\theta)+\varepsilon$ for a fixed $\varepsilon>0$. There exists $\varepsilon'>0$, a rational interval $\mathscr{I}$ and a rational step function $\mathscr{Z}$ such that w.h.p. for all $1\leq i<\ln n$, we have*

$$\|\boldsymbol{\sigma}-\tau^{(i+1)}\|_1\leq\frac{1}{3}\|\boldsymbol{\sigma}-\tau^{(i)}\|_1.$$

We prove Proposition 2.8 in Section 4.2.

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Algorithm 3 The SPEX algorithm.

Input: $\mathbf{G}_{\text{sc}}, \boldsymbol{\sigma}''$
Output: an estimate of $\boldsymbol{\sigma}$
1 Let $\tau^{(1)}$ be the output of SPARC($\mathbf{G}_{\text{sc}}, \boldsymbol{\sigma}''$);
2 **for** $i = 1, \ldots, \lceil \ln n \rceil$ **do**
3 For all $x \in V[s+1] \cup \cdots \cup V[\ell]$ calculate
4 $$Y_x(\tau^{(i)}) = \sum_{a \in \partial x \setminus F[0]} \mathbb{1}\left\{\forall y \in \partial a \setminus \{x\} : \tau_y^{(i)} = 0\right\}, \quad Z_x(\tau^{(i)}) = \sum_{a \in \partial x \setminus F[0] : \boldsymbol{\sigma}''_a = 1} \mathbb{1}\left\{\forall y \in \partial a \setminus \{x\} : \tau_y^{(i)} = 0\right\};$$
5 Let $$\tau_x^{(i+1)} = \begin{cases} \tau_x^{(i)} & \text{if } x \in V[1] \cup \cdots \cup V[s], \\ \mathbb{1}\{Y_x(\tau^{(i)})/\Delta \in \mathscr{I} \text{ and } Z_x(\tau^{(i)})/\Delta > \mathcal{Z}(Y_x(\tau^{(i)})/\Delta)\} & \text{otherwise} \end{cases};$$
6 **return** $\tau^{(\lceil \ln n \rceil)}$

Proof of Theorem 1.3 (upper bound on exact recovery by SPEX). Since Proposition 2.8 shows that the number of misclassified individuals decreases geometrically as we iterate Steps 3–5 of SPEX, we have $\tau^{(\lceil \ln n \rceil)} = \boldsymbol{\sigma}$ w.h.p. Furthermore, thanks to Theorem 1.1 and Proposition 2.7, SPEX is indeed a polynomial time algorithm. $\square$

Remark 2.9. In the noiseless case $p_{00} = p_{11} = 1$, Theorem 1.3 reproduces the analysis of the SPIV algorithm from [9]. One key difference between SPARC and SPEX on the one hand and SPIV on the other is that the former are based on BP, while the latter relies on combinatorial intuition. More precisely, the SPIV algorithm infers from positive tests by means of a weighted sum identical to $\boldsymbol{W}_x^+(\tau)$ from (2.30) with the special values $p_{00} = p_{11} = 1$ and $d = \ln 2$. In the noiseless case, the weights w_j^+ were 'guessed' based on combinatorial intuition. Furthermore, in noiseless case, we can be certain that any individual contained in a negative test is healthy, and therefore, the SPIV algorithm only takes negative tests into account in this simple, deterministic manner. By contrast, in the noisy case, the negative tests give rise to a second weighted sum $\boldsymbol{W}_x^-(\tau)$. An important novelty is that rather than 'guessing' the weights $w_j^\pm$, here we discovered how they can be derived systematically from the BP formalism. Apart from shedding new light on the noiseless case as well, we expect that this type of approach can be generalised to other inference problems as well. A second novelty in the design of SPEX is the use of the threshold function $\mathcal{Z}(\,\cdot\,)$ that depends on the number untainted tests. The need for such a threshold function is encoded in the optimisation problem (1.9) that gives rise to a nontrivial choice of the coefficient d that governs the size of the tests. This type of optimisation is unnecessary in the noiseless case, where simply $d = \ln 2$ is the optimal choice for all $\theta \in (0, 1)$.

2.6. The constant column lower bound

Having completed the discussion of the algorithms SPARC and SPEX for Theorems 1.1 and 1.3, we turn to the proof of Theorem 1.4 on the information-theoretic lower bound for the constant column design. The goal is to show that for $m < (c_{\text{ex}}(\theta) - \varepsilon)k \ln(n/k)$, no algorithm, efficient or otherwise, can exactly recover $\boldsymbol{\sigma}$ on $\mathbf{G}_{\text{cc}}$.

To this end, we are going to estimate the probability of the actual ground truth $\boldsymbol{\sigma}$ under the posterior given the test results. We recall from Fact 2.2 that the posterior coincides with the Boltzmann distribution from (2.4). The following proposition, whose proof can be found in Section 5.1, estimates the Boltzmann weight $\psi_{\mathbf{G}_{\text{cc}}, \boldsymbol{\sigma}''}(\boldsymbol{\sigma})$ of the ground truth. Recall from (2.16) that $d = k\Delta/m$. Also recall the weight function from (2.3).

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Proposition 2.10. *W.h.p., the weight of $\boldsymbol{\sigma}$ satisfies*

$$\frac{1}{m}\ln\psi_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''}(\boldsymbol{\sigma}) = -\exp(-d)h(p_{00}) - (1-\exp(-d))h(p_{11}) + O(m^{-1/2}\ln^3). \tag{2.40}$$

We are now going to argue that for $c < c_{\mathrm{ex}}(\theta)$ the partition function $Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''}$ dwarfs the Boltzmann weight (2.40) w.h.p. The definition (1.11) of $c_{\mathrm{ex}}(\theta)$ suggests that there are two possible ways how this may come about. The first occurs when $c = m/(k\ln(n/k))$ is smaller than the local stability bound $c_{\mathrm{ex},1}(d,\theta)$ from (1.9). In this case, we will essentially put the analysis of SPEX into reverse gear. That is, we will show that there are plenty of individuals x whose status could be flipped from infected to healthy or vice versa without reducing the posterior likelihood. In effect, the partition function will be far greater than the Boltzmann weight of $\boldsymbol{\sigma}$.

Proposition 2.11. *Assume that there exists $y \in \mathscr{Y}(c,d,\theta)$ such that there exists $z \in (p_{01},p_{11})$ such that*

$$cd(1-\theta)\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z\|p_{11}\right)\right) < \theta \qquad \text{and} \tag{2.41}$$

$$cd(1-\theta)\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z\|p_{01}\right)\right) < 1. \tag{2.42}$$

Then w.h.p., there exist $n^{\Omega(1)}$ pairs $(v,w) \in V_1 \times V_0$ such that for the configuration $\boldsymbol{\sigma}^{[v,w]}$ obtained from $\boldsymbol{\sigma}$ by inverting the v,w-entries, we have

$$\mu_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''}(\boldsymbol{\sigma}^{[v,w]}) = \mu_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''}(\boldsymbol{\sigma}). \tag{2.43}$$

We prove Proposition 2.11 in Section 5.2.

The second case is that c is smaller than the entropy bound $c_{\mathrm{ex},2}(d)$ from (1.10). In this case, we will show by way of a moment computation that $Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''}$ exceeds the Boltzmann weight of $\boldsymbol{\sigma}$ w.h.p. More precisely, in Section 5.3, we prove the following.

Proposition 2.12. *Let $\varepsilon > 0$. If $c < c_{\mathrm{ex},2}(d) - \varepsilon$, then*

$$\mathbb{P}\left[\ln Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''} \geq k\ln(n/k)\left[1 - c/c_{\mathrm{ex},2}(d) + o(1)\right]\right] > 1 - \varepsilon + o(1).$$

Proof of Theorem 1.4 *(lower bound on exact recovery on G_{cc}).* Proposition 2.11 readily implies that $\mu_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''}(\boldsymbol{\sigma}) = o(1)$ w.h.p. if $c < c_{\mathrm{ex},1}(d,\theta)$. Furthermore, in the case $c < c_{\mathrm{ex},2}(d)$, the assertion follows from Propositions 2.10 and 2.12. □

3. Analysis of the approximate recovery algorithm

In this section, we carry out the proofs of the various propositions leading up to the proof of Theorem 1.1 (Propositions 2.3, 2.5, and 2.6).

Proposition 2.3 contains some basic properties about $\mathbf{G}_{\mathrm{sc}}$ that we need further on. Further Proposition 2.5 justifies to only consider V^+ in the main step of SPARC. The most important part of the analysis is contained in the proof of Proposition 2.6 that shows the correctness for the main step of SPARC. Here we consider three different sets of possibly misclassified individuals (Lemmas 3.1, 3.2, and 3.3) that we analyse separately to show that the in total the number of misclassified individuals is small.

Throughout, we work with the spatially coupled test design $\mathbf{G}_{\mathrm{sc}}$ from Section 2.3. Hence, we keep the notation and assumptions from (2.13) to (2.17). We also make a note of the fact that for any $x \in V[i]$ and any $a \in F[i+j-1]$, $j = 1,\ldots,s$,

$$\mathbb{P}[x \in \partial a] = 1 - \mathbb{P}[x \notin \partial a] = 1 - \binom{|F[i+j-1]|-1}{\Delta/s}\binom{|F[i+j-1]|}{\Delta/s}^{-1} = \frac{\Delta\ell}{ms} + O\left(\left(\frac{\Delta\ell}{ms}\right)^2\right); \tag{3.1}$$

this is because the construction of $\mathbf{G}_{sc}$ ensures that x joins Δ/s random tests in each of $F[i+j-1]$, drawn uniformly without replacement.

3.1. Proof of Proposition 2.3 (properties of $\mathbf{G}_{sc}$)

Property **G1** is a routine consequence of the Chernoff bound and was previously established as part of (9, Proposition 4.1). With respect to **G2,** we may condition on the event $\mathfrak{E}$ that the bound (2.18) from **G1** is satisfied. Consider a test $a \in F[i]$. Recall that a comprises individuals from the compartments $V[i-s+1], \ldots, V[i]$. Since the probability that a specific individual joins a specific test is given by (3.1) and since individuals choose the tests that they join independently, on $\mathfrak{E}$ for each $i-s+1 \le h \le i$, we have

$$|V_1[h] \cap \partial a| \sim \mathrm{Bin}\left(\frac{k}{\ell} + O\left(\sqrt{\frac{k}{\ell}}\ln n\right), \frac{\Delta\ell}{ms} + O\left(\left(\frac{\Delta\ell}{ms}\right)^2\right)\right). \tag{3.2}$$

Combining (2.16) and (3.2), we obtain

$$\mathbb{E}[|V_1[h] \cap \partial a| \mid \mathfrak{E}] = \frac{\Delta k}{ms} + O\left(\sqrt{\frac{\ell}{k}}\ln n\right) = \frac{d}{s} + O\left(k^{-1/2}\ln^{3/2} n\right). \tag{3.3}$$

Further, combining (3.2) and (3.3), we get

$$\begin{aligned}\mathbb{P}\left[V_1[h] \cap \partial a = \emptyset \mid \mathfrak{E}\right] &= \exp\left[\left(\frac{k}{\ell} + O\left(\sqrt{\frac{k}{\ell}}\ln n\right)\right)\ln\left(1 - \frac{\Delta\ell}{ms} + O\left(\left(\frac{\Delta\ell}{ms}\right)^2\right)\right)\right] \\ &= \exp\left(-\frac{d}{s} + O\left(k^{-1/2}\ln^{3/2} n\right)\right).\end{aligned}$$

Multiplying these probabilities up on $i-s+1 \le h \le i$, we arrive at the estimate

$$\mathbb{P}\left[V_1[h] \cap \partial a = \emptyset \mid \mathfrak{E}\right] = \exp\left(-d + O\left(k^{-1/2}\ln^{8/5} n\right)\right).$$

Hence,

$$\mathbb{E}[\,|F_0[i]| \mid \mathfrak{E}] = \frac{m}{\ell}\exp(-d) + O\left(\sqrt{m}\ln^2 n\right). \tag{3.4}$$

To establish concentration, observe that the set ∂x of tests that a specific infected individual $x \in V_1[h]$ joins can change $|F_0[i]|$ by Δ at the most. Moreover, changing the neighbourhood ∂x of a healthy individual cannot change the actual test results at all. Therefore, by Azuma–Hoeffding,

$$\mathbb{P}\left[||F_0[i]| - \mathbb{E}\left[|F_0[i]| \mid \mathscr{E}\right]| \ge \sqrt{m}\ln^2 n \mid \mathscr{E}\right] \le 2\exp\left(-\frac{m\ln^4 n}{2k\Delta^2}\right) = o(n^{-3}). \tag{3.5}$$

Thus, (3.4), (3.5), and **G1** show that with probability $1-o(n^{-2})$ for all $1 \le i \le \ell$, we have

$$|F_0[i]| = \frac{m}{\ell}\exp(-d) + O\left(\sqrt{m}\ln^2 n\right), \quad \text{and thus} \quad |F_1[i]| = \frac{m}{\ell}(1-\exp(-d)) + O\left(\sqrt{m}\ln^2 n\right). \tag{3.6}$$

Since the actual test results are subjected to the $\boldsymbol{p}$-channel independently to obtain the displayed test results, the distributions of $|F^{\pm}_{0/1}[i]$ given $F_{0/1}[i]$ read

$$\left|F_0^-[i]\right| = \mathrm{Bin}(|F_0[i]|, p_{00}), \qquad \left|F_0^+[i]\right| = \mathrm{Bin}(|F_0[i]|, p_{01}), \tag{3.7}$$

$$\left|F_1^-[i]\right| = \mathrm{Bin}(|F_1[i]|, p_{10}), \qquad \left|F_1^+[i]\right| = \mathrm{Bin}(|F_1[i]|, p_{11}). \tag{3.8}$$

Thus, **G2** follows from (3.6), (3.7), and Lemma 1.5 (the Chernoff bound).

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3.2. Proof of Proposition 2.5 (plausible number of positive tests)

Any $x \in V_1[i]$ has a total of Δ/s neighbours in each of $F[i], \ldots, F[i+s-1]$. Moreover, all tests $a \in \partial x$ are actually positive. Since the displayed result is obtained via the $\boldsymbol{p}$-channel independently for every a, the number of displayed positive neighbours $|\partial x \cap F^+[i+j-1]|$ is a binomial variable with distribution $\mathrm{Bin}(\Delta/s, p_{11})$. Since $\Delta = \Theta(\ln n)$ and $s = \Theta(\ln\ln n)$, the first assertion (2.24) is immediate from Lemma 1.5.

Moving on to the second assertion, we condition on the event $\mathfrak{E}$ that the bounds (2.19)–(2.22) hold for all $i \in [\ell]$. Then Proposition 2.3 shows that $\mathbb{P}[\mathfrak{E}] = 1 - o(n^{-2})$. Given $\mathfrak{E}$, we know that

$$|F^+[i]| = \frac{q_0^+ m}{\ell} + O\left(\sqrt{m}\ln^3 n\right), \qquad |F^-[i]| = \frac{(1-q_0^+)m}{\ell} + O\left(\sqrt{m}\ln^3 n\right). \tag{3.9}$$

Now consider an individual $x \in V_0[i]$. Also consider any test $a \in F[i+j-1]$ for $j \in [s]$. Then the actual result $\boldsymbol{\sigma}'_a$ of a is independent of the event $\{x \in a\}$ because x is uninfected. Since the displayed result $\boldsymbol{\sigma}''_a$ depends solely on $\boldsymbol{\sigma}'_a$, we conclude that $\boldsymbol{\sigma}''_a$ is independent of $\{x \in a\}$ as well. Therefore, (3.9) shows that on the event $\mathfrak{E}$, the number of displayed positive tests that x is a member has conditional distribution

$$|\partial x \cap F^+[i+j-1]| \sim \mathrm{Hyp}\left(\frac{m}{\ell}, \frac{q_0^+ m}{\ell} + O\left(\sqrt{m}\ln^3 n\right), \frac{\Delta}{s}\right). \tag{3.10}$$

Since the random variables $(|\partial x \cap F^+[i+j-1]|)_{1 \le j \le s}$ are mutually independent, (2.25) follows from Lemma 1.6.

3.3. Proof of Proposition 2.6 (correctness of SPARC for V^+)

We reduce the proof of Proposition 2.6 to three lemmas. The first two estimate the sums from (2.30) when evaluated at the actual ground truth $\boldsymbol{\sigma}$.

Lemma 3.1. *Assume that (2.34) is satisfied. w.h.p., we have*

$$\sum_{s<i\le\ell}\sum_{x\in V_1^+[i]} \mathbb{1}\left\{\boldsymbol{W}_x^+(\boldsymbol{\sigma}) < (1-\zeta/2)W^+ \text{ or } \boldsymbol{W}_x^-(\boldsymbol{\sigma}) > (1+\zeta/2)W^-\right\} \le k\exp\left(-\Omega\left(\frac{\ln n}{(\ln\ln n)^4}\right)\right). \tag{3.11}$$

Lemma 3.2. *Assume that (2.34) is satisfied. w.h.p., we have*

$$\sum_{s\le i<\ell}\sum_{x\in V_0^+[i]} \mathbb{1}\left\{\boldsymbol{W}_x^+(\boldsymbol{\sigma}) \ge (1-2\zeta)W^+ \text{ and } \boldsymbol{W}_x^-(\boldsymbol{\sigma}) \le (1+2\zeta)W^-\right\} \le k^{1-\Omega(1)}. \tag{3.12}$$

We defer the proofs of Lemmas 3.1 and 3.2 to Sections 3.4 and 3.5. While the proof of Lemma 3.1 is fairly routine, the proof of Lemma 3.2 is the linchpin of the entire analysis of SPARC, as it effectively vindicates the BP heuristics that we have been invoking so liberally in designing the algorithm.

Additionally, to compare $\boldsymbol{W}_x^{\pm}(\boldsymbol{\sigma})$ with the algorithm's estimate $\boldsymbol{W}_x^{\pm}(\tau)$, we resort to the following expansion property of $\mathbf{G}_{\mathrm{sc}}$.

Lemma 3.3. *Let $0 < \alpha, \beta < 1$ be such that $\alpha + \beta > 1$. Then w.h.p. for any $T \subseteq V$ of size $|T| \le \exp(-\ln^\alpha n)k$,*

$$\left|\left\{x \in V : \sum_{a \in \partial x \setminus F[0]} \mathbb{1}\{T \cap \partial a \setminus \{x\} \ne \emptyset\} \ge \ln^\beta n\right\}\right| \le \frac{|T|}{8\ln\ln n}.$$

In a nutshell, Lemma 3.3 posits that for any 'small' set T of individuals, there are even fewer individuals that share many tests with individuals from T. Lemma 3.3 is an generalisation of

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(9, Lemma 4.16). The proof, based on a routine union bound argument, is included in Appendix C for completeness.

Proof of Proposition 2.6 *(correctness of SPARC for V^+).* Proposition 2.4 shows that for all individuals x in the seed $V[1], \dots, V[s]$, we have $\tau_x = \boldsymbol{\sigma}_x$ w.h.p. Let $\mathcal{M}[i] = \left\{x \in V^+[i] : \tau_x \neq \boldsymbol{\sigma}_x\right\}$ be the set of misclassified individuals in $V^+[i]$. We are going to show that w.h.p. for all $1 \le i \le \ell$ and for large enough n,

$$|\mathcal{M}[i]| \le k \exp\left(-\frac{\ln n}{(\ln\ln n)^5}\right). \tag{3.13}$$

We proceed by induction on i. As mentioned above, Proposition 2.4 ensures that $\mathcal{M}[1] \cup \dots \cup \mathcal{M}[s] = \emptyset$ w.h.p. Now assume that (3.13) is correct for all $i < h \le \ell$; we are going to show that (3.13) holds for $i = h$ as well. To this end, recalling ζ from (2.33) and $W^{\pm}$ from (2.32), we define for $p_{10} > 0$

$$\begin{aligned}
\mathcal{M}_1[h] &= \left\{x \in V_1^+[h] : \boldsymbol{W}_x^+(\boldsymbol{\sigma}) < (1-\zeta/2)W^+ \text{ or } \boldsymbol{W}_x^-(\boldsymbol{\sigma}) > (1+\zeta/2)W^-\right\},\\
\mathcal{M}_2[h] &= \left\{x \in V_0^+[h] : \boldsymbol{W}_x^+(\boldsymbol{\sigma}) > (1-2\zeta)W^+ \text{ and } \boldsymbol{W}_x^-(\boldsymbol{\sigma}) < (1+2\zeta)W^-\right\},\\
\mathcal{M}_3[h] &= \left\{x \in V^+[h] : |\boldsymbol{W}_x^+(\boldsymbol{\sigma}) - \boldsymbol{W}_x^+(\tau)| + |\boldsymbol{W}_x^-(\boldsymbol{\sigma}) - \boldsymbol{W}_x^-(\tau)| > \zeta(W^+ \wedge W^-)/8\right\}
\end{aligned}$$

and further for $p_{10} = 0$

$$\begin{aligned}
\mathcal{M}_1[h] &= \left\{x \in V_1^+[h] : \boldsymbol{W}_x^+(\boldsymbol{\sigma}) < (1-\zeta/2)W^+\right\},\\
\mathcal{M}_2[h] &= \left\{x \in V_0^+[h] : \boldsymbol{W}_x^+(\boldsymbol{\sigma}) > (1-2\zeta)W^+ \text{ and } \boldsymbol{W}_x^-(\boldsymbol{\sigma}) = 0\right\},\\
\mathcal{M}_3[h] &= \left\{x \in V^+[h] : |\boldsymbol{W}_x^+(\boldsymbol{\sigma}) - \boldsymbol{W}_x^+(\tau)| > \zeta(W^+)/8\right\}.
\end{aligned}$$

We claim that $\mathcal{M}[h] \subseteq \mathcal{M}_1[h] \cup \mathcal{M}_2[h] \cup \mathcal{M}_3[h]$. To see this, assume that $x \in \mathcal{M}[h] \setminus (\mathcal{M}_1[h] \cup \mathcal{M}_2[h])$. Then for SPARC to misclassify x, it must be the case that

$$\left|\boldsymbol{W}_x^+(\tau) - \boldsymbol{W}_x^+(\boldsymbol{\sigma})\right| + |\boldsymbol{W}_x^-(\tau) - \boldsymbol{W}_x^-(\boldsymbol{\sigma})| > \frac{\zeta}{8}(W^+ \wedge W^-),$$

and thus $x \in \mathcal{M}_3[h]$.

Thus, to complete the proof, we need to estimate $|\mathcal{M}_1[h]|, |\mathcal{M}_2[h]|, |\mathcal{M}_3[h]|$. Lemmas 3.1 and 3.2 readily show that

$$|\mathcal{M}_1[h]| + |\mathcal{M}_2[h]| \le k \exp\left(-\Omega\left(\frac{\ln n}{(\ln\ln n)^4}\right)\right). \tag{3.14}$$

Furthermore, in order to bound $|\mathcal{M}_3[h]|$, we will employ Lemma 3.3. Specifically, consider $x \in \mathcal{M}_3[h]$. Since $W^+ \wedge W^- = \Omega(\Delta) = \Omega(\ln n)$ for $p_{10} > 0$ by (2.29) and (2.32), there exists $j \in [s]$ such that

$$\left|\,|\boldsymbol{W}_{x,j}^+(\tau)| - |\boldsymbol{W}_{x,j}^+(\boldsymbol{\sigma})|\,\right| > \ln^{1/2} n \qquad \text{or} \qquad \left|\,|\boldsymbol{W}_{x,j}^-(\tau)| - |\boldsymbol{W}_{x,j}^-(\boldsymbol{\sigma})|\,\right| > \ln^{1/2} n.$$

Since $W^+ = \Omega(\ln n)$ for $p_{10} = 0$, there exists $j \in [s]$ such that

$$\left|\,|\boldsymbol{W}_{x,j}^+(\tau)| - |\boldsymbol{W}_{x,j}^+(\boldsymbol{\sigma})|\,\right| > \ln^{1/2} n$$

Assume without loss of generality that the left inequality holds. Then there are at least $\ln^{1/2} n$ tests $a \in \partial x \cap F^+[h+j-1]$ such that $\partial a \cap (\mathcal{M}[1] \cup \dots \cup \mathcal{M}[h-1]) \neq \emptyset$. Therefore, Lemma 3.3 shows that

$$|\mathcal{M}_3[h]| \le \frac{|\mathcal{M}[h-s] \cup \dots \cup \mathcal{M}[h-1]|}{8 \ln\ln n}.$$

Hence, using induction to bound $|\mathcal{M}[h-s]|, \dots, |\mathcal{M}[h-1]|$ and recalling from (2.14) that $s \le 1 + \ln\ln n$, we obtain (3.13) for $i = h$. □

3.4. Proof of Lemma 3.1 (large deviations for infected individuals)

The thrust of Lemma 3.1 is to verify that the definition (2.23) of the set $V^+[i]$ faithfully reflects the typical statistics of positively/negatively displayed tests in the neighbourhood of an infected individual $x \in V_1[i]$ with $s < i \leq \ell$. Recall from the definition of $\mathbf{G}_{sc}$ that such an individual x joins tests in $F[i+j-1]$ for $j \in [s]$. Moreover, apart from x itself, a test $a \in F[i+j-1] \cap \partial x$ recruits from $V[i+j-s], \ldots, V[i+j-1]$. In particular, a recruits participants from the $s-j$ compartments $V[i+j-s], \ldots, V[i-1]$ preceding $V[i]$. Let

$$\mathscr{U}_{i,j} = \left\{a \in F[i+j-1] : (V_1[1] \cup \cdots \cup V_1[i-1]) \cap \partial a = \emptyset\right\}$$
$$= \left\{a \in F[i+j-1] : \bigcup_{h=i+j-s}^{i-1} V_1[h] \cap \partial a = \emptyset\right\} \tag{3.15}$$

be the set of tests in $F[i+j-1]$ that do not contain an infected individual from $V[i+j-s], \ldots, V[i-1]$.

Claim 3.4. *With probability* $1 - o(n^{-2})$ *for all* $s \leq i < \ell, j \in [s]$, *we have*

$$|\mathscr{U}_{i,j}| = \left(1 + O(n^{-\Omega(1)})\right) \frac{m}{\ell} \cdot \exp\left(d(j-s)/s\right).$$

Proof. We condition on the event $\mathfrak{E}$ that **G1** from Proposition 2.3 holds. Then for any $a \in F[i+j-1]$ and any $i+j-s \leq h < i$, the number $V_1[h] \cap \partial a$ of infected individuals in a from $V[h]$ is a binomial variable as in (3.2). Since the random variables $(V_1[h] \cap \partial a)_h$ are mutually independent, we therefore obtain

$$\mathbb{P}\left[(V_1[i+j-s] \cup \cdots V_1[i-1]) \cap \partial a = \emptyset \mid \mathfrak{E}\right] = \exp\left(d(j-s)/s\right) + O(n^{-\Omega(1)}). \tag{3.16}$$

Hence,

$$\mathbb{E}\left[|\mathscr{U}_{i,j}| \mid \mathfrak{E}\right] = \frac{m}{\ell} \exp\left[d(j-s)/s + O(n^{-\Omega(1)})\right]. \tag{3.17}$$

Further, changing the set ∂x of tests that a single $x \in V_1$ joins can alter $|\mathscr{U}_{i,j}|$ by Δ at the most, while changing the set of neighbours of any $x \in V_0$ does not change $|\mathscr{U}_{i,j}|$ at all. Therefore, Azuma–Hoeffding shows that

$$\mathbb{P}\left[\left||\mathscr{U}_{i,j}| - \mathbb{E}[|\mathscr{U}_{i,j}| \mid \mathfrak{E}]\right| > \sqrt{m}\ln^2 n \mid \mathfrak{E}\right] \leq 2\exp\left(-\frac{m\ln^4 n}{2k\Delta^2}\right) = o(n^{-2}). \tag{3.18}$$

The assertion follows from (3.17) to (3.18). □

Let

$$q_{1,j}^- = p_{01}\exp\left(d(j-s)/s\right), \qquad q_{1,j}^+ = p_{11}\exp\left(d(j-s)/s\right). \tag{3.19}$$

Claim 3.5. *For all* $s < i \leq \ell$, $x \in V_1[i]$, *and* $j \in [s]$, *we have*

$$\mathbb{P}\left[|\boldsymbol{W}_{x,j}^+| < (1-\zeta/2)q_{1,j}^+\frac{\Delta}{s}\right] + \mathbb{P}\left[|\boldsymbol{W}_{x,j}^-| > (1+\zeta/2)q_{1,j}^-\frac{\Delta}{s}\right] \leq \exp\left(-\Omega\left(\frac{\ln n}{(\ln\ln n)^4}\right)\right).$$

Proof. Fix $s \leq i < \ell$, $1 \leq j \leq s$ and $x \in V_1[i]$. In light of Proposition 2.3 and Claim 3.4, the event $\mathfrak{E}$ that **G1** from Proposition 2.3 holds and that $|\mathscr{U}_{i,j}| = \exp\left(d(j-s)/s\right)\frac{m}{\ell}(1 + O(n^{-\Omega(1)}))$ has probability

$$\mathbb{P}[\mathfrak{E}] = 1 - o(n^{-2}). \tag{3.20}$$

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Let $\boldsymbol{U}_{i,j} = |\partial x \cap \mathscr{U}_{i,j}|$. Given $\mathscr{U}_{i,j}$, we have

$$\boldsymbol{U}_{i,j} \sim \mathrm{Hyp}\left(\frac{m}{\ell} + O(1), \exp\left(\frac{d(j-s)}{s}\right)\frac{m}{\ell}(1+O(n^{-\Omega(1)})), \frac{\Delta}{s}\right).$$

Therefore, Lemma 1.6 shows that the event

$$\mathfrak{E}' = \left\{\left|U_{i,j} - \frac{\Delta}{s}\exp\left(\frac{d(j-s)}{s}\right)\right| \geq \frac{\ln n}{(\ln\ln n)^2}\right\}$$

has conditional probability

$$\mathbb{P}\left[\mathfrak{E}' \mid \mathfrak{E}\right] \leq \exp\left(-\Omega\left(\frac{\ln n}{(\ln\ln n)^4}\right)\right). \tag{3.21}$$

Finally, given $\boldsymbol{U}_{i,j}$ the number $|\boldsymbol{W}^+_{x,j}|$ of positively displayed $a \in \partial x \cap \mathscr{U}_{i,j}$ has distribution $\mathrm{Bin}(\boldsymbol{U}_{i,j}, p_{11})$; similarly, $|\boldsymbol{W}^-_{x,j}|$ has distribution $\mathrm{Bin}(\boldsymbol{U}_{i,j}, p_{01})$. Thus, since $\Delta = \Theta(\ln n)$ while $s = O(\ln\ln n)$ by (2.14), the assertion follows from (3.20), (3.21), and Lemma 1.5. □

Proof of Lemma 3.1 *(large deviations for infected individuals).* The lemma follows from Claim 3.5 and Markov's inequality. □

3.5. Proof of Lemma 3.2 (large deviations for healthy individuals)

To prove Lemma 3.2, we need to estimate the probability that an uninfected individual $x \in V[i]$, $s < i \leq \ell$, 'disguises' itself to look like an infected individual. In addition to the sets $\mathscr{U}_{i,j}$ from (3.15) towards the proof of Lemma 3.2, we also need to consider the sets

$$\mathscr{P}_{i,j} = F_1[i+j-1] \cap \mathscr{U}_{i,j}, \qquad \mathscr{N}_{i,j} = F_0[i+j-1] \cap \mathscr{U}_{i,j},$$
$$\mathscr{P}^{\pm}_{i,j} = F^{\pm}_1[i+j-1] \cap \mathscr{U}_{i,j}, \qquad \mathscr{N}^{\pm}_{i,j} = F^{\pm}_0[i+j-1] \cap \mathscr{U}_{i,j}.$$

In words, $\mathscr{P}_{i,j}$ and $\mathscr{N}_{i,j}$ are the sets of actually positive/negative tests in $\mathscr{U}_{i,j}$, that is, actually positive/negative tests $a \in F[i+j-1]$ that do not contain an infected individual from $V[i+j-s] \cup \cdots \cup V[i-1]$. Additionally, $\mathscr{P}^{\pm}_{i,j}$, $\mathscr{N}^{\pm}_{i,j}$ discriminate based on the displayed test results. We begin by estimating the likely sizes of these sets.

Claim 3.6. *Let $s < i \leq \ell$ and $j \in [s]$. Then with probability $1 - o(n^{-2})$, we have*

$$|\mathscr{P}_{i,j}| = \left(1 + O(n^{-\Omega(1)})\right)\frac{m}{\ell} \cdot \frac{\exp(dj/s) - 1}{\exp(d)}, \quad |\mathscr{N}_{i,j}| = \left(1 + O(n^{-\Omega(1)})\right)\frac{m}{\ell} \cdot \exp(-d), \tag{3.22}$$

selectfont

$$|\mathscr{P}^{+}_{i,j}| = (1 + O(n^{-\Omega(1)}))\frac{m}{\ell} \cdot \frac{p_{11}(\exp(dj/s) - 1)}{\exp(d)}, \quad |\mathscr{P}^{-}_{i,j}| = (1 + O(n^{-\Omega(1)}))\frac{m}{\ell} \cdot \frac{p_{10}(\exp(dj/s) - 1)}{\exp(d)}, \tag{3.23}$$

$$|\mathscr{N}^{+}_{i,j}| = \left(1 + O(n^{-\Omega(1)})\right)\frac{m}{\ell} \cdot p_{01}\exp(-d), \quad |\mathscr{N}^{-}_{i,j}| = \left(1 + O(n^{-\Omega(1)})\right)\frac{m}{\ell} \cdot p_{00}\exp(-d). \tag{3.24}$$

Proof. Since $\mathscr{N}_{i,j} = F_0[i+j-1]$, the second equation in (3.22) just follows from Proposition 2.3, **G2**. Furthermore, since $\mathscr{P}_{i,j} = \mathscr{U}_{i,j} \setminus \mathscr{N}_{i,j}$, the first equation (3.22) is an immediate consequence of Claim 3.4. Finally, to obtain (3.23)–(3.24), we simply notice that given $|\mathscr{P}_{i,j}|, |\mathscr{N}_{i,j}|$, we have

$$|\mathscr{P}^{+}_{i,j}| = \mathrm{Bin}(|\mathscr{P}_{i,j}|, p_{11}), \ |\mathscr{P}^{-}_{i,j}| = \mathrm{Bin}(|\mathscr{P}_{i,j}|, p_{10}), \ |\mathscr{N}^{+}_{i,j}| = \mathrm{Bin}(|\mathscr{N}_{i,j}|, p_{01}), \ |\mathscr{N}^{-}_{i,j}| = \mathrm{Bin}(|\mathscr{N}_{i,j}|, p_{00}).$$

Hence, (3.23)–(3.24) just follow from (3.22) and Lemma 1.5. □

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Let $\mathfrak{U}$ be the event that **G1–G2** from Proposition 2.3 hold and that the estimates (3.22)–(3.24) hold. Then by Proposition 2.3 and Claim 3.6, we have

$$\mathbb{P}\left[\mathfrak{U}\right] = 1 - o(n^{-2}). \tag{3.25}$$

To facilitate the following computations, we let

$$q_{0,j}^{-} = \exp(-d)p_{00} + (\exp(d(j-s)/s) - \exp(-d))p_{10}, \qquad q_{0,j}^{+} = \exp(-d)p_{01} + (\exp(d(j-s)/s) - \exp(-d))p_{11}. \tag{3.26}$$

Additionally, we introduce the shorthand $\lambda = (\ln\ln n)/\ln^{3/7} n$ for the error term from the definition (2.23) of $V^+[i]$. Our next step is to determine the distribution of the random variables $|\boldsymbol{W}_{x,j}^{\pm}|$ that contribute to $\boldsymbol{W}_x^{\pm}(\boldsymbol{\sigma})$ from (2.30).

Claim 3.7. *Let $s < i \leq \ell$ and $j \in [s]$. Given $\mathfrak{U}$ for every $x \in V_0^+[i]$, we have*

$$|\boldsymbol{W}_{x,j}^{-}| \sim \mathrm{Hyp}\left(\left(1 + O(n^{-\Omega(1)})\right)\frac{mq_0^-}{\ell}, \left(1 + O(n^{-\Omega(1)})\right)\frac{mq_{0,j}^-}{2\ell}, (1 + O(\lambda))\frac{\Delta}{s}p_{10}\right), \tag{3.27}$$

$$|\boldsymbol{W}_{x,j}^{+}| \sim \mathrm{Hyp}\left(\left(1 + O(n^{-\Omega(1)})\right)\frac{mq_0^+}{\ell}, \left(1 + O(n^{-\Omega(1)})\right)\frac{mq_{0,j}^+}{2\ell}, (1 + O(\lambda))\frac{\Delta s}{p_{11}}\right). \tag{3.28}$$

Proof. The definition (2.23) of $V^+[i]$ prescribes that for any $x \in V_0^+[i]$, we have $\partial x \cap F^+[i+j-1] = (p_{11} + O(\lambda))\Delta/s$. The absence or presence of $x \in V_0[i]$ in any test a does not affect the displayed results of a because x is uninfected. Therefore, the conditional distributions of $|\boldsymbol{W}_{x,j}^{\pm}|$ read

$$|\boldsymbol{W}_{x,j}^{\pm}| \sim \mathrm{Hyp}\left(|F^{\pm}[i+j-1]|, |\mathcal{N}_{i,j}^{\pm}| + |\mathcal{P}_{i,j}^{\pm}|, (1 + O(\lambda))\frac{\Delta}{s}p_{10}\right).$$

Since on $\mathfrak{U}$ the bounds (2.19)–(2.22) and (3.22)–(3.24) hold, the assertion follows. □

We are now ready to derive an exact expression for the probability that for $x \in V_0[i]$, the value $\boldsymbol{W}_x^+$ gets (almost) as large as the value W^+ that we would expected to see for an infected individual. Recall the values $q_{1,j}^{\pm}$ from (3.19).

Claim 3.8. *Let*

$$\mathcal{M}^+ = \min \frac{1}{s}\sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j \| \frac{q_{0,j}^+}{q_0^+}\right) \qquad \text{s.t.} \quad \sum_{j=1}^{s}\left(z_j - \frac{q_{1,j}^+}{p_{11}}\right)w_j^+ = 0, \qquad z_1, \ldots, z_s \in (0,1). \tag{3.29}$$

Then for all $s < i \leq \ell$ and all $x \in V[i]$, we have

$$\mathbb{P}\left[\boldsymbol{W}_x^+ \geq (1-2\zeta)W^+ \mid \mathfrak{U},\ x \in V_0^+[i]\right] \leq \exp(-(1+o(1))p_{11}\Delta\mathcal{M}^+).$$

Proof. Since $\Delta = O(\ln n)$ and $s = O(\ln\ln n)$ by (2.14), we can write

$$\begin{aligned}
&\mathbb{P}\left[\boldsymbol{W}_x^+(\boldsymbol{\sigma}) \geq (1-2\zeta)W^+ \mid \mathfrak{U},\ x \in V_0^+[i]\right] \\
&\leq \sum_{0 \leq v_1,\ldots,v_s \leq \Delta/s} \mathbb{1}\left\{\sum_{j=1}^{n} v_j w_j^+ \geq (1-2\zeta)W^+\right\}\mathbb{P}\left[\forall j \in [s] : \boldsymbol{W}_{x,j}^+(\boldsymbol{\sigma}) = v_j \mid \mathfrak{U},\ x \in V_0^+[i]\right] \\
&\leq \exp(o(\Delta)) \max_{0 \leq v_1,\ldots,v_s \leq \Delta/s} \mathbb{1}\left\{\sum_{j=1}^{n} v_j w_j^+ \geq (1-2\zeta)W^+\right\}\mathbb{P}\left[\forall j \in [s] : \boldsymbol{W}_{x,j}^+(\boldsymbol{\sigma}) = v_j \mid \mathfrak{U},\ x \in V_0^+[i]\right].
\end{aligned} \tag{3.30}$$

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Further, given $\mathfrak{U}$ and $x \in V_0^+[i]$, the random variables $(\boldsymbol{W}_{x,j}^+)_{j\in[s]}$ are mutually independent because x joins tests in the compartments $F[i+j-1]$, $j \in [s]$, independently. Hence, Claim 3.7 shows that for any $v_1, \ldots, v_s$,

$$\mathbb{P}\left[\forall j \in [s] : \boldsymbol{W}_{x,j}^+(\boldsymbol{\sigma}) = v_j \mid \mathfrak{U},\ x \in V_0^+[i]\right] = \prod_{j=1}^{s} \mathbb{P}\left[\boldsymbol{W}_{x,j}^+(\boldsymbol{\sigma}) = v_j \mid \mathfrak{U},\ x \in V_0^+[i]\right]. \tag{3.31}$$

Thus, consider the optimisation problem

$$\mathscr{M}_t^+ = \min \frac{1}{s}\sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j \| \frac{q_{0,j}^+}{q_0^+}\right) \qquad \text{s.t.} \quad \sum_{j=1}^{s} z_j w_j^+ \geq (1-t)W^+, \quad z_1, \ldots, z_s \in [q_{0,j}^+/q_0^+, 1]. \tag{3.32}$$

Then combining (3.30) and (3.31) with Claim 3.7 and Lemma 1.6 and using the substitution $z_j = v_j/(p_{11}\Delta)$, we obtain

$$\mathbb{P}\left[\boldsymbol{W}_x^+ \geq (1-2\zeta)W^+ \mid \mathfrak{U},\ x \in V_0^+[i]\right] \leq \exp\left(-\Delta p_{11}\mathscr{M}_{2\zeta}^+ + o(\Delta)\right). \tag{3.33}$$

We claim that (3.33) can be sharpened to

$$\mathbb{P}\left[\boldsymbol{W}_x^+ \geq (1-2\zeta)W^+ \mid \mathfrak{U},\ x \in V_0^+[i]\right] \leq \exp\left(-\Delta p_{11}\mathscr{M}_{0}^+ + o(\Delta)\right). \tag{3.34}$$

Indeed, consider any feasible solution $z_1, \ldots, z_s$ to $\mathscr{M}_\zeta^+$ such that $\sum_{j\geq s} z_j w_j^+ < W^+$. Obtain $z_1', \ldots, z_s'$ by increasing some of the values $z_1, \ldots, z_s$ such that $\sum_{j\leq s} z_j' w_j^+ = W^+$. Then because the functions $z \mapsto D_{\mathrm{KL}}(z \| q_{0,j}^+/q_0^+)$ with $j \in [s]$ are equicontinuous on the compact interval $[0, 1]$, we see that

$$\frac{1}{s}\sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j \| q_{0,j}^+/q_0^+\right) \geq \frac{1}{s}\sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j' \| q_{0,j}^+/q_0^+\right) + o(1)$$

uniformly for all $z_1, \ldots, z_s$ and $z_1', \ldots, z_s'$. Thus, (3.34) follows from (3.33).

Finally, we notice that the condition $z_j \geq q_{0,j}^+/q_0^+$ in (3.32) is superfluous. Indeed, since $D_{\mathrm{KL}}(q_{0,j}^+/q_0^+ \| q_{0,j}^+/q_0^+) = 0$, there is nothing to be gained from choosing $z_j < q_{0,j}^+/q_0^+$. Furthermore, since the Kullback–Leibler divergence is strictly convex and (1.1) ensures that $q_{0,j}^+/q_0^+ < q_{1,j}^+/p_{11}$ for all j, the optimisation problem $\mathscr{M}_0^+$ attains a unique minimum, which is not situated at the boundary of the intervals $[q_{0,j}^+/q_0^+, 1]$. That said, the unique minimiser satisfies the weight constraint $\sum_{j\geq s} z_j w_j^+$ with equality; otherwise, we could reduce some z_j, thereby decreasing the objective function value. In summary, we conclude that $\mathscr{M}_0^+ = \mathscr{M}^+$. Thus, the assertion follows from (3.34). □

Claim 3.9. *Let*

$$\mathscr{M}^- = \min \frac{1}{s}\sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j \| \frac{q_{0,j}^-}{q_0^-}\right) \qquad \text{s.t.} \quad \sum_{j=1}^{s}\left(z_j - \frac{q_{1,j}^-}{p_{01}}\right) w_j^- = 0, \qquad z_1, \ldots, z_s \in (0, 1). \tag{3.35}$$

Then for all $s < i \leq \ell$ and all $x \in V[i]$, we have

$$\mathbb{P}\left[\boldsymbol{W}_x^- \leq (1+2\zeta)W^- \mid \mathfrak{U},\ x \in V_0^+[i]\right] \leq \exp\left(-(1+o(1))p_{10}\Delta\mathscr{M}^-\right).$$

Recall that by convention $0 \cdot \infty = 0$. Thus for $p_{10} = 0$, the condition of (3.35) boils down to $z_j = q_{1,j}^-/p_{01}$, and the optimisation becomes trivial.

Proof. Analogous to the proof of Claim 3.8. □

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Clearly, as a next step, we need to solve the optimisation problems (3.29) and (3.35). We defer this task to Section 3.6, where we prove the following.

Lemma 3.10. *Let* $\boldsymbol{X}$ *have distribution* $Be(\exp(-d))$, *and let* $\boldsymbol{Y}$ *be the (random) channel output given input* $\boldsymbol{X}$. *Then*

$$p_{11}\mathscr{M}^+ + p_{10}\mathscr{M}^- = \frac{I(\boldsymbol{X},\boldsymbol{Y})}{d} - D_{\mathrm{KL}}\left(p_{11}\|q_0^+\right).$$

Proof of Lemma 3.2 *(large deviations for healthy individuals).* In light of Claims 3.8 and 3.9 and Lemma 3.10 to work out that all but $o(k)$ positive individuals are identified correctly, using Markov's inequality, we need to verify that

$$\left|V_0^+\right| \exp\left(-\Delta\left(p_{11}\mathscr{M}^+ + p_{10}\mathscr{M}^-\right)\right) < k \tag{3.36}$$

Taking the logarithm of (3.36) and simplifying, we arrive at

$$\ln(n)\left(1 - cd(1-\theta)\left(D_{\mathrm{KL}}\left(p_{11}\|(1-\exp(-d))p_{11} + \exp(-d)p_{01}\right) + p_{11}\mathscr{M}^+ + p_{10}\mathscr{M}^-\right)\right) < \theta \ln(n). \tag{3.37}$$

Thus, we need to show that

$$cd\left[D_{\mathrm{KL}}\left(p_{11}\|(1-\exp(-d))p_{11} + \exp(-d)p_{01}\right) + p_{11}\mathscr{M}^+ + p_{10}\mathscr{M}^-\right] > 1. \tag{3.38}$$

This boils down to $cI(\boldsymbol{X},\boldsymbol{Y}) > 1$, which in turn is identical to (2.34). □

3.6. Proof of Lemma 3.10

We tackle the optimisation problems $\mathscr{M}^\pm$ via the method of Lagrange multipliers. Since the objective functions are strictly convex, these optimisation problems possess unique stationary points. As the parameters from (3.19) satisfy $q_{1,j}^+/p_{11} = \exp(d(j-s)/s)$, the optimisation problem (3.29) gives rise to the following Lagrangian.

Claim 3.11. *The Lagrangian*

$$\mathscr{L}^+ = \sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j \Big\| \frac{q_{0,j}^+}{q_0^+}\right) + \lambda w_j^+\left(z_j - \exp(-d(s-j)/s)\right)$$

has the unique stationary point $z_j = \exp(-d(s-j)/s)$, $\lambda = -1$.

Proof. Since the objective function $\sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j\|q_{1,j}^+/p_{11}\right)$ is strictly convex, we just need to verify that $\lambda = -1$ and $z_j = \exp(-d(s-j)/s)$ is a stationary point. To this end, we calculate

$$\frac{\partial \mathscr{L}^+}{\partial z_j} = \ln\frac{z_j^+}{1-z_j^+} - \ln\frac{q_{1,j}^+}{p_{11}-q_{1,j}^+} + \lambda w_j^+, \qquad \frac{\partial \mathscr{L}^+}{\partial \lambda} = \sum_{j=1}^{s}\left(z_j - \exp\left(-d(s-j)/s\right)\right) w_j^+. \tag{3.39}$$

Substituting in the definition (2.29) of the weights w_j^+ and the definition (3.19) of $p_{11}, q_{1,j}^+$ and simplifying, we obtain

$$\left.\frac{\partial L^+}{\partial z_j}\right|_{\substack{z_j=\exp(-d(s-j)/s)\\ \lambda=-1}} = \ln\frac{\exp(-d(s-j)/s)}{1-\exp(-d(s-j)/s)} - \ln\frac{p_{11}(\exp(dj/s)-1)+p_{01}}{p_{11}(\exp(d)-1)+p_{01}} + \ln\left(1 - \frac{p_{11}(\exp(dj/s)-1)+p_{01}}{p_{11}(\exp(d)-1)+p_{01}}\right) - \ln\frac{p_{11}}{p_{11}+(p_{01}-p_{11})\exp(-dj/s)} = 0.$$

Finally, (3.39) shows that setting $z_j = \exp(-d(s-j)/s)$ ensures that $\partial\mathscr{L}^+/\partial\lambda = 0$ as well. □

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Analogous steps prove the corresponding statement for $\mathscr{M}^-$.

Claim 3.12. *Assume $p_{10} > 0$. The Lagrangian*

$$\mathscr{L}^- = \sum_{j=1}^{s} D_{\mathrm{KL}}\left(z_j \| \frac{q_{0,j}^-}{p_{01}}\right) + \lambda w_j^- \left(z_j - \exp\left(-d(s-j)/s\right)\right)$$

has the unique stationary point $z_j = \exp\left(-d(s-j)/s\right), \lambda = -1$.

Having identified the minimisers of $\mathscr{M}^{\pm}$, we proceed to calculate the optimal objective function values. Note that for $\mathscr{M}^-$, the minimisers z_j for the cases $p_{10} > 0$ and $p_{10} = 0$ coincide.

Claim 3.13. *Let*

$$\begin{aligned}\lambda^+ &= \ln\left(q_0^+\right) = \ln\left(p_{01}\exp\left(-d\right) + p_{11}(1-\exp\left(-d\right))\right), \lambda^- = \ln\left(q_0^-\right) \\ &= \ln\left(p_{00}\exp\left(-d\right) + p_{10}(1-\exp\left(-d\right))\right).\end{aligned}$$

Then

$$\begin{aligned}p_{11} d\exp\left(d\right)\mathscr{M}^+ =& (\lambda^+ + d)\left(p_{11}\left((d-1)\exp\left(d\right)+1\right) - p_{01}\right) + p_{01}\ln\left(p_{01}\right) \\ &- p_{11}\left((d-1)\exp\left(d\right)+1\right)\ln\left(p_{11}\right) - d(d-1)\exp\left(d\right)p_{11} + O(1/s), \\ p_{10} d\exp\left(d\right)\mathscr{M}^- =& (\lambda^- + d)\left(p_{10}\left((d-1)\exp\left(d\right)+1\right) - p_{00}\right) + p_{00}\ln\left(p_{00}\right) \\ &- p_{10}\left((d-1)\exp\left(d\right)+1\right)\ln\left(p_{10}\right) - d(d-1)\exp\left(d\right)p_{10} + O(1/s) \quad \text{if } p_{10} > 0.\end{aligned}$$

Proof. We perform the calculation for $\mathscr{M}^+$ in detail. Syntactically identical steps yield the expression for $\mathscr{M}^-$, the only required change being the obvious modification of the indices of the channel probabilities. Substituting the optimal solution $z_j = \exp\left(-d(s-j)/s\right)$ and the definitions (3.19) and (2.1), (3.19), and (3.26) of $q_{1,j}^+, q_0^+, q_{0,j}^+$ into (3.29), we obtain

$$\mathscr{M}^+ = \frac{1}{s}\sum_{j=1}^{s} D_{\mathrm{KL}}\left(\exp\left(-d(s-j)/s\right) \| \frac{p_{01} + (\exp(dj/s)-1)p_{11}}{p_{01} + (\exp(d)-1)p_{11}}\right) = \mathscr{I}^+ + O(1/s), \qquad \text{where} \tag{3.40}$$

$$\mathscr{I}^+ = \int_0^1 D_{\mathrm{KL}}\left(\exp\left(d(x-1)\right) \| \frac{p_{11}(\exp(dx)-1) + p_{01}}{p_{11}(\exp(d)-1) + p_{01}}\right) \mathrm{d}x;$$

the $O(1/s)$-bound in (3.40) holds because the derivative of the integrand $x \mapsto D_{\mathrm{KL}}\left(\exp\left(d(x-1)\right) \| \frac{p_{11}(\exp(dx)-1)+p_{01}}{p_{11}(\exp(d)-1)+p_{01}}\right)$ is bounded on $[0,1]$. Replacing the Kullback–Leibler divergence by its definition, we obtain $\mathscr{I}^+ = \mathscr{I}_1^+ + \mathscr{I}_2^+$, where

$$\begin{aligned}\mathscr{I}_1^+ &= \int_0^1 \exp\left(d(x-1)\right)\ln\frac{\exp\left(d(x-1)\right)\left(p_{11}(\exp\left(d\right)-1)+p_{01}\right)}{p_{11}(\exp\left(dx\right)-1)+p_{01}}\mathrm{d}x, \\ \mathscr{I}_2^+ &= \int_0^1 \left(1-\exp\left(d(x-1)\right)\right)\ln\left[\frac{1-\exp\left(d(x-1)\right)}{1-\frac{p_{11}(\exp(dx)-1)+p_{01}}{p_{11}(\exp(d)-1)+p_{01}}}\right]\mathrm{d}x.\end{aligned}$$

Splitting the logarithm in the first integrand, we further obtain $\mathscr{I}_1^+ = \mathscr{I}_{11}^+ + \mathscr{I}_{12}^+$, where

$$\begin{aligned}\mathscr{I}_{11}^+ &= \int_0^1 \exp\left(d(x-1)\right)\ln\left[\exp\left(d(x-1)\right)\left(p_{11}(\exp\left(d\right)-1)+p_{01}\right)\right]\mathrm{d}x, \\ \mathscr{I}_{12}^+ &= -\int_0^1 \exp\left(d(x-1)\right)\ln\left[p_{11}(\exp\left(dx\right)-1)+p_{01}\right]\mathrm{d}x.\end{aligned}$$

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Setting $\Lambda^+ = \ln\left(p_{11}(\exp\left(d\right) - 1) + p_{01}\right) = \lambda^+ + d$ and introducing $u = \exp\left(d(x-1)\right)$, we calculate

$$\mathscr{I}_{11}^+ = \frac{1}{d}\left[\int_{\exp(-d)}^{1} \ln(u)\mathrm{d}u + \int_{\exp(-d)}^{1} \Lambda^+\mathrm{d}u\right] = \frac{1}{d}\left[(d+1)\exp(-d) - 1 + (1-\exp\left(-d\right))\Lambda^+\right]. \tag{3.41}$$

Concerning $\mathscr{I}_{12}^+$, we once again substitute $u = \exp\left(d(x-1)\right)$ to obtain

$$\begin{aligned}\mathscr{I}_{12}^+ &= -\frac{1}{d}\int_{\exp(-d)}^{1} \ln\left(p_{11}\exp(d)u + p_{01} - p_{11}\right)\mathrm{d}u\\ &= -\frac{1}{d}\left[\left(\frac{p_{01}}{p_{11}}\exp\left(-d\right) - \exp\left(-d\right) + 1\right)\Lambda^+ - \exp\left(-d\right)\frac{p_{01}}{p_{11}}\ln(p_{01}) + \exp\left(-d\right) - 1\right]\end{aligned} \tag{3.42}$$

Proceeding to $\mathscr{I}_2$, we obtain

$$\begin{aligned}\mathscr{I}_2^+ &= \int_0^1 (1-\exp\left(d(x-1)\right))\ln\frac{p_{11}(\exp\left(d\right)-1)+p_{01}}{p_{11}\exp(d)}\mathrm{d}x\\ &= \int_0^1 (1-\exp\left(d(x-1)\right))\ln\left(p_{11}(\exp(d)-1)+p_{01}\right)\mathrm{d}x\\ &\quad - \int_0^1 (1-\exp\left(d(x-1)\right))\ln(p_{11}\exp(d))\mathrm{d}x\\ &= \frac{(\Lambda^+ - \ln(p_{11}) - d)(d-1+\exp\left(-d\right))}{d}.\end{aligned} \tag{3.43}$$

Finally, recalling that $\mathscr{I}^+ = \mathscr{I}_1 + \mathscr{I}_2 = \mathscr{I}_{11}^+ + \mathscr{I}_{12}^+ + \mathscr{I}_2^+$ and combining (3.40)–(3.43) and simplifying, we arrive at the desired expression for $\mathscr{M}^+$. □

Proof of Lemma 3.10. We have

$$\begin{aligned}I(\boldsymbol{X},\boldsymbol{Y}) = H(\boldsymbol{Y}) - H(\boldsymbol{Y}|\boldsymbol{X}) &= h(p_{00}\exp(-d) + p_{10}(1-\exp(-d))) - \exp(-d)h(p_{00})\\ &\quad - (1-\exp(-d))h(p_{11}).\end{aligned}$$

Hence, Claim 3.13 yields

$$\begin{aligned}p_{11}\mathscr{M}^+ + p_{10}\mathscr{M}^- &= -\frac{h(p_{00})}{d\exp(d)} - \frac{(1-\exp(-d))h(p_{11})}{d} + h(p_{11})\\ &\quad + \frac{(p_{11}-p_{01})\lambda^+ + (p_{10}-p_{00})\lambda^-}{d\exp(d)} + \frac{d-1}{d}\left[p_{11}\lambda^+ + p_{10}\lambda^-\right]\\ &= -\frac{h(p_{00})}{d\exp(d)} - \frac{(1-\exp(-d))h(p_{11})}{d} - D_{\mathrm{KL}}\left(p_{11}\|q_0^+\right) - \frac{\lambda^+}{d}\exp(\lambda^+)\\ &\quad - \frac{\lambda^-}{d}\exp(\lambda^-)\\ &= \frac{I(\boldsymbol{X},\boldsymbol{Y})}{d} - D_{\mathrm{KL}}\left(p_{11}\|q_0^+\right),\end{aligned}$$

as desired. □

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4. Analysis of the exact recovery algorithm

In this section, we establish Propositions 2.7 and 2.8, which are the building blocks of the proof of Theorem 1.3. Remember that SPEX uses the result of SPARC and performs $O(\ln(n))$ cleanup steps to fix possible mistakes. Each of these cleanup steps updates the estimate via thresholding for every individual. Proposition 2.7 ensures that this thresholding is algorithmically possible as intended. Proposition 2.8 then guarantees that every single cleanup step decreases the number of mistakes enough. We continue to work with the spatially coupled design $\mathbf{G}_{\mathrm{sc}}$ from Section 2.3 and keep the notation and assumptions from (2.13)–(2.17).

4.1. Proof of Proposition 2.7

Assume that $c > c_{\mathrm{ex},1}(d,\theta) + \varepsilon$, and let $c' = c_{\mathrm{ex},1}(d,\theta) + \varepsilon/2$. Since $c > c' + \varepsilon/2$, the definitions (1.11) of $c_{\mathrm{ex},1}(d,\theta)$ and (1.6) of $\mathscr{Y}(c', d, \theta)$ ensure that for small enough $\delta > 0$, we can find an open interval $\mathscr{I} \subseteq \mathscr{Y}(c', d, \theta)$ with rational boundary points such that **Z1** is satisfied.

Let $\bar{\mathscr{I}}$ be the closure of $\mathscr{I}$. Then by the definition of $c_{\mathrm{ex},1}(d,\theta)$, there exists a function $z : \bar{\mathscr{I}} \to [p_{01}, p_{11}]$ such that for all $y \in \bar{\mathscr{I}}$ we have

$$cd(1-\theta)\left[D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z(y)\|p_{11}\right)\right] = \theta. \tag{4.1}$$

In fact, because the Kullback–Leibler divergence is strictly convex, the equation (4.1) defines $z(y)$ uniquely. The inverse function theorem implies that the function $z(y)$ is continuous and therefore uniformly continuous on $\bar{\mathscr{I}}$. Additionally, once again, because the Kullback–Leibler divergence is convex and $c > c_{\mathrm{ex},1}(d,\theta)$, for all $y \in \bar{\mathscr{I}}$, we have

$$cd(1-\theta)\left[D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z(y)\|p_{01}\right)\right] > 1.$$

Therefore, there exists $\hat{\delta} = \hat{\delta}(c, d, \theta) > 0$ such that for all $y \in \bar{\mathscr{I}}$, we have

$$cd(1-\theta)\left[D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z(y)\|p_{01}\right)\right] > 1 + \hat{\delta}. \tag{4.2}$$

Combining (4.1) and (4.2), we find a continuous $\hat{z} : \bar{\mathscr{I}} \to [p_{01}, p_{11}]$ such that for small enough $\delta > 0$ for all $y \in [0, 1]$, we have

$$cd(1-\theta)\left[D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(\hat{z}(y)\|p_{11}\right)\right] > \theta + 2\delta, \qquad \text{and} \tag{4.3}$$

$$cd(1-\theta)\left[D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(\hat{z}(y)\|p_{01}\right)\right] > 1 + 2\delta. \tag{4.4}$$

Additionally, by uniform continuity for any given $0 < \varepsilon' < \delta/2$ (which may depend arbitrarily on δ and $\mathscr{I}$), we can choose $\delta' > 0$ small enough so that

$$|\hat{z}(y) - \hat{z}(y')| < \varepsilon'/2 \qquad \text{for all } y, y' \in \bar{\mathscr{I}} \text{ with} |y - y'| < \delta'. \tag{4.5}$$

Finally, let $y_0, \ldots, y_\nu$ with $\nu = \nu(\delta', \varepsilon') > 0$ be a large enough number of equally spaced points in $\bar{\mathscr{I}} = [y_0, y_\nu]$. Then for each i, we pick $\mathscr{Z}(y_i) \in [p_{01}, p_{11}] \cap \mathbb{Q}$ such that $|\hat{z}(y_i) - \mathscr{Z}(y_i)|$ is small enough. Extend $\mathscr{Z}$ to a step function $\bar{\mathscr{I}} \to \mathbb{Q} \cap [0, 1]$ by letting $\mathscr{Z}(y) = \mathscr{Z}(y_{i-1})$ for all $y \in (y_{i-1}, y_i)$ for $1 \le i \le \nu$. Since $y \mapsto \hat{z}(y)$ is uniformly continuous, we can choose ν large enough so that (4.3)–(4.5) imply

$$\begin{aligned}
&cd(1-\theta)\left[D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(\mathscr{Z}(y)\|p_{11}\right)\right] > \theta + \delta && \text{for all } y \in \bar{\mathscr{I}},\\
&cd(1-\theta)\left[D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(\mathscr{Z}(y)\|p_{01}\right)\right] > 1 + \delta && \text{for all } y \in \bar{\mathscr{I}},\\
&\qquad\qquad |\mathscr{Z}(y) - \mathscr{Z}(y')| < \varepsilon' && \text{for all } y, y' \in \bar{\mathscr{I}} \text{ such that } |y - y'| < \delta',
\end{aligned}$$

as claimed.

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4.2. Proof of Proposition 2.8

As in the proof of Proposition 2.6 in Section 3.3, we will first investigate an idealised scenario where we assume that the ground truth $\boldsymbol{\sigma}$ is known. Then we will use the expansion property provided by Lemma 3.3 to connect this idealised scenario with the actual steps of the algorithm.

In order to study the idealised scenario, for $x \in V[i]$ and $j \in [s]$, we introduce

$$\boldsymbol{Y}_{x,j} = \left|\left\{a \in F[i+j-1] \cap \partial x : V_1 \cap \partial a \subseteq \{x\}\right\}\right|, \quad \boldsymbol{Z}_{x,j} = \left|\left\{a \in F^+[i+j-1] \cap \partial x : V_1 \cap \partial a \subseteq \{x\}\right\}\right|,$$
$$\boldsymbol{Y}_x = \sum_{j=1}^{s} \boldsymbol{Y}_{x,j}, \qquad \boldsymbol{Z}_x = \sum_{j=1}^{s} \boldsymbol{Z}_{x,j}.$$

Thus, $\boldsymbol{Y}_{x,j}$ is the number of untainted tests in compartment $F[i+j-1]$ that contain x, that is, test that does not contain another infected individual. Moreover, $\boldsymbol{Z}_{x,j}$ is the number of positively displayed untainted tests. Finally, $\boldsymbol{Y}_x, \boldsymbol{Z}_x$ are the sums of these quantities on $j \in [s]$. The following lemma provides an estimate of the number of individuals x with a certain value of $\boldsymbol{Y}_x$.

Lemma 4.1. *w.h.p., for all $0 \le Y \le \Delta$ and all $i \in [\ell]$, we have*

$$\sum_{x \in V_0[i]} \mathbb{1}\{\boldsymbol{Y}_x = Y\} \le n \exp\left(-\Delta D_{\mathrm{KL}}\left(Y/\Delta \| \exp(-d)\right) + o(\Delta)\right), \tag{4.6}$$
$$\sum_{x \in V_1[i]} \mathbb{1}\{\boldsymbol{Y}_x = Y\} \le k \exp\left(-\Delta D_{\mathrm{KL}}\left(Y/\Delta \| \exp(-d)\right) + o(\Delta)\right). \tag{4.7}$$

Proof. Let $1 \le i \le \ell$ and consider any $x \in V[i]$. Further, obtain $\mathbf{G}_{\mathrm{sc}} - x$ from $\mathbf{G}_{\mathrm{sc}}$ by deleting individual x (and, of course, removing x from all tests). Additionally, obtain $\mathbf{G}'_{\mathrm{sc}}$ from $\mathbf{G}_{\mathrm{sc}} - x$ by re-inserting x and assigning x to Δ/s random tests in the compartments $F[i+j-1]$ for $j \in [s]$ as per the construction of the spatially coupled test design. Then the random test designs $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}'_{\mathrm{sc}}$ are identically distributed.

Let $\mathfrak{E}$ be the event that $\mathbf{G}_{\mathrm{sc}}$ enjoys properties **G1** and **G2** from Proposition 2.3. Then Proposition 2.3 shows that

$$\mathbb{P}\left[\mathfrak{E}\right] = 1 - o(n^{-2}). \tag{4.8}$$

Moreover, given $\mathfrak{E}$ for every $j \in [s]$, the number of tests in $F[i+j-1]$ that contain no infected individual aside from x satisfies

$$\sum_{a \in F[i+j-1]} \mathbb{1}\{\partial a \cap V_1 \setminus \{x\} = \emptyset\} = (1 + O(n^{-\Omega(1)}))\frac{m}{\ell}\exp(-d); \tag{4.9}$$

this follows from the bounds on $F_0[i+j-1]$ provided by **G2** and the fact that discarding x can change the numbers of actually positive/negative tests by no more than Δ.

Now consider the process of re-inserting x to obtain $\mathbf{G}'_{\mathrm{sc}}$. Then (4.9) shows that given $\mathfrak{E}$, we have

$$\boldsymbol{Y}_{x,j} \sim \mathrm{Hyp}\left(\frac{m}{\ell} + O(1), (1 + O(n^{-\Omega(1)}))\frac{m}{\ell}\exp(-d), \frac{\Delta}{s}\right) \qquad (j \in [s]).$$

These hypergeometric variables are mutually independent given $\mathbf{G}_{\mathrm{sc}} - x$. Therefore, Lemma 1.6 implies that on $\mathfrak{E}$,

$$\mathbb{P}\left[\boldsymbol{Y}_x = Y \mid \mathbf{G}_{\mathrm{sc}} - x\right] \le \exp\left(-\Delta D_{\mathrm{KL}}\left(Y/\Delta \| \exp(-d)\right) + o(\Delta)\right). \tag{4.10}$$

This estimate holds independently of the infection status $\boldsymbol{\sigma}_x$. Thus, the assertion follows from (4.8), (4.10), and Markov's inequality. □

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As a next step, we argue that for c beyond the threshold $c_{\mathrm{ex},1}(d,\theta)$, the function $\mathscr{X}$ from Proposition 2.7 separates the infected from the uninfected individuals w.h.p.

Lemma 4.2. *Assume that* $c > c^*(d,\theta)+\varepsilon$. *Let* $\mathscr{I}=(l,r)$, $\delta>0$ *be the interval and the number from Proposition 2.7, choose* $\varepsilon'>0$ *sufficiently small, and let* δ', $\mathscr{X}$ *be such that* ***Z1–Z4*** *are satisfied.*

- *(i) For all* $x\in V_1$, *we have* $\mathbf{Y}_x/\Delta\in(l+\varepsilon', r-\varepsilon')$ *and* $\mathbf{Z}_x/\Delta>\mathscr{X}(\mathbf{Y}_x/\Delta)+3\varepsilon'$.
- *(ii) For all* $x\in V_0$ *with* $\mathbf{Y}_x/\Delta\in\mathscr{I}$, *we have* $\mathbf{Z}_x/\Delta<\mathscr{X}(\mathbf{Y}_x/\Delta)-3\varepsilon'$.

Proof. Let $\mathfrak{E}$ be the event that the bounds (4.6)–(4.7) hold for all $0\le Y\le\Delta$. Then (4.7) and Proposition 2.7, **Z1** show that w.h.p. $\mathbf{Y}_x/\Delta\in(l+\varepsilon',r-\varepsilon')$ for all $x\in V_1$, provided $\varepsilon'>0$ is small enough. Moreover, for a fixed $0\le Y\le\Delta$ such that $Y/\Delta\in\mathscr{I}$ and $i\in[\ell]$, let $\boldsymbol{X}_1(Y)$ be the number of variables $x\in V_1[i]$ such that $\mathbf{Y}_x=Y$ and $\mathbf{Z}_x\le\Delta\mathscr{X}(Y/\Delta)+3\varepsilon'\Delta$. Since x itself is infected, all tests $a\in\partial x$ are actually positive. Therefore, a is displayed positively with probability p_{11}. As a consequence, Lemma 1.5 shows that

$$\mathbb{P}\left[\mathbf{Z}_x\le\Delta\mathscr{X}(\mathbf{Y}_x/\Delta)+3\varepsilon'\Delta\mid\mathbf{Y}_x=Y\right]\le\exp\left(-YD_{\mathrm{KL}}\left(\mathscr{X}(Y/\Delta)+3\varepsilon'\Delta/Y\|p_{11}\right)+o(\Delta)\right). \tag{4.11}$$

Combining (4.7) and (4.11), recalling that $k=\lceil n^\theta\rceil$ and choosing $\varepsilon'>0$ sufficiently small, we obtain

$$\begin{aligned}&\mathbb{E}\left[\sum_{x\in V_1[i]}\mathbb{1}\left\{\mathbf{Y}_x=Y,\ \mathbf{Z}_x\le\Delta\mathscr{X}(\mathbf{Y}_x/\Delta)+3\varepsilon'\Delta\right\}\mid\mathfrak{E}\right]\\&\quad\le k\exp\left(-\Delta D_{\mathrm{KL}}\left(Y/\Delta\|\exp(-d)\right)-YD_{\mathrm{KL}}\left(\mathscr{X}(Y/\Delta)+3\varepsilon'\Delta/Y\|p_{11}\right)+o(\Delta)\right) &(4.12)\\&\quad\le n^{-\Omega(1)}\qquad[\text{ due to Proposition 2.7, }\mathbf{Z2}]. &(4.13)\end{aligned}$$

Taking a union bound on the $O(\ln^2 n)$ possible combinations (i,Y), we see that (i) follows from (4.13). A similar argument based on Proposition 2.7, **Z3** yields (ii). □

Proof of Proposition 2.8. For $t=1\ldots\lceil\ln n\rceil$, consider the set of misclassified individuals after $t-1$ iterations:

$$\mathscr{M}_t=\left\{x\in V[s+1]\cup\cdots V[\ell]:\tau_x^{(t)}\ne\boldsymbol{\sigma}_x\right\}.$$

Propositions 2.5 and 2.6 show that w.h.p., the size of the initial set satisfies

$$|\mathscr{M}_1|\le k\exp\left(-\Omega(\ln^{1/8}n)\right). \tag{4.14}$$

We are going to argue by induction that $|\mathscr{M}_t|$ decays geometrically. Apart from the bound (4.14), this argument depends on only two conditions. First, that the random graph $\mathbf{G}_{\mathrm{sc}}$ indeed enjoys the expansion property from Lemma 3.3. Second, that (i)–(ii) from Lemma 4.2 hold. Let $\mathfrak{E}$ be the event that these two conditions are satisfied and that (4.14) holds. Propositions 2.5 and 2.6 and Lemmas 3.3 and 4.2 show that $\mathbb{P}[\mathfrak{E}]=1-o(1)$.

To complete the proof, we are going to show by induction on $t\ge 2$ that on $\mathfrak{E}$,

$$|\mathscr{M}_t|\le|\mathscr{M}_{t-1}|/3. \tag{4.15}$$

Indeed, consider the set

$$\mathscr{M}_t^*=\left\{x\in V[s+1]\cup\cdots V[\ell]:\sum_{a\in\partial x\setminus F[0]}|\partial a\cap\mathscr{M}_{t-1}\setminus\{x\}|\ge\Delta/\ln\ln n\right\}.$$

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Since by (4.14) and induction we know that $|\mathcal{M}_{t-1}| \le k \exp\left(-\Omega(\ln^{1/8} n)\right)$, the expansion property from Lemma 3.3 implies that $\mathcal{M}_t^* \le \mathcal{M}_{t-1}/3$. Therefore, to complete the proof of (4.15), it suffices to show that $\mathcal{M}_t \subseteq \mathcal{M}_t^*$.

To see this, suppose that $x \in \mathcal{M}_t$.

Case 1: $x \in V_1$ but $Y_x(\tau^{(t-1)})/\Delta \notin \mathcal{I}$ Lemma 4.2 (i) ensures that $\boldsymbol{Y}_x/\Delta \in (l+\varepsilon', r-\varepsilon')$. Therefore, the case $Y_x(\tau^{(t-1)})/\Delta \notin \mathcal{I}$ can occur only if at least $\varepsilon'\Delta$ tests $a \in \partial x$ contain a misclassified individual $x' \in \mathcal{M}_{t-1}$. Hence, $x \in \mathcal{M}_t^*$.

Case 2: $x \in V_1$ and $Y_x(\tau^{(t-1)})/\Delta \notin \mathcal{I}$ but $Z_x(\tau^{(i)})/\Delta \le \mathcal{Z}(Y_x(\tau^{(i)})/\Delta)$ by Lemma 4.2, (i) we have $\boldsymbol{Z}_x/\Delta > \mathcal{Z}(\boldsymbol{Y}_x/\Delta) + 2\varepsilon'$. Thus, if $Z_x(\tau^{(t-1)})/\Delta \le \mathcal{Z}(Y_x(\tau^{(t-1)})/\Delta)$, then by the continuity property **Z4**, we have $|Y_x(\tau^{(t-1)}) - \boldsymbol{Y}_x| > \varepsilon'\Delta$. Consequently, as in Case 1 we have $x \in \mathcal{M}_t^*$.

Case 3: $x \in V_0$ as in the previous cases, due to **Z4** and Lemma 4.2, (ii) the event $x \in \mathcal{M}_t$ can occur only if $|\boldsymbol{Y}_x - Y_v(\tau^{(t-1)})| > \varepsilon'\Delta$. Thus, $x \in \mathcal{M}_t^*$.

Hence, $\mathcal{M}_t \subseteq \mathcal{M}_t^*$, which completes the proof. □

5. Lower bound on the constant column design

5.1. Proof of Proposition 2.10

The following lemma is an adaptation of Proposition 2.3 (**G2**) to $\mathbf{G}_{cc}$.

Lemma 5.1. *The random graph* $\mathbf{G}_{cc}$ *enjoys the following properties with probability* $1 - o(n^{-2})$*:*

$$m\exp(-d)p_{00} - \sqrt{m}\ln^3 n \le \left|F_0^-\right| \le m\exp(-d)p_{00} + \sqrt{m}\ln^3 n, \tag{5.1}$$

$$m\exp(-d)p_{01} - \sqrt{m}\ln^3 n \le \left|F_0^+\right| \le m\exp(-d)p_{01} + \sqrt{m}\ln^3 n, \tag{5.2}$$

$$m(1-\exp(-d))p_{10} - \sqrt{m}\ln^3 n \le \left|F_1^-\right| \le m(1-\exp(-d))p_{10} + \sqrt{m}\ln^3 n, \tag{5.3}$$

$$m(1-\exp(-d))p_{11} - \sqrt{m}\ln^3 n \le \left|F_1^+\right| \le m(1-\exp(-d))p_{11} + \sqrt{m}\ln^3 n. \tag{5.4}$$

The proof of Lemma 5.1 is similar to that of Proposition 2.3 (see Section 3.1); the details are thus omitted.

Proof of Proposition 2.10. The definition (2.2) of the weight functions ensures that

$$\ln \psi_{\mathbf{G}_{cc},\boldsymbol{\sigma}''}(\boldsymbol{\sigma}) = |F_0^-|\ln p_{00} + |F_0^+|\ln p_{01} + |F_1^-|\ln p_{10} + |F_1^+|\ln p_{11}.$$

Substituting in the estimates from (5.1) to (5.4) completes the proof. □

5.2. Proof of Proposition 2.11 (indistinguishable configurations with high overlap)

Let $\mathcal{X}_r(Y)$ be the set of individuals $x \in V_r$ such that

$$\sum_{a \in \partial x} \mathbb{1}\{\partial a \setminus \{x\} \subseteq V_0\} = Y.$$

Hence, x participates in precisely Y tests that do not contain another infected individual.

Lemma 5.2. *Let* $y \in \mathcal{Y}(c, d, \theta)$ *be such that* $y\Delta$ *is an integer. Then w.h.p., we have*

$$|\mathcal{X}_0(y\Delta)| = n\exp\left(-\Delta D_{KL}\left(y \| \exp(-d)\right) + o(\Delta)\right), \tag{5.5}$$

$$|\mathcal{X}_1(y\Delta)| = k\exp\left(-\Delta D_{KL}\left(y \| \exp(-d)\right) + o(\Delta)\right). \tag{5.6}$$

Proof. Let $Y = y\Delta$, and let $\mathcal{E}$ be the event that the bounds (5.1)–(5.4) hold. We begin by computing $\mathbb{E}|\mathcal{X}_1(Y)|$. By exchangeability, we may condition on the event $\mathcal{S} = \{\boldsymbol{\sigma}_{x_1} = \cdots = \boldsymbol{\sigma}_{x_k} = 1\}$, that is, precisely the first k individuals are infected. Hence, by the linearity of expectation, it suffices to

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prove that

$$\mathbb{P}\left[x_1 \in \mathscr{X}_1(Y) \mid \mathscr{E}, \mathscr{S}\right] = \exp\left(-\Delta D_{\mathrm{KL}}\left(y \| \exp(-d)\right) + o(\Delta)\right). \tag{5.7}$$

Let $\boldsymbol{G}' = \mathbf{G}_{\mathrm{cc}} - x_1$ be the random design without x_1, and let F_0' be the set of actually negative tests of $\boldsymbol{G}'$. Given $\boldsymbol{m}_0' = |F_0'|$, the number of tests $a \in \partial x_1$ such that $\partial a \setminus \{x_1\} \subseteq V_0$ has distribution $\mathrm{Hyp}(m, \boldsymbol{m}_0', \Delta)$ because x_1 joins precisely Δ tests independently of all other individuals. Hence, (1.19) yields

$$\mathbb{P}\left[x_1 \in \mathscr{X}_1(Y) \mid \mathscr{S}, \boldsymbol{m}_0'\right] = \binom{\boldsymbol{m}_0'}{Y}\binom{m - \boldsymbol{m}_0'}{\Delta - Y}\binom{m}{\Delta}^{-1}. \tag{5.8}$$

Expanding (5.8) via Stirling's formula and using the bounds (5.1)–(5.2), we obtain (5.7), which implies that

$$\mathbb{E}\left[|\mathscr{X}_1(y\Delta)| \mid \mathscr{E}\right] = k\exp\left(-\Delta D_{\mathrm{KL}}\left(y \| \exp(-d)\right) + o(\Delta)\right). \tag{5.9}$$

Since the above argument does not depend on the infection status of x_1, analogously we obtain

$$\mathbb{E}\left[|\mathscr{X}_0(y\Delta)| \mid \mathscr{E}\right] = (n-k)\exp\left(-\Delta D_{\mathrm{KL}}\left(y \| \exp(-d)\right) + o(\Delta)\right). \tag{5.10}$$

To turn (5.9)–(5.10) into "with high probability" bounds, we resort to the second moment method. Specifically, we are going to show that

$$\mathbb{E}\left[|\mathscr{X}_1(y\Delta)|(|\mathscr{X}_1(y\Delta)| - 1) \mid \mathscr{E}\right] \sim \mathbb{E}\left[|\mathscr{X}_1(y\Delta)| \mid \mathscr{E}\right]^2, \tag{5.11}$$

$$\mathbb{E}\left[|\mathscr{X}_0(y\Delta)|(|\mathscr{X}_0(y\Delta)| - 1) \mid \mathscr{E}\right] \sim \mathbb{E}\left[|\mathscr{X}_0(y\Delta)| \mid \mathscr{E}\right]^2. \tag{5.12}$$

Then the assertion is an immediate consequence of (5.9)–(5.12) and Lemma 5.1.

For similar reasons as above, it suffices to prove (5.11). More precisely, we merely need to show that

$$\mathbb{P}\left[x_1, x_2 \in \mathscr{X}_1(Y) \mid \mathscr{E}, \mathscr{S}\right] \sim \mathbb{P}\left[x_1 \in \mathscr{X}_1(Y) \mid \mathscr{E}, \mathscr{S}\right]^2. \tag{5.13}$$

To compute the probability on the l.h.s., obtain $\boldsymbol{G}'' = \mathbf{G}_{\mathrm{cc}} - x_1 - x_2$ by removing x_1, x_2. Let $\boldsymbol{m}_0''$ be the number of actually negative tests of $\boldsymbol{G}''$. We claim that on $\mathscr{E}$,

$$\mathbb{P}\left[x_1, x_2 \in \mathscr{X}_1(Y) \mid \mathscr{S}, \boldsymbol{m}_0''\right] = \sum_{I=0}^{\Delta - Y}\binom{\boldsymbol{m}_0''}{I}\binom{\boldsymbol{m}_0''}{Y}\binom{\boldsymbol{m}_0'' - Y}{Y}\binom{m - \boldsymbol{m}_0''}{\Delta - Y - I}\binom{m - \boldsymbol{m}_0''}{\Delta - Y - I}\binom{m}{\Delta}^{-2}. \tag{5.14}$$

Indeed, we first choose $0 \le I \le \Delta - Y$ tests that are actually negative in $\boldsymbol{G}''$ that both x_1, x_2 will join. Observe that these tests a do *not* satisfy $\partial a \setminus \{x_{1/2}\} \subseteq V_0$. Then we choose Y distinct actually negative tests for x_1 and x_2 to join. Finally, we choose the remaining $\Delta - Y - I$ tests for x_1, x_2 among the actually positive tests of $\boldsymbol{G}''$.

Since on $\mathscr{E}$, the total number $\boldsymbol{m}_0''$ is much bigger than Δ, it is easily verified that the sum (5.14) is dominated by the term $I = 0$; thus, on $\mathscr{E}$, we have

$$\mathbb{P}\left[x_1, x_2 \in \mathscr{X}_1(Y) \mid \mathscr{S}, \boldsymbol{m}_0''\right] = (1 + O(\Delta^2/m))\binom{\boldsymbol{m}_0''}{Y}\binom{\boldsymbol{m}_0'' - Y}{Y}\binom{m - \boldsymbol{m}_0''}{\Delta - Y}^2\binom{m}{\Delta}^{-2}; \tag{5.15}$$

the error term $O(\Delta^2/m)$ hides the terms for $I > 0$. Furthermore, a careful expansion of the binomial coefficients from (5.15) shows that uniformly for all $m_0', m_0'' = m\exp(-d) + O\left(\sqrt{m}\ln^3 n\right)$, we have

$$\frac{\mathbb{P}\left[x_1, x_2 \in \mathscr{X}_1(Y) \mid \mathscr{S}, \boldsymbol{m}_0'' = m_0''\right]}{\mathbb{P}\left[x_1 \in \mathscr{X}_1(Y) \mid \mathscr{S}, \boldsymbol{m}_0' = m_0'\right]^2} \sim 1,$$

whence we obtain (5.13). A similar argument applies to $|\mathscr{X}_0(Y)|$. □

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As a next step, consider the set $\mathscr{X}_r(Y,Z)$ of all $x \in \mathscr{X}_r(Y)$ such that

$$\sum_{a \in \partial x \cap F^+} \mathbb{1}\{\partial a \setminus \{x\} \subseteq V_0\} = Z.$$

Corollary 5.3. *Let $y \in \mathscr{Y}(c,d,\theta)$ be such that $y\Delta$ is an integer, and let $z \in (p_{01},p_{11})$ be such that $z\Delta$ is an integer and such that (2.41)–(2.42) are satisfied. Then w.h.p., we have*

$$|\mathscr{X}_0(y\Delta, z\Delta)| = n \exp\left(-\Delta\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z\|p_{01}\right)\right) + o(\Delta)\right), \tag{5.16}$$

$$|\mathscr{X}_1(y\Delta, z\Delta)| = k \exp\left(-\Delta\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z\|p_{11}\right)\right) + o(\Delta)\right). \tag{5.17}$$

Proof. Let $Y = y\Delta$ and $Z = z\Delta$. We deal with $|\mathscr{X}_0(y\Delta, z\Delta)|$ and $|\mathscr{X}_1(y\Delta, z\Delta)|$ by two related but slightly different arguments. The computation of $|\mathscr{X}_1(y\Delta, z\Delta)|$ is pretty straightforward. Indeed, Lemma 5.2 shows that w.h.p., the set $\mathscr{X}_1(Y)$ has the size displayed in (5.6). Furthermore, since (1.2) provides that tests are subjected to noise independently, Lemma 1.5 shows that

$$\mathbb{E}\left[|\mathscr{X}_1(Y,Z)| \mid |\mathscr{X}_1(Y)|\right] = |\mathscr{X}_1(Y)| \exp\left(-\Delta\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z\|p_{11}\right)\right) + o(\Delta)\right). \tag{5.18}$$

Moreover, as we saw in the proof of Lemma 5.2, any $x_1, x_2 \in \mathscr{X}_1(Y)$ have disjoint sets of untainted tests. Hence, in perfect analogy to (5.18), we obtain

$$\mathbb{E}\left[|\mathscr{X}_1(Y,Z)|(|\mathscr{X}_1(Y,Z)|-1) \mid |\mathscr{X}_1(Y)|\right] = |\mathscr{X}_1(Y)|(|\mathscr{X}_1(Y)|-1) \exp\left(-2\Delta\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right)\right.\right.$$
$$\left.\left. + yD_{\mathrm{KL}}\left(z\|p_{11}\right)\right) + o(\Delta)\right). \tag{5.19}$$

Thus, (5.17) follows from (5.18) to (5.19) and Chebyshev's inequality.

Let us proceed to prove (5.16). As in the case of $|\mathscr{X}_1(y\Delta, z\Delta)|$, we obtain

$$\mathbb{E}\left[|\mathscr{X}_0(Y,Z)| \mid |\mathscr{X}_0(Y)|\right] = |\mathscr{X}_0(Y)| \exp\left(-\Delta\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z\|p_{01}\right)\right) + o(\Delta)\right), \tag{5.20}$$

Hence, as in (5.10) from the proof of Lemma 5.10,

$$\mathbb{E}\left[|\mathscr{X}_0(Y,Z)| \mid \mathscr{E}\right] = n \exp\left(-\Delta\left(D_{\mathrm{KL}}\left(y\|\exp(-d)\right) + yD_{\mathrm{KL}}\left(z\|p_{01}\right)\right) + o(\Delta)\right). \tag{5.21}$$

With respect to the second moment calculation, it is not necessarily true that $x_i, x_j \in \mathscr{X}_0(Y,Z)$ with $k < i < j \leq n$ have disjoint sets of untainted tests. Thus, as in the expression (5.14), let $\boldsymbol{m}_0''$ be the number of actually negative tests of $\boldsymbol{G}'' = \mathbf{G}_{\mathrm{cc}} - x_i - x_j$ and introduce $0 \leq I \leq \Delta$ to count the untainted tests that x_i, x_j have in common. Additionally, write $0 \leq I_1 \leq \min\{I, Z\}$ for the number of common untainted tests that display a negative result. Then

$$\mathbb{P}\left[x_i, x_j \in \mathscr{X}_0(Y,Z) \mid \mathscr{S}, \boldsymbol{m}_0''\right] = \sum_{I, I_1} \binom{\boldsymbol{m}_0''}{I}\binom{\boldsymbol{m}_0'' - I}{Y - I}\binom{\boldsymbol{m}_0'' - Y}{Y - I}\binom{m - \boldsymbol{m}_0''}{\Delta - Y}^2 \binom{m}{\Delta}^{-2}$$
$$\cdot \binom{I}{I_1} p_{00}^{I - I_1} p_{01}^{I_1} \left[\binom{Y - I}{Z - I_1} p_{00}^{Y - Z - I + I_1} p_{01}^{Z - I_1}\right]^2. \tag{5.22}$$

As in the proof of Lemma 5.2, it is easily checked that the summand $I = I_1 = 0$ dominates (5.22) and that therefore

$$\mathbb{E}\left[|\mathscr{X}_0(Y,Z)|(|\mathscr{X}_0(Y,Z)| - 1) \mid \mathscr{E}\right] \sim \mathbb{E}\left[|\mathscr{X}_0(Y,Z)| \mid \mathscr{E}\right]^2. \tag{5.23}$$

Thus, (5.16) follows from (5.21), (5.23), and Chebyshev's inequality. □

Proof of Proposition 2.11. By continuity, we can find y, z that satisfy (2.41)–(2.42) such that $y\Delta$, $z\Delta$ are integers, provided that n is large enough. Now, if (2.41)–(2.42) are satisfied, then Corollary 5.3 shows that

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$$|\mathscr{X}_1(y\Delta, z\Delta) \times \mathscr{X}_0(y\Delta, z\Delta)| = n^{\Omega(1)}.$$

Hence, take any pair $(v, w) \in \mathscr{X}_1(y\Delta, z\Delta) \times \mathscr{X}_0(y\Delta, z\Delta)$. Then $\{a \in \partial v : \partial a \setminus \{v\} \subseteq V_0\}$ and $\{a \in \partial w : a \in F_0\}$ are disjoint because $v \in V_1$. Therefore, any such pair (v, w) satisfies (2.43). □

5.3. *Proof of Proposition 2.12*

We are going to lower bound the partition function $Z_{\mathbf{G}_{cc},\boldsymbol{\sigma}''}$ by way of a moment computation. To this end, we are going to couple the constant column design $(\mathbf{G}_{cc}, \boldsymbol{\sigma}'')$ with the displayed test results $\boldsymbol{\sigma}''$ with another random pair $(\mathbf{G}_{cc}, \boldsymbol{\sigma}''')$ where the test results indicated by the vector $\boldsymbol{\sigma}'''$ are purely random, that is, do not derive from an actual vector $\boldsymbol{\sigma}$ of infected individuals. One can think of $(\mathbf{G}_{cc}, \boldsymbol{\sigma}''')$ as a 'null model'. Conversely, in the language of random constraint satisfaction problems [12], ultimately $(\mathbf{G}_{cc}, \boldsymbol{\sigma}'')$ will turn out to be the 'planted model' associated with $(\mathbf{G}_{cc}, \boldsymbol{\sigma}''')$.

Hence, let $\boldsymbol{m}^+$ be the number $\|\boldsymbol{\sigma}''\|_1$ of positively displayed tests of $(\mathbf{G}_{cc}, \boldsymbol{\sigma}'')$. Moreover, for a given integer $0 \le m^+ \le m$, let $\boldsymbol{\sigma}''' \in \{0, 1\}^F$ be a uniformly random vector of Hamming weight m^+, drawn independently of $\mathbf{G}_{cc}, \boldsymbol{\sigma}, \boldsymbol{\sigma}''$. In other words, in the null model $(\mathbf{G}_{cc}, \boldsymbol{\sigma}''')$, we simply choose a set of uniformly random tests to display positively.

Let $\hat{F}^+ = \{a \in F : \boldsymbol{\sigma}'''_a = 1\}$ and $\hat{F}^- = F \setminus F^+$. Moreover, just as in (2.2) define weight functions

$$\psi_{\mathbf{G}_{cc},\boldsymbol{\sigma}''',a} : \{0, 1\}^{\partial a} \to \mathbb{R}_{\ge 0}, \quad \sigma_{\partial a_i} \mapsto \begin{cases} \mathbb{1}\{\|\sigma\|_1 = 0\}p_{00} + \mathbb{1}\{\|\sigma\|_1 > 0\}p_{10} & \text{if } \boldsymbol{\sigma}'''_a = 0, \\ \mathbb{1}\{\|\sigma\|_1 = 0\}p_{01} + \mathbb{1}\{\|\sigma\|_1 > 0\}p_{11} & \text{if } \boldsymbol{\sigma}'''_a = 1. \end{cases} \tag{5.24}$$

In addition, exactly as in (2.3)–(2.4), let

$$\psi_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''}(\sigma) = \mathbb{1}\left\{\|\sigma\|_1 = k\right\} \prod_{a \in F} \psi_{\mathbf{G}_{cc},\boldsymbol{\sigma}''',a}(\sigma_{\partial a}), Z_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''} = \sum_{\sigma \in \{0,1\}^V} \psi_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''}(\sigma), \mu_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''}(\sigma)$$
$$= \psi_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''}(\sigma)/Z_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''}. \tag{5.25}$$

We begin by computing the mean of the partition function (aka the 'annealed average').

Lemma 5.4. *For any* $0 \le m^+ \le m$*, we have* $\mathbb{E}\left[Z_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''}\right] = \binom{n}{k}\binom{m}{m^+}^{-1}\mathbb{P}\left[\boldsymbol{m}^+ = m^+\right]$.

Proof. Writing out the definitions of $\mathbf{G}_{cc}, \boldsymbol{\sigma}'''$, we obtain

$$\mathbb{E}\left[Z_{\mathbf{G}_{cc},\boldsymbol{\sigma}'''}\right] = \binom{m}{m^+}^{-1} \sum_{G} \sum_{\substack{\sigma'' \in \{0,1\}^F : \|\sigma''\|_1 = \boldsymbol{m}^+ \\ \sigma \in \{0,1\} : \|\sigma\|_1 = k}} \mathbb{P}\left[\mathbf{G}_{cc} = G\right] \psi_{\mathbf{G}_{cc},\sigma''}(\sigma)$$
$$= \binom{m}{m^+}^{-1} \sum_{G,\sigma'',\sigma : \|\sigma''\|_1 = m^+, \|\sigma\|_1 = k} \mathbb{P}\left[\mathbf{G}_{cc} = G\right] \mathbb{P}$$
$$\left[\boldsymbol{\sigma}'' = \sigma'' \mid \mathbf{G}_{cc} = G, \ \boldsymbol{\sigma} = \sigma\right] \qquad \text{[by (5.24)-(5.25)]}$$
$$\binom{n}{k}\binom{m}{m^+}^{-1} \sum_{G,\sigma'',\sigma : \|\sigma''\|_1 = m^+, \|\sigma\|_1 = k} \mathbb{P}\left[\mathbf{G}_{cc} = G\right] \mathbb{P}\left[\boldsymbol{\sigma} = \sigma\right] \mathbb{P}\left[\boldsymbol{\sigma}'' = \sigma'' \mid \mathbf{G}_{cc} = G, \ \boldsymbol{\sigma} = \sigma\right]$$
$$\binom{n}{k}\binom{m}{m^+}^{-1} \sum_{\sigma'' : \|\sigma''\|_1 = m^+} \mathbb{P}\left[\boldsymbol{\sigma}'' = \sigma''\right] = \binom{n}{k}\binom{m}{m^+}^{-1} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right],$$

as claimed. □

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As a next step, we sort out the relationship of the null model and of the 'real' group testing instance.

Lemma 5.5. *Let* $0 \le m^+ \le m$ *be an integer. Then for any* G *and any* $\sigma'' \in \{0,1\}^F$ *with* $\|\sigma''\|_1 = m^+$, *we have*

$$\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G, \boldsymbol{\sigma}'' = \sigma'' \mid \boldsymbol{m}^+ = m^+\right] = \frac{\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G, \boldsymbol{\sigma}''' = \sigma''\right] Z_{G,\sigma''}}{\mathbb{E}\left[Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}'''}\right]}. \tag{5.26}$$

Proof. We have

$$\begin{aligned}
\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G, \boldsymbol{\sigma}'' = \sigma'' \mid \boldsymbol{m}^+ = m^+\right] &= \sum_{\sigma : \|\sigma\|_1 = k} \frac{\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G\right] \mathbb{P}\left[\boldsymbol{\sigma}'' = \sigma'' \mid \mathbf{G}_{\mathrm{cc}} = G, \boldsymbol{\sigma} = \sigma\right]}{\binom{n}{k} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right]} \\
&= \sum_{\sigma : \|\sigma\|_1 = k} \frac{\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G\right] \psi_{G,\sigma''}(\sigma)}{\binom{n}{k} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right]} && \text{[by (2.2)-(2.3)]} \\
&= \frac{\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G\right] Z_{G,\sigma''}}{\binom{n}{k} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right]} && \text{[by (2.4)]} \\
&= \frac{\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G\right] \mathbb{P}\left[\boldsymbol{\sigma}''' = \sigma''\right] Z_{G,\sigma''}}{\binom{n}{k} \binom{m}{m^+}^{-1} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right]} && \text{[as } \boldsymbol{\sigma}''' \text{ is uniformly random]} \\
&= \frac{\mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G, \boldsymbol{\sigma}''' = \sigma''\right] Z_{G,\sigma''}}{\mathbb{E}\left[Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}'''}\right]} && \text{[by Lemma 5.4],}
\end{aligned}$$

as claimed. □

Combining Lemmas 5.4–5.5, we obtain the following lower bound on $Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''}$.

Corollary 5.6. *Let* $0 \le m^+ \le m$ *be an integer. For any* $\delta > 0$, *we have*

$$\mathbb{P}\left[Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''} < \delta \binom{n}{k} \binom{m}{m^+}^{-1} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right] \mid \boldsymbol{m}^+ = m^+\right] < \delta.$$

Proof. Lemmas 5.4 and 5.5 yield

$$\begin{aligned}
&\mathbb{P}\left[Z_{\mathbf{G}_{\mathrm{cc}},\boldsymbol{\sigma}''} < \delta \binom{n}{k} \binom{m}{m^+}^{-1} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right] \mid \boldsymbol{m}^+ = m^+\right] \\
&= \sum_{G,\sigma'' : \|\sigma''\|_1 = m^+} \frac{\mathbb{1}\{Z_{G,\sigma''} < \delta \binom{n}{k}\binom{m}{m^+}^{-1} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right]\} Z_{G,\sigma''} \mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G, \boldsymbol{\sigma}''' = \sigma'' \mid \boldsymbol{m}^+ = m^+\right]}{\binom{n}{k}\binom{m}{m^+}^{-1} \mathbb{P}\left[\boldsymbol{m}^+ = m^+\right]} \\
&< \sum_{G,\sigma'' : \|\sigma''\|_1 = m^+} \delta \cdot \mathbb{P}\left[\mathbf{G}_{\mathrm{cc}} = G, \boldsymbol{\sigma}''' = \sigma'' \mid \boldsymbol{m}^+ = m^+\right] \le \delta,
\end{aligned}$$

as desired. □

Proof of Proposition 2.12. Since Proposition 2.3 shows that $\mathbb{P}\left[\boldsymbol{m}^+ = mq_0^+ + O\left(\sqrt{m} \ln^3 n\right)\right] = 1 - o(1)$, the proposition follows immediately from Corollary 5.6. □

Acknowledgement

We thank Petra Berenbrink and Jonathan Scarlett for helpful comments and discussions.

https://doi.org/10.1017/S0963548324000336 Published online by Cambridge University Press

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Appendix A. Table of notation

Table 1. Overview of notation

Symbol	Definition or domain	Meaning
n		Number of individuals
m		Number of total tests
G	$G=(V\cup F, E)$	Test design represented as a bipartite graph
F	$\|F\|=m$	Vertices that represent the test
V	$\|V\|=m$	Vertices that represent the individuals
∂v	$\partial v=\partial_G v=\{u: \{u,v\}\in E\}$	Set of neighbours of v in test design G
k	$k\sim n^{\theta}$	Number of infected individuals
θ		Infection rate
$\boldsymbol{p}$	$\boldsymbol{p}=(p_{00},p_{01},p_{10},p_{11})$	Noise channel
p_{ij}		Probability to observe an actual i test as j
$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}\in\{0,1\}^V$	Vector with hamming weight k that represent the true infection states
$\boldsymbol{\sigma}'$	$\boldsymbol{\sigma}'\in\{0,1\}^F$ with $\boldsymbol{\sigma}'_a=1$ iff $V_1\cap\partial a\neq\emptyset$	Actual test result
$\boldsymbol{\sigma}''$	$\mathbb{P}\left[\boldsymbol{\sigma}''_a=\sigma''\mid G,\boldsymbol{\sigma}'_a=\sigma'\right]=p_{\sigma'\sigma''}$	Displayed test results
$\mathbf{G}_{\mathrm{cc}}$		Constant column test design
$\mathbf{G}_{\mathrm{sc}}$		Spatially coupled test design
c	$m=ck\ln(n/k)$	Constant factor for the number of tests
d		Density parameter of $\mathbf{G}_{\mathrm{cc}}$ and $\mathbf{G}_{\mathrm{sc}}$ that varies the variable and test degrees
Δ	$\Delta=cd\ln(n/k)$	Degree of variable vertices in $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{cc}}$
Γ	$\Gamma=dn/k$	Expected degree test vertices in $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{cc}}$
q_0^-	$q_0^-=\exp(-d)p_{00}+(1-\exp(-d))p_{10}$	Probability that a test is displayed negative
q_0^+	$q_0^-=\exp(-d)p_{01}+(1-\exp(-d))p_{11}$	Probability that a test is displayed positive
τ	$\tau=(\tau_y)_{y\in V}$	Estimate of $\boldsymbol{\sigma}$ of SPARC

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Table 1. Continued

Symbol	Definition or domain	Meaning
$\tau^{(i)}$	$\tau^{(i)} = (\tau_y^{(i)})_{y \in V}$	Estimate of $\boldsymbol{\sigma}$ of the ith round of SPEX
V_r	$\{v \in V : \boldsymbol{\sigma}_v = r\}$	Vertices with infection state r
$V[i]$		In $\mathbf{G}_{cc}$ the variable nodes of compartment i
$V^+[i]$	$V^+[i] = \left\{x \in V[i] : \sum_{j=1}^{s} \left\lvert \lvert \partial x \cap F^+[i+j-1] \rvert - \Delta p_{11}/s \right\rvert \le \ln^{4/7} n \right\}$	Individuals in $V[i]$ with a typical number of pos. tests for a pos. individual
$V_r^+[i]$	$V_r^+[i] = V^+[i] \cap V_r$	Individuals in comp. i with status r that "seem" positive
V^+	$V^+ = \bigcup_{s<i\le\ell} V^+[i]$	
s	$\lceil \ln \ln n \rceil$	Sliding window
ℓ	$\lceil \sqrt{\ln n} \rceil$	Number of compartments
F_r	$F_r = \{a \in F : \boldsymbol{\sigma}'_a = r\}$	Tests with actual test result r
F^+	$F^+ = \{a \in F : \boldsymbol{\sigma}'_a = 1\}$	Tests that are displayed positive
F^-	$F^- = \{a \in F : \boldsymbol{\sigma}'_a = 0\}$	Tests that are displayed negative
$F_r^\pm$	$F_r^\pm = F^\pm \cap F_r$	Tests with actual result r that are displayed positive resp. negative
$F[j]$		In $\mathbf{G}_{sc}$ tests of comp. j
$F_r[j]$	$F[j] \cap F_r$	In $\mathbf{G}_{sc}$ tests of comp. j with actual result r
$F_r^\pm[j]$	$F_r[j] \cap F^\pm$	In $\mathbf{G}_{sc}$ tests of comp. j with actual result r, disp. as $\pm$

Appendix B. Proof of Theorem 1.2 (lower bound for approximate recovery)

The basic idea is to compute the mutual information of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}''$. What makes matters tricky is that we are dealing with the *adaptive* scenario where tests may be conducted one by one. To deal with this issue, we closely follow the arguments from [19]. Furthermore, the displayed test results are obtained by putting the actual test results through the noisy channel.

As a first step, we bound the mutual information between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}''$ from above under the assumption that the statistician applies an adaptive scheme where the next test to be conducted depends deterministically on the previously displayed test results. Let m be the fixed total number of tests that are conducted.

Lemma B.1. *For a deterministic adaptive algorithm, we have* $I(\boldsymbol{\sigma}, \boldsymbol{\sigma}'') \le m/c_{\mathrm{Sh}}$.

Proof. Let $\boldsymbol{\sigma}'$ be the vector of actual test results. Then

$$\begin{aligned} I(\boldsymbol{\sigma}, \boldsymbol{\sigma}'') &= \sum_{s,s''} \mathbb{P}\left[\boldsymbol{\sigma}'' = s'' \mid \boldsymbol{\sigma} = s\right] \mathbb{P}\left[\boldsymbol{\sigma} = s\right] \ln \frac{\mathbb{P}\left[\boldsymbol{\sigma}'' = s'' \mid \boldsymbol{\sigma} = s\right]}{\mathbb{P}\left[\boldsymbol{\sigma}'' = s''\right]} \\ &= \sum_{s,s',s''} \mathbb{P}\left[\boldsymbol{\sigma}'' = s'' \mid \boldsymbol{\sigma}' = s'\right] \mathbb{P}\left[\boldsymbol{\sigma}' = s' \mid \boldsymbol{\sigma} = s\right] \mathbb{P}\left[\boldsymbol{\sigma} = s\right] \\ &\qquad \ln \frac{\mathbb{P}\left[\boldsymbol{\sigma}'' = s'' \mid \boldsymbol{\sigma}' = s'\right] \mathbb{P}\left[\boldsymbol{\sigma}' = s' \mid \boldsymbol{\sigma} = s\right]}{\mathbb{P}\left[\boldsymbol{\sigma}'' = s''\right]} \\ &= I(\boldsymbol{\sigma}'', \boldsymbol{\sigma}') - H(\boldsymbol{\sigma}' \mid \boldsymbol{\sigma}) \le I(\boldsymbol{\sigma}'', \boldsymbol{\sigma}'). \end{aligned}$$

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Furthermore,

$$I(\boldsymbol{\sigma}', \boldsymbol{\sigma}'') = \sum_{s'', s'} \mathbb{P}\left[\boldsymbol{\sigma}'' = s'', \boldsymbol{\sigma}' = s'\right] \ln \frac{\mathbb{P}\left[\boldsymbol{\sigma}'' = s'', \boldsymbol{\sigma}' = s'\right]}{\mathbb{P}\left[\boldsymbol{\sigma}'' = s''\right] \mathbb{P}\left[\boldsymbol{\sigma}' = s'\right]}.$$

Since the tests are conducted adaptively, we obtain

$$\mathbb{P}\left[\boldsymbol{\sigma}'' = s'' \mid \boldsymbol{\sigma}' = s'\right] = \prod_{i=1}^{m} \mathbb{P}\left[\boldsymbol{\sigma}''_i = s''_i \mid \forall j < i : \boldsymbol{\sigma}''_j = s''_j,\ \boldsymbol{\sigma}'_i = s'_i\right].$$

Hence,

$$I(\boldsymbol{\sigma}', \boldsymbol{\sigma}'') = \sum_{i=1}^{m} \sum_{s''_1, \ldots, s''_i, s'_i} \mathbb{P}\left[\forall j < i : \boldsymbol{\sigma}''_j = s''_j, \boldsymbol{\sigma}'_i = s'_i\right]$$
$$\cdot\, \mathbb{P}\left[\boldsymbol{\sigma}''_i = s''_i \mid \forall j < i : \boldsymbol{\sigma}''_j = s''_j, \boldsymbol{\sigma}'_i = s'_i\right] \ln \frac{\mathbb{P}\left[\boldsymbol{\sigma}''_i = s''_i \mid \forall j < i : \boldsymbol{\sigma}''_j = s''_j, \boldsymbol{\sigma}'_i = s'_i\right]}{\mathbb{P}\left[\boldsymbol{\sigma}''_i = s''_i \mid \forall j < i : \boldsymbol{\sigma}''_j = s''_j\right]}.$$

In the last term, $\boldsymbol{\sigma}'_i$ is a Bernoulli random variable (whose distribution is determined by $\boldsymbol{\sigma}''_j$ for $j < i$), and $\boldsymbol{\sigma}''_i$ is the output of that variable upon transmission through our channel. Furthermore, the expression in the second line above is the mutual information of these quantities. Hence, the definition of the channel capacity implies that $I(\boldsymbol{\sigma}', \boldsymbol{\sigma}'') \leq m/c_{\mathrm{Sh}}$. □

Proof of Theorem 1.2 (lower bound for approximate recovery). As a first step, we argue that it suffices to investigate deterministic adaptive group testing algorithms (under the assumption that the ground truth $\boldsymbol{\sigma}$ is random). Indeed, a randomised adaptive algorithm $\mathscr{A}(\,\cdot\,)$ can be modelled as having access to a (single) sample $\boldsymbol{\omega}$ from a random source that is independent of $\boldsymbol{\sigma}$. Now, if we assume that for an arbitrarily small $\delta > 0$, we have

$$\mathbb{E}\left\|\mathscr{A}(\boldsymbol{\sigma}'', \boldsymbol{\omega}) - \boldsymbol{\sigma}\right\|_1 < \delta k,$$

where the expectation is on both $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$, then there exists some outcome ω such that

$$\mathbb{E}\left\|\mathscr{A}(\boldsymbol{\sigma}'', \omega) - \boldsymbol{\sigma}\right\|_1 < \delta k,$$

where the expectation is on $\boldsymbol{\sigma}$ only.

Thus, assume that $\mathscr{A}(\,\cdot\,)$ is deterministic. We have $I(\boldsymbol{\sigma}, \boldsymbol{\sigma}'') = H(\boldsymbol{\sigma}) - H(\boldsymbol{\sigma} \mid \boldsymbol{\sigma}'')$. Furthermore, $H(\boldsymbol{\sigma}) \sim k \ln(n/k)$. Hence, Lemma B.1 yields

$$H(\boldsymbol{\sigma} \mid \boldsymbol{\sigma}'') = H(\boldsymbol{\sigma}) - I(\boldsymbol{\sigma}, \boldsymbol{\sigma}'') \geq H(\boldsymbol{\sigma}) - m/c_{\mathrm{Sh}},$$

which implies the assertion. □

Remark B.2. *Alternatively, as pointed out to us by Jonathan Scarlett, Theorem 1.2 could be derived from Fano's inequality via a straightforward adaptation of the proof of (33, Theorem 5).*

Appendix C. Proof of Lemma 3.3

This proof is a straightforward adaption of (9, proof of Lemma 4.16). Fix $T \subseteq V$ of size $t = |T| \leq \exp(-\ln^{\alpha} n)k$ as well as a set $R \subseteq V$ of size $r = \lceil t/\lambda \rceil$ with $\lambda = 8 \ln \ln n$. Let $\gamma = \lceil \ln^{\beta} n \rceil$. Further, let $U \subseteq F[1] \cup \cdots \cup F[\ell]$ be a set of size $\gamma r \leq u \leq \Delta t$. Additionally, let $\mathscr{E}(R, T, U)$ be the event that every test $a \in U$ contains two individuals from $R \cup T$. Then

$$\mathbb{P}\left[R \subseteq \left\{x \in V : \sum_{a \in \partial x \setminus F[0]} \mathbb{1}\left\{T \cap \partial a \setminus \{x\} \neq \emptyset\right\} \geq \gamma\right\}\right] \leq \mathbb{P}\left[\mathscr{E}(R, T, U)\right]. \tag{C.1}$$

Hence, it suffices to bound $\mathbb{P}\left[\mathscr{E}(R, T, U)\right]$.

For a test $a \in U$, there are no more than $\binom{r+t}{2}$ ways to choose distinct individuals $x_a, x'_a \in R \cup T$. Moreover, (3.1) shows that the probability of the event $\{x_a, x'_a \in \partial a\}$ is bounded by $(1+o(1))(\Delta\ell/(ms))^2$; in fact, this probability might be zero if we choose an individual that cannot join a due to the spatially coupled construction of $\mathbf{G}_{\mathrm{sc}}$. Hence, due to negative association [16, Lemma 2]

$$\mathbb{P}\left[\mathscr{E}(R, T, U)\right] \leq \left[\binom{r+t}{2}\left(\frac{(1+o(1))\Delta\ell}{ms}\right)^2\right]^u.$$

Consequently, by the union bound the event $\mathfrak{E}(r, t, u)$ that there exist sets R, T, U of sizes $|R| = r, |T| = t, |U| = u$ such that $\mathscr{E}(R, T, U)$ occurs has probability

$$\mathbb{P}\left[\mathfrak{E}(r, t, u)\right] \leq \binom{n}{r}\binom{n}{t}\binom{m}{u}\left[\binom{r+t}{2}\left(\frac{(1+o(1))\Delta\ell}{ms}\right)^2\right]^u.$$

Hence, the bounds $\gamma t/\lambda \leq \gamma r \leq u \leq \Delta t$ yield

$$\mathbb{P}\left[\mathfrak{E}(r, t, u)\right] \leq \binom{n}{t}^2\binom{m}{u}\left[\binom{2t}{2}\left(\frac{(1+o(1))\Delta\ell}{ms}\right)^2\right]^u \leq \left(\frac{en}{t}\right)^{2t}\left(\frac{2e\Delta^2\ell^2t^2}{ms^2u}\right)^u$$
$$\leq \left[\left(\frac{en}{t}\right)^{\lambda/\gamma}\frac{2e\lambda\Delta^2\ell^2t}{\gamma ms^2}\right]^u \leq \left[\left(\frac{en}{t}\right)^{\lambda/\gamma}\cdot\frac{t\ln^4 n}{m}\right]^u \qquad \text{[due to (2.13), (2.14)]}.$$

Further, since $\gamma = \Omega(\ln^{\beta} n)$ and $m = \Omega(k \ln n)$ while $t \leq \exp(-\ln^{\alpha} n)k$ and $\alpha + \beta > 1$, we obtain

$$\mathbb{P}\left[\mathfrak{E}(r, t, u)\right] \leq \exp\left(-u \ln^{\Omega(1)} n\right).$$

Thus, summing on $1 \leq t \leq \exp(-\ln^{\alpha} n)k$, $\gamma r \leq u \leq \Delta t$ and recalling $r = \lceil t/\lambda \rceil$, we obtain

$$\sum_{t,u}\mathbb{P}\left[\mathfrak{E}(r, t, u)\right] \leq \sum_{u \geq 1} u \exp\left(-u \ln^{\Omega(1)} n\right) = o(1). \tag{C.2}$$

Finally, the assertion follows from (C.1) and (C.2).

Appendix D. Parameter optimisation for the binary symmetric channel

Let $\mathscr{D}(\theta, \boldsymbol{p}) = \{d > 0 : \max\{c_{\mathrm{ex},1}(d, \theta), c_{\mathrm{ex},2}(d)\} = c_{\mathrm{ex}}(\theta)\}$ be the set of those d where the minimum in the optimisation problem (1.10) is attained. The goal in this section is to show that for the binary symmetric channel, this minimum is not always attained at the information-theoretic value $d_{\mathrm{Sh}} = \ln 2$ that minimises the term $c_{\mathrm{ex},2}(d)$ from (1.10).

Proposition D.1. *For any binary symmetric channel $\boldsymbol{p}$ given by $0 < p_{01} = p_{10} < 1/2$, there is $\hat{\theta}(p_{01})$ such that for all $\theta > \hat{\theta}(p_{01})$, we have $d_{\mathrm{Sh}} = \ln(2) \notin \mathscr{D}(\theta, \boldsymbol{p})$.*

In order to show that d_{Sh} is suboptimal, we will use the following analytic bound on $c_{\mathrm{ex},1}(d, \theta)$ for binary symmetric channels.

Lemma D.2. *For a binary symmetric channel $\boldsymbol{p}$ given by $p_{01} < 1/2$, we have*

$$\frac{\theta}{-(1-\theta)d\ln\left(1-(1-a)\exp(-d)\right)} < c_{\mathrm{ex},1}(d, \theta) \leq \frac{1}{-(1-\theta)d\ln\left(1-(1-a)\exp(-d)\right)}, \quad \text{where} \tag{D.1}$$

$$a = \exp\left(-D_{\mathrm{KL}}\left(1/2 \| p_{01}\right)\right) = 2\sqrt{p_{01}(1-p_{01})}. \tag{D.2}$$

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The proof of Lemma D.2 uses the following fact, which can be verified by elementary calculus.

Fact D.3. *For all $d > 0$, $0 < p < 1$, and $0 \le z \le 1$, we have*

$$\text{argmin}_{0\le y\le 1}\left\{D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(z\|p\right)\right\} = \frac{a\exp(-d)}{1-(1-a)\exp(-d)},$$

$$\text{where } a = \exp\left(-D_{\text{KL}}\left(z\|p\right)\right), \quad \text{and}$$

$$\min_{0\le y\le 1}\left\{D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(z\|p\right)\right\} = -\ln\left(1-(1-a)\exp(-d)\right).$$

□

Proof of Lemma D.2 *(bounding $c_{\text{ex},1}(d,\theta)$).* Fact D.3 shows that

$$\min_{0\le y\le 1}\left\{D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(1/2\|p_{01}\right)\right\} = -\ln\left(1-(1-a)\exp(-d)\right).$$

with a as in (D.2). Hence, it is sufficient to show that

$$\theta < \min_{0\le y\le 1}\left\{c_{\text{ex},1}(d,\theta)\cdot d(1-\theta)\left(D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(1/2\|p_{01}\right)\right)\right\} \le 1. \tag{D.3}$$

Let us first prove the upper bound. Choose $\hat{c}(d,\theta)$ such that

$$\min_{0\le y\le 1}\left\{\hat{c}(d,\theta)d(1-\theta)\left(D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(1/2\|p_{01}\right)\right)\right\} = 1;$$

we need to show that $c_{\text{ex},1}(d,\theta) \le \hat{c}(d,\theta)$. By the channel symmetry and because $p_{01} < 1/2 < p_{11}$, we have

$$D_{\text{KL}}\left(1/2\|p_{01}\right) = D_{\text{KL}}\left(1/2\|p_{11}\right) \le D_{\text{KL}}\left(p_{01}\|p_{11}\right).$$

Therefore, the definition of $\hat{c}(d,\theta)$ ensures that for all $y \in [0, 1]$, we have

$$\hat{c}(d,\theta)\cdot d(1-\theta)\left(D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(p_{01}\|p_{11}\right)\right) \ge \hat{c}(d,\theta)\cdot d(1-\theta)\left(D_{\text{KL}}\left(y\|\exp(-d)\right)\right.$$
$$\left. + yD_{\text{KL}}\left(1/2\|p_{01}\right)\right) \ge 1 > \theta,$$

and thus $\hat{c}(d,\theta) \ge c_{\text{ex},0}(d,\theta)$. Once again by channel symmetry and the definitions of $\hat{c}(d,\theta)$ and $\mathfrak{z}(y)$ (see [1.8]), we see that $D_{\text{KL}}\left(\mathfrak{z}(y)\|p_{11}\right) \le D_{\text{KL}}\left(1/2\|p_{11}\right) = D_{\text{KL}}\left(1/2\|p_{01}\right)$ and hence $D_{\text{KL}}\left(\mathfrak{z}(y)\|p_{01}\right) \ge D_{\text{KL}}\left(1/2\|p_{01}\right)$ for all y. Consequently, the definition of $\hat{c}(d,\theta)$ ensures we see that for all $y \in [0, 1]$,

$$\hat{c}(d,\theta)\cdot d(1-\theta)\left(D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(\mathfrak{z}(y)\|p_{01}\right)\right) \ge \hat{c}(d,\theta)\cdot d(1-\theta)\left(D_{\text{KL}}\left(y\|\exp(-d)\right)\right.$$
$$\left. + yD_{\text{KL}}\left(1/2\|p_{01}\right)\right) \ge 1.$$

Hence, $\hat{c}(d,\theta) \ge c_{\text{ex},1}(d,\theta)$, which is the right inequality in (D.3).

Moving on to the lower bound, choose $\check{c}(d,\theta)$ such that

$$\min_{0\le y\le 1}\left\{\check{c}(d,\theta)d(1-\theta)\left(D_{\text{KL}}\left(y\|\exp(-d)\right) + yD_{\text{KL}}\left(1/2\|p_{11}\right)\right)\right\} = \theta; \tag{D.4}$$

we need to show that $c_{\text{ex},1}(d,\theta) > \check{c}(d,\theta)$. The $y = \hat{y}$ where the minimum (D.4) is attained satisfies $\hat{y} \in \mathscr{Y}(c,d,\theta)$ because $D_{\text{KL}}\left(1/2\|p_{11}\right) > 0$ (due to $p_{11} > 1/2$). Moreover, $\mathfrak{z}(\hat{y}) = 1/2$ by the definition of $\mathfrak{z}(\,\cdot\,)$. But since $D_{\text{KL}}\left(1/2\|p_{01}\right) = D_{\text{KL}}\left(1/2\|p_{11}\right)$ by symmetry of the channel, we have

$$\check{c}(d,\theta)\cdot d(1-\theta)\left(D_{\text{KL}}\left(\hat{y}\|\exp(-d)\right) + \hat{y}D_{\text{KL}}\left(\mathfrak{z}(\hat{y})\|p_{01}\right)\right) = \theta < 1.$$

Hence, we obtain $\check{c}(d,\theta) < c_{\text{ex},1}(d,\theta)$, which is the left inequality in (D.3). □

To complete the proof of Proposition D.1, we need a second elementary fact.

Fact D.4. *The function $f(x,p) = \ln(x)\ln(1-px)$ is concave in its first argument for $x, p \in (0, 1)$, and for any given $p \in (0, 1)$, any x maximising $f(x,p)$ is strictly less than $\frac{1}{2}$.* □

https://doi.org/10.1017/S0963548324000336 Published online by Cambridge University Press

Proof of Proposition D.1 (d = ln [2] can be suboptimal). We reparameterise the bounds on $c_{\mathrm{ex},1}(d,\theta)$ from Lemma D.2 in terms of exp $(-d)$, obtaining

$$\frac{\theta}{-(1-\theta)f(\exp(-d),1-a)} < c_{\mathrm{ex},1}(d,\theta) \le \frac{1}{-(1-\theta)f(\exp(-d),1-a)}, \quad \text{where}$$
$$f(x,p) = \ln(x)\ln(1-px), \quad a = \exp\left(-D_{\mathrm{KL}}\left(1/2\|p_{01}\right)\right).$$

As $0<a<1$ Fact D.4 shows that any x maximising $f(x,1-a)$ is strictly less than 1/2. Hence, for $\hat{d} > \ln(2)$ minimising $f(\exp(-d),1-a)$, we have $f(\exp(-d),1-a) < f(1/2,1-a) = f(\exp(-d_{\mathrm{Sh}}),1-a)$. In particular, the value

$$\hat{\theta}(p_{01}) = \inf\left\{0<\theta<1 : \frac{c_{\mathrm{ex},2}(\hat{d})}{\theta} < \frac{1}{-\hat{d}(1-\theta)\ln(1-(1-a)\exp(-\hat{d}))} < \frac{\theta}{-d_{\mathrm{Sh}}(1-\theta)\ln(1-(1-a)\exp(-d_{\mathrm{Sh}}))}\right\}$$

is well defined. Hence, for all $\theta > \hat{\theta}(p_{01})$, we have

$$\max\left\{c_{\mathrm{ex},1}(\hat{d},\theta), c_{\mathrm{ex},2}(\hat{d})\right\} = c_{\mathrm{ex},1}(\hat{d},\theta) < c_{\mathrm{ex},1}(d_{\mathrm{Sh}},\theta) = \max\left\{c_{\mathrm{ex},1}(d_{\mathrm{Sh}},\theta), c_{\mathrm{ex},2}(d_{\mathrm{Sh}})\right\},$$

and thus $d_{\mathrm{Sh}} \notin \mathscr{D}(\theta,\boldsymbol{p})$. □

Appendix E. Parameter optimisation for the *Z*-channel

Much as in Appendix D, the goal here is to show that also for the Z-channel, the value d_{Sh} from (1.17) at which $c_{\mathrm{ex},2}(d)$ from (1.10) attains its minimum is not generally the optimal choice to minimise $\max\{c_{\mathrm{ex},1}(d,\theta), c_{\mathrm{ex},2}(d)\}$ and thus obtain the optimal bound $c_{\mathrm{ex}}(\theta)$. To this end, we derive the explicit formula (1.15) for $c_{\mathrm{ex},1}(d,\theta)$; the derivation of the second formula (1.16) is elementary.

Proposition E.1. *For a Z-channel $\boldsymbol{p}$ given by $p_{01}=0$ and $0<p_{11}<1$, we have* $c_{\mathrm{ex},1}(d,\theta) = \frac{\theta}{-(1-\theta)d\ln(1-\exp(-d)p_{11})}$.

Proof. We observe that for a Z-channel, we have $c_{\mathrm{ex},1}(d,\theta) = c_{\mathrm{ex},0}(d,\theta)$. Indeed, fix any $c > c_{\mathrm{ex},0}(d,\theta)$. Then by the definitions (1.7) of $c_{\mathrm{ex},0}(d,\theta)$ and of $\mathfrak{z}(y)$, we have $\mathfrak{z}(y) > p_{01}$. Since the Z-channel satisfies $p_{01}=0$, the value $D_{\mathrm{KL}}\left(\mathfrak{z}(y)\|p_{01}\right)$ diverges for all $c > c_{\mathrm{ex},0}(d,\theta)$, rendering the condition in the definition of $c_{\mathrm{ex},1}(d,\theta)$ void for all $y>0$. Moreover, since $c > c_{\mathrm{ex},0}(d,\theta)$, we also have $0 \notin \mathscr{Y}(c,d,\theta)$ and thus $c \ge c_{\mathrm{ex},1}(d,\theta)$. Since $c_{\mathrm{ex},1}(d,\theta) \ge c_{\mathrm{ex},0}(d,\theta)$ by definition, this implies that $c_{\mathrm{ex},1}(d,\theta) = c_{\mathrm{ex},0}(d,\theta)$ on the Z-channel.

Hence, it remains to verify that $c_{\mathrm{ex},1}(d,\theta) = c_{\mathrm{ex},0}(d,\theta)$ has the claimed value. This is a direct consequence of Fact D.3 (with $z = p_{01}$ and $p = p_{11}$) in combination with the fact that $D_{\mathrm{KL}}\left(p_{01}\|p_{11}\right) = D_{\mathrm{KL}}\left(0\|p_{11}\right) = -\ln(p_{10})$ and thus $1-\exp\left(-D_{\mathrm{KL}}\left(p_{01}\|p_{11}\right)\right) = 1-p_{10} = p_{11}$. □

The following proposition shows that indeed $d = d_{\mathrm{Sh}}$ is not generally the optimal choice. Recall that $\mathscr{D}(\theta,\boldsymbol{p})$ is the set of d where the minimum in the optimisation problem defining $c_{\mathrm{ex}}(\theta)$ is attained for a given channel $\boldsymbol{p}$.

Proposition E.2. *For a Z-channel $\boldsymbol{p}$ given by $p_{01}=0$ and any $0<p_{11}<1$, there is a $\hat{\theta}(p_{11}) < 1$ such that for all $\theta > \hat{\theta}(p_{11})$, we have $d_{\mathrm{Sh}} \notin \mathscr{D}(\theta,\boldsymbol{p})$.*

Towards the proof of Proposition E.2, we state the following fact whose proof comes down to basic calculus.

Fact E.3. *For a Z-channel $\boldsymbol{p}$ where $p_{01}=0$ and $0<p_{10}<1$, we have* $\exp(-d_{\mathrm{Sh}}(\boldsymbol{p})) > \frac{1}{2}$. □

https://doi.org/10.1017/S0963548324000336 Published online by Cambridge University Press

Proof of Proposition E.2 (d_{Sh} is suboptimal for the z-channel). First we show that $d = d_{Sh}$ does not minimise $c_{ex,1}(d, \theta)$ for all $0 < p_{11} < 1$. We reparameterise the expression for $c_{ex,1}(d, \theta)$ from Proposition E.1 in terms of $\exp(-d)$, obtaining

$$c_{ex,1}(d,\theta) = \frac{\theta}{(1-\theta)f(\exp(-d), p_{11})}, \quad \text{where} \quad f(x, p_{11}) = \ln(x)\ln(1 - p_{11}x).$$

Hence, for any given p_{11}, the value of $c_{ex,1}(d, \theta)$ is minimised when $f(\exp(-d), p_{11})$ is minimised. Using elementary calculus, we check that f is concave in its first argument on the interval (0, 1) and that for all $0 < p_{11} < 1$, the value of x maximising $f(x, p_{11})$ is strictly smaller than $\frac{1}{2}$ (see Fact D.4). Now for any Z-channel with $0 < p_{10} < 1$, we have $\exp(-d_{Sh}) > \frac{1}{2}$ (using Fact E.3). Hence, for the Z-channel, d_{Sh} does not minimise $c_{ex,1}(d, \theta)$.

Now let d_1 be a d minimising $c_{ex,1}(d, \theta)$; in particular, $d_1 \neq d_{Sh}$. Since $\frac{\theta}{1-\theta}$ is increasing in θ and unbounded as $\theta \to 1$, the same holds for $c_{ex,1}(d_1, \theta)$ and $c_{ex,1}(d_{Sh}, \theta)$. Hence, we may consider

$$\hat{\theta}(p_{11}) = \inf\left\{0 < \theta < 1 : c_{ex,1}(d_1,\theta) > c_{ex,2}(d_1), c_{ex,1}(d_{Sh},\theta) > c_{ex,2}(d_{Sh})\right\},$$

check that it is strictly less than 1, and that by definition of $\hat{\theta}(p_{11})$, it holds for all $\theta > \hat{\theta}(p_{11})$ that

$$\max\left\{c_{ex,1}(d_1,\theta), c_{ex,2}(d_1)\right\} = c_{ex,1}(d_1,\theta) < c_{ex,1}(d_{Sh},\theta) = \max\left\{c_{ex,1}(d_{Sh},\theta), c_{ex,2}(d_{Sh})\right\}.$$

Consequently, $d_{Sh} \notin \mathscr{D}(\theta, \boldsymbol{p})$. □

Appendix F. Comparison with the results of Chen and Scarlett on the symmetric channel

Chen and Scarlett [7] recently derive the preci,se information-theoretic threshold of the constant column design $\mathbf{G}_{cc}$ for the symmetric channel (i.e. $p_{11} = p_{00}$). The aim of this section is to verify that their threshold coincides with $m \sim c_{ex}(\theta)k\ln(n/k)$, with $c_{ex}(\theta)$ from (1.11) on the symmetric channel. The threshold quoted in (7, Theorems 3 and 4) reads $m \sim \min_{d>0} c_{CS}(d, \theta)k\ln(n/k)$, where $c_{CS}(d, \theta)$ is the solution to the following optimisation problem:

$$c_{CS}(d,\theta) = \max\{c_{ex,2}(d,\theta), c_{ls}(d,\theta)\}, \qquad \text{where} \tag{F.1}$$

$$c_{ls}(d,\theta) = \left[(1-\theta)d \min_{y\in(0,1),z\in(0,1)} \max\left\{\frac{1}{\theta}\left(D_{KL}\left(y\|e^{-d} + yD_{KL}\left(z\|e^{-d}\right)\right)\right),\right.\right.$$
$$\left.\left.\min_{y'\in[|y(2z-1)|,1]}\left(D_{KL}\left(y'\|e^{-d}\right) + y'D_{KL}\left(z'(z,y,y')\|p_{01}\right)\right)\right\}\right]^{-1} \tag{F.2}$$

$$z'(z,y,y') = \frac{1}{2} + \frac{y(2z-1)}{2y'}.$$

Lemma F.1. *For symmetric noise* ($p_{00} = p_{11}$) *and any* $d > 0$, *we have* $c_{CS}(d, \theta) = \max\{c_{ex,1}(d, \theta), c_{ex,2}(d)\}$.

Proof. The definition (F.2) of $c_{ls}(d)$ can be equivalently rephrased as follows:

$$\begin{aligned} c_{ls}(d,\theta) = \inf&\left\{c > 0 : \forall y, z \in [0,1]\ \forall y' \in \left[\left|y(2z-1)\right|, 1\right] :\right.\\ &\left.\left(f_1(c,d,\theta,y,z) \geq \theta \vee f_2(c,d,\theta,y,z,y',z'(z,y,y')) \geq 1\right)\right\}, \qquad \text{where}\\ f_1(c,d,\theta,y,z) =& cd(1-\theta)\left(D_{KL}\left(y\|\exp(-d)\right) + yD_{KL}\left(z\|p_{11}\right)\right),\\ f_2(c,d,\theta,y,z,y',z') =& cd(1-\theta)\left(D_{KL}\left(y'\|\exp(-d)\right) + y'D_{KL}\left(z'(z,y,y')\|p_{01}\right)\right). \end{aligned}$$

Consequently, $c < c_{ls}(d, \theta)$ iff

$$\exists y, z, y' \text{ s.t. } f_1(c,d,\theta,y,z) < \theta \text{ and } f_2(c,d,\theta,y,z,y',z'(z,y,y')) < 1.$$

Recall that $c < c_{\text{ex},1}(d, \theta)$ iff

$$\exists y, z \text{ s.t } f_1(c, d, \theta, y, z) < \theta \text{ and } f_2(c, d, \theta, y, z, y, z) < 1.$$

Since $z'(z, y, y) = z$, we conclude that if $c < c_{\text{ex},1}(d, \theta)$, then $c < c_{\text{ls}}(d, \theta)$. Hence, $c_{\text{ex},1}(d, \theta) \leq c_{\text{ls}}(d, \theta)$.

To prove the converse inequality, we are going to show that any $c < c_{\text{ls}}(d, \theta)$ also satisfies $c < c_{\text{ex},1}(d, \theta)$. Hence, assume for contradiction that $c < c_{\text{ls}}(d, \theta)$ and $c \geq c_{\text{ex},1}(d, \theta)$. Then the following four inequalities hold:

$$cd(1-\theta)\left(D_{\text{KL}}\left(y\|\mathrm{e}^{-d}\right) + yD_{\text{KL}}\left(z\|p_{11}\right)\right) < \theta, cd(1-\theta)\left(D_{\text{KL}}\left(y\|\mathrm{e}^{-d}\right) + yD_{\text{KL}}\left(z\|p_{01}\right)\right) \geq 1, \tag{F.3}$$

$$\begin{aligned} cd(1-\theta)\left(D_{\text{KL}}\left(y'\|\mathrm{e}^{-d}\right) + y'D_{\text{KL}}\left(z'(z, y, y')\|p_{01}\right)\right) < 1, \\ cd(1-\theta)\left(D_{\text{KL}}\left(y'\|\mathrm{e}^{-d}\right) + y'D_{\text{KL}}\left(z'\|p_{11}\right)\right) \geq \theta. \end{aligned} \tag{F.4}$$

The two inequalities on the left are a direct consequence of $c < c_{\text{ls}}(d, \theta)$. Note that if one of the right two inequalities is violated, then $c < c_{\text{ex},1}(d, \theta)$. Combining the inequalities of (F.3) leads us to

$$\begin{aligned} &cd(1-\theta)\left(D_{\text{KL}}\left(y\|\mathrm{e}^{-d}\right) + yD_{\text{KL}}\left(z\|p_{01}\right)\right) - cd(1-\theta)\left(D_{\text{KL}}\left(y\|\mathrm{e}^{-d}\right) + yD_{\text{KL}}\left(z\|p_{11}\right)\right) \\ &= cd(1-\theta)y\left(D_{\text{KL}}\left(z\|p_{01}\right) - D_{\text{KL}}\left(z\|1-p_{01}\right)\right) = cd(1-\theta)y(1-2z)\ln\left(\frac{p_{01}}{1-p_{01}}\right) > 1-\theta. \end{aligned} \tag{F.5}$$

The remaining inequalities given by (F.4) imply

$$\begin{aligned} &cd(1-\theta)\left(D_{\text{KL}}\left(y'\|\mathrm{e}^{-d}\right) + y'D_{\text{KL}}\left(z'(z, y, y')\|p_{01}\right)\right) - cd(1-\theta)\left(D_{\text{KL}}\left(y'\|\mathrm{e}^{-d}\right) + y'D_{\text{KL}}\left(z'\|p_{11}\right)\right) \\ &\quad = cd(1-\theta)y'\left(D_{\text{KL}}\left(z'(z, y, y')\|p_{01}\right) - D_{\text{KL}}\left(z'(z, y, y')\|1-p_{01}\right)\right) \\ &\quad = cd(1-\theta)y'\frac{y}{y'}(1-2z)\ln\left(\frac{p_{01}}{1-p_{01}}\right) < 1-\theta, \end{aligned}$$

which contradicts (F.5). □

Cite this article: Coja-Oghlan A, Hahn-Klimroth M, Hintze L, Kaaser D, Krieg L, Rolvien M, and Scheftelowitsch O (2024). Noisy group testing via spatial coupling. *Combinatorics, Probability and Computing*. https://doi.org/10.1017/S0963548324000336