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Convergence Factors and Compactness in Weighted Convolution Algebras

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Abstract. We study convergence in weighted convolution algebras $L^1(\omega)$ on R^+, with the weights chosen such that the corresponding weighted space $M(\omega)$ of measures is also a Banach algebra and is the dual space of a natural related space of continuous functions. We determine convergence factor η for which weak*-convergence of $\{\lambda_n\}$ to λ in $M(\omega)$ implies norm convergence of $\lambda_n * f$ to $\lambda * f$ in $L^1(\omega\eta)$. We find necessary and sufficent conditions which depend on ω and f and also find necessary and sufficent conditions for η to be a convergence factor for all $L^1(\omega)$ and all f in $L^1(\omega)$. We also give some applications to the structure of weighted convolution algebras. As a preliminary result we observe that η is a convergence factor for ω and f if and only if convolution by f is a compact operator from $M(\omega)$ (or $L^1(\omega)$) to $L^1(\omega\eta)$.

1 Introduction

Suppose that $\{\lambda_n\}$ is a sequence of locally integrable functions on $\mathbf{R}^+ = [0, \infty)$ and that f also belongs to $L^1_{\mathrm{loc}}(\mathbf{R}^+)$. In this paper, we answer a form of the following question: If $\lambda_n * f$ converges to f in some "very weak" manner, then in which norms is $\lim_{n\to\infty} \lambda_n * f = f$? Such convergence questions occur frequently, for instance in determining whether a sequence is a summability kernel or approximate identity. For us these questions arose in our studies of the structure of weighted convolution algebras. We will give some applications of our convergence results in Section 5. For our applications, we need to allow the λ_n to be locally finite complex measures, that is, complex linear combinations of sigma-finite regular Borel measures on $\mathbf{R}^+$, and it is convenient to allow $\lambda_n * f$ to converge to $\lambda * f$, where λ also belongs to the space $M_{\mathrm{loc}}(\mathbf{R}^+)$ of locally finite measures. It is also sometimes convenient to extend functions and measures to all of $\mathbf{R}$ by making them equal to 0 off $\mathbf{R}^+$, so that, for instance, we can write

$$\lambda * f(x) = \int_{\mathbf{R}} f(x-t)\, d\lambda(t) \quad \text{for all } x \geq 0.$$

Also, we usually will identify the function f with the measure $f(t)\,dt$.

Since we are interested in convolutions, the most natural spaces to consider are weighted L^1 spaces and algebras. For us a *weight* will be a positive Borel function ω which is locally bounded, and locally bounded away from 0 on $\mathbf{R}^+$. Then $L^1(\omega)$ is the Banach space of (equivalence classes of) locally integrable functions f for which $f\omega$

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is integrable. We give $L^1(\omega)$ the inherited norm

$$\|f\| = \|f\|_\omega = \|f\omega\|_1 = \int_0^\infty |f(t)|\omega(t)\,dt.$$

In a similar way, we define the space $M(\omega)$ of measures for which the norm is

$$\|\mu\| = \|\mu\|_\omega = \int_{\mathbf{R}^+} \omega(t)\,d|\mu|(t) < \infty.$$

We say that the weight ω is an *algebra weight* if ω is always positive, is submultiplicative (that is, $\omega(x+y) \le \omega(x)\omega(y)$), is everywhere right continuous and has $\omega(0) = 1$. When ω is an algebra weight, it is easy to see that both $L^1(\omega)$ and $M(\omega)$ are Banach algebras under convolution, and $L^1(\omega)$ is a closed ideal of $M(\omega)$. The most important algebra weights are $\omega(t) \equiv 1$, which gives the classical algebra $L^1(\mathbf{R}^+)$, and weights with $\lim_{t\to\infty} \omega(t)^{1/t} = 0$, which give radical algebras.

Requiring that ω be an algebra weight in our sense is just a normalization; whenever ω is a weight for which $L^1(\omega)$ is an algebra under convolution, we can always replace the given weight by an algebra weight which gives rise to the same space with an equivalent norm [Gr2, Theorem 2.1, p. 591]. The importance of this normalization is that when ω is an algebra weight, we can then identify $M(\omega)$ as a dual space so that it has a weak* topology. Explicitly, let $L^\infty(1/\omega)$ be the space of locally integrable functions g for which g/ω is essentially bounded, and let $C_0(1/\omega)$ be its subspace of continuous functions with $\lim_{t\to\infty} g(t)/\omega(t) = 0$. We give g the inherited norm of g/ω in $L^\infty(\mathbf{R}^+)$. Under the natural duality $\langle \mu, h\rangle = \int_{\mathbf{R}^+} h(t)\,d\mu(t)$, $L^\infty(1/\omega)$ is the dual space of $L^1(\omega)$. For algebra weights, $M(\omega)$ is also the dual space of $C_0(1/\omega)$ [Gr2, Theorem 2.2, p. 592]. Thus we can speak of the weak*-topology on $M(\omega)$, and on $L^1(\omega)$ considered as a subspace of $M(\omega)$.

The natural convergence in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$ is convergence in the Fréchet topology given by the L^1 seminorms on finite intervals. So we say that $\{f_n\}$ *converges to f in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$* if

$$\lim_{n\to\infty} \int_0^b |f_n - f|\,dt = 0, \quad \text{for all } b \ge 0.$$

This is one of a number of types of "very weak convergence". In previous papers, particularly [GG1, Section 3], we have shown that all these types of very weak convergence are equivalent.

The following result lists some of the equivalences that we will see in the paper.

Theorem 1.1 *Suppose that $\{\lambda_n\}$ is a sequence of locally finite measures on $\mathbf{R}^+$ and that ω is any algebra weight for which $\{\lambda_n\}$ is a norm-bounded sequence in $M(\omega)$. Then the following are equivalent:*

(a) *For some (equivalently all) non-zero f in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$, $\{\lambda_n * f\}$ converges to $\lambda * f$ in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$;*
(b) *$\lambda_n \to \lambda$ weak* in $M(\omega)$;*
(c) *for some (equivalently all) non-zero f in $L^1(\omega)$, $\{\lambda_n * f\}$ converges to $\lambda * f$ weak* in $M(\omega)$.*

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Proof The equivalence of (b) and (c) is [Gr2, Lemma 3.2, p. 595]. That (b) implies (a) is [GG1, Corollary 3.3, p. 513]. For the convenience of the reader we prove (a) $\Rightarrow$ (b), which is proved in the same way as [Gr3, Theorem 4.1, p. 183]. Since $M(\omega)$ is the dual space of a separable Banach space, every norm-bounded sequence has a weak* convergent subsequence. Thus it is enough to show that if $\{\lambda_n'\}$ is a subsequence of $\{\lambda_n\}$ and has a weak* limit μ, then $\lambda = \mu$. But (b) $\Rightarrow$ (a) shows that for all f in $L^1(\omega)$, we have $\lambda_n' * f$ converges to $\mu * f$ in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$. Hence $\lambda * f = \mu * f$, so that $\lambda = \mu$, since M_{loc} is an integral domain by Titchmarsh's convolution theorem.

There are several other conditions equivalent to weak* convergence. For instance [GG1, Theorem 3.1(a), p. 511] if f is continuous with $f(0) = 0$, then $\lambda_n * f(x)$ converges pointwise to $\lambda * f(x)$. The existence of some $M(\omega)$ in which $\{\lambda_n\}$ is bounded is equivalent to the sequence $\{|\lambda_n|[0, b)\}$ being bounded for each fixed b.

The main purpose of this paper is to answer the following question. We also prove related results and give some applications of our answer.

Question 1 Suppose that ω is an algebra weight and that f belongs to $L^1(\omega)$. For which bounded weights η does $\lambda_n \to \lambda$ weak* in $M(\omega)$ imply that $\lambda_n * f \to \lambda * f$ in norm in $L^1(\omega\eta)$?

We give two different answers to Question 1. In Theorem 3.1, we give necessary and sufficient conditions in terms of ω and f. In Theorem 4.1 we give necessary and sufficient conditions for η to work for all ω and f. The condition of Theorem 3.1 (c) on η for an arbitrary fixed ω looks technical, but it can be applied effectively. It shows, in particular, that η does not depend strongly on f, but only on

$$\alpha(f) = \inf(\text{support } f).$$

If η works for one f in $L^1(\omega)$, then it will turn out (Theorem 3.1 (b)) that η works for all g with $\alpha(g) \geq \alpha(f)$, so we define:

Definition 1.2 Let ω be an algebra weight and η be a bounded weight. Then η is a *convergence factor for ω at $a \geq 0$*, provided that $\lambda_n \to \lambda$ weak* in $M(\omega)$ and $f \in L^1(\omega)$ with $\alpha(f) \geq a$ together imply that $\lambda_n * f \to \lambda * f$, in the norm of $L^1(\omega\eta)$. If this holds for all algebra weights ω and all f in $L^1(\omega)$, we say that η is a *universal convergence factor*.

Thus Question 1 is really asking for necessary and sufficient conditions for η to be a convergence factor. The first step is to concentrate on a fixed f. For this purpose it is convenient to reformulate the property in terms of compactness of convolution operators. For f in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$, we let $T_f(\mu) = f * \mu$ be the operator of convolution by f on $M_{\mathrm{loc}}(\mathbf{R}^+)$, or any restriction of this operator to maps between subspaces of $M(\omega)$. Then we obtain:

Theorem 1.3 *For ω an algebra weight, η a bounded weight and f in $L^1(\omega)$, the following are equivalent:*

(a) *Whenever $\lambda_n \to \lambda$ weak $*$ in $M(\omega)$, then $\lambda_n * f \to \lambda * f$ in norm in $L^1(\omega\eta)$;*
(b) *T_f is a compact operator from $M(\omega)$ to $L^1(\omega\eta)$;*

(c) *T_f is a compact operator from $L^1(\omega)$ to $L^1(\omega\eta)$;*
(d) *T_f is a weakly compact operator from $L^1(\omega)$ to $L^1(\omega\eta)$.*

Proof We first prove the equivalence of (a) and (b). Suppose that (a) holds and let $\{\lambda_n\}$ be a bounded sequence in $M(\omega)$. Then $\{\lambda_n\}$ has a weak* convergent subsequence $\{\lambda'_n\}$. By (a) $\{T_f(\lambda'_n)\}$ is a convergent subsequence of $\{T_f(\lambda_n)\}$ in $L^1(\omega\eta)$, and so $T_f\colon M(\omega) \to L^1(\omega\eta)$ is compact. Conversely, suppose that T_f is compact and that $\lambda_n \to \lambda$ weak* in $M(\omega)$. Since T_f is compact we need only show that if $\{\lambda_n * f\}$ has a subsequence $\{\lambda'_n * f\}$ which converges in norm in $L^1(\omega\eta)$ to some g in $L^1(\omega\eta)$, then $\lambda * f = g$. Suppose that h is a continuous function with compact support. Since $\lambda'_n * f$ converges weak* to $\lambda * f$ in $M(\omega)$, by Theorem 1.1 (c), we have that $\langle \lambda'_n * f, h\rangle \to \langle \lambda * f, h\rangle$. Similarly, $\lambda'_n * f \to g$ in norm in $L^1(\omega\eta)$ also implies $\langle \lambda'_n * f, h\rangle \to \langle g, h\rangle$, since $1/(\omega\eta)$ is bounded on compact sets. Hence $\langle \lambda * f, h\rangle = \langle g, h\rangle$, for all continuous functions h with compact support in $\mathbf{R}^+$. It then follows from the uniqueness in the Riesz Representation Theorem that $(\lambda * f)\, dt$ and $g dt$ are the same measures on $\mathbf{R}^+$, so $\lambda * f = g$ almost everywhere, as required.

It is clear that (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

The proofs that (c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (c) are similar to the proofs we have given for the special case that $\eta \equiv 1$ in [GGM, Lemma 3.1, p. 283] and [GG1, Theorem 4.1, p. 515], respectively. For the convenience of the reader, we sketch the arguments.

Since compactness of an operator is defined in terms of the closure of the image of the unit ball, to prove (c) $\Rightarrow$ (b) it suffices to show that if μ belongs to the unit ball of $M(\omega)$, then $T_f(\mu)$ is a limit of a sequence $\{T_f(g_n)\}$ with each g_n in the unit ball of $L^1(\omega)$. Since ω is an algebra weight, there is an approximate identity $\{e_n\}$ with $\|e_n\| = 1$ in $L^1(\omega)$ [Gr.2, Theorem 2.2(A), p. 592]. Let $g_n = e_n * \mu$. Then $\|g_n\| \le \|\mu\|$ and $T_f(g_n) = (e_n * f) * \mu$ has limit $f * \mu = T_f(\mu)$ in $L^1(\omega)$ and hence in $L^1(\omega\eta)$.

To prove (d) $\Rightarrow$ (c) it will be enough to show that whenever $\lambda_n \to 0$ weak* in $M(\omega)$ and $\lambda_n * f \to 0$ weakly in $L^1(\omega\eta)$, then $\lambda_n * f \to 0$ in norm in $L^1(\omega\eta)$. We will use a variant of the Dunford-Pettis Theorem for the measure $\omega\eta dt$ on $\mathbf{R}^+$. It follows from [DS, Theorem IV.8.9, and Corollary IV.8.10, pp. 292–293] that $\lambda_n * f \to 0$ weakly implies that

$$\lim_{b\to\infty} \int_b^\infty |\lambda_n * f(t)|\omega(t)\eta(t)\, dt = 0,$$

uniformly in n. On the other hand it follows from Theorem 1.1 and the fact that $\omega\eta$ is locally bounded that

$$\lim_{n\to\infty} \int_0^b |\lambda_n * f(t)|\omega(t)\eta(t)\, dt = 0,$$

for each fixed b. Thus $\lambda_n * f \to 0$ in norm in $L^1(\omega\eta)$, as required.

The requirements that the bounded weight η be positive with $1/\eta$ locally bounded is a technical convenience to simplify proofs, like the proof that (b) $\Rightarrow$ (a) in Theorem 1.3. All our results can be extended to bounded $\eta \ge 0$ in the following manner: Let $\eta'(t)$ be a universal convergence factor which is also a continuous algebra weight.

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For instance $\eta'(t) = e^{-t}$ will do (see Theorem 4.1 below, or the weaker result in [GG1, Theorem 3.2, p. 512]). Let $\eta_1(t) = \eta(t) + \eta'(t)$, which is bounded and locally bounded away from 0. Since all our results work for $\eta'(t)$, a given condition will hold for η if and only if it holds for η_1. For an algebra weight ω, the assumption $\omega > 0$ forces the weight ω to be locally bounded away from 0 [HP, Theorem 7.4.1, p. 241].

The assumption in most of our results that η is bounded is essential. It is equivalent to saying that $L^1(\omega) \subseteq L^1(\omega\eta)$, with the embedding continuous. Thus any convergent net in $L^1(\omega)$ also converges in $L^1(\omega\eta)$ norm, and any continuous map into $L^1(\omega)$ can also be considered as a continuous map into $L^1(\omega\eta)$.

The fundamental paper on compactness of convolution operators on $L^1(\omega)$, that is, the case $\eta \equiv 1$, is [BD1]. When $\eta = 1$, they prove the equivalence of Theorem 1.3 (c) and (d), and show that if $\alpha(f) = a$, then T_f is compact on $L^1(\omega)$ if and only if $\lim_{t\to\infty} \frac{\omega(t+b)}{\omega(t)} = 0$ for all $b > a$. Such weights are said to be *regulated* at a. In [GGM] we prove Theorem 1.3 (a) and (b) for regulated weights, and use 1.3 (c) to prove that homomorphisms in $L^1(\omega)$, when ω is regulated, have many nice properties. We continue the study of various types of convergence of $\{\lambda_n * f\}$ in [GG1]. In [GG1, Theorem 3.2, p. 512] we give sufficient conditions for η to be what we now call a universal convergence factor. Another interesting study of compactness is in Detre's thesis [D], which unfortunately was never published. Although we do not make direct use of Detre's results, some of our arguments are influenced by his techniques.

From the original paper of Bade and Dales [BD1] onwards, the study of compactness, and later convergence, was done in the context of studying ideals and homomorphisms of $L^1(\omega)$, particularly in the radical case. For these purposes, the key question about convergence, which we repeat from [GG1, Question 3, p. 507], seems to be:

Question 2 Suppose that ω is an algebra weight and $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$ for which the ideal $I = \{f \in L^1(\omega) \colon \lambda_n * f \to \lambda * f \text{ in norm}\}$ contains some g with $\alpha(g) = 0$. Must I be all of $L^1(\omega)$?

In [GG1, Theorem 2.5, p. 511] we show that the answer to Question 2 above is positive when ω is regulated. As we have pointed out in [GG1], a negative answer to Question 2 for a radical $L^1(\omega)$ would provide a long-sought example with $\alpha(g) = 0$ but $L^1(\omega) * g$ not dense. A positive answer, even with the additional assumption that the closed ideal J is weak*-dense, would show that all homomorphisms into $L^1(\omega)$ have the nice properties we have listed in [GGM, Theorem 2.2. p. 280] and elsewhere.

In the next section we give an integral formula equivalent to T_f being compact from $M(\omega)$ to $L^1(\omega\eta)$. In Section 3, we use this formula to characterize when η is a convergence factor for a fixed ω. In Section 4 we give characterizations of universal convergence factors. In Section 6 we apply our results to give conditions on when a closed ideal in $L^1(\omega)$ must be weak*-closed, generalizing results in [BD2]. We also give some applications of our characterizations of convergence factors, which extend results we proved in [GG1]. In Section 5 we characterize compactness of $T_f \colon L^1(\omega) \to L^1(\omega\eta)$ in terms of properties of its adjoint, and we give $L^p(\omega)$ analogues of our convergence factor results. These generalize and extend results for the

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case $\eta \equiv 1$ given in [BD1] and [GG2].

2 Compactness of Convolution by f

In this section we fix an algebra weight ω, a bounded weight η, and a function f in $L^1(\omega)$, and we describe when $T_f \colon M(\omega) \to L^1(\omega\eta)$ is compact. Recall that for $s \geq 0$, δ_s is the point mass at s. When g is locally integrable, then $\delta_s * g(t) = g(t-s)$ for $t \geq s$, so that $\delta_s * g$ is the right translate of g by s. The compactness of T_f will depend on

$$D(s) = D(s, f) = \left\| \frac{1}{\omega(s)} \delta_s * f \right\|_{\omega\eta} = \int_0^\infty |f(t)| \frac{\omega(t+s)}{\omega(s)} \eta(t+s)\, dt \tag{2.1}$$

in the following way.

Theorem 2.1 *The map $T_f \colon \mu \mapsto \mu * f$ is compact from $M(\omega)$ into $L^1(\omega\eta)$ if and only if* $\lim_{s\to\infty} D(s) = 0$.

This generalizes the result of Bade and Dales [BD1, Theorem 2.2 p. 85] for the case $\eta = 1$, but the proof we give is closer in spirit to Detre's [D, pp. 42–43].

For one direction of the proof, we will need to calculate the norms of certain restrictions of T_f. Recall that $\alpha(\mu) = \inf(\text{support } \mu)$. For a subspace M of $M_{\text{loc}}(\mathbf{R}^+)$ and $a \geq 0$, we define

$$M_a = \{\mu \in M \colon \alpha(\mu) \geq a\}. \tag{2.2}$$

We then obtain:

Lemma 2.2 *The restriction of $T_f \colon M(\omega) \to L^1(\omega\eta)$ to the subspace $M(\omega)_a$ has a norm* $\sup\{D(s) \colon s \geq a\}$.

Proof Since $\frac{1}{\omega(s)}\delta_s$ is a unit vector in $M(\omega)_a$, the restriction must have norm at least

$$\left\| T_f \left(\frac{1}{\omega(s)} \delta_s \right) \right\| = \left\| \frac{1}{\omega(s)} \delta_s * f \right\| = D(s),$$

for each $s \geq a$. Probably the easiest way to get the reverse inequality is to use the standard Bochner integral formula for convolution. For μ in $M(\omega)_a$, we have

$$T_f(\mu) = f * \mu = \int_{[a,\infty)} \delta_t * f \, d\mu(t),$$

where the integral is a Bochner integral in $L^1(\omega)$, and hence in the larger space $L^1(\omega\eta)$. Hence we have that

$$\begin{aligned} \|T_f(\mu)\| = \|\mu * f\|_{\omega\eta} &\leq \int_{[a,\infty)} \left\| \frac{1}{\omega(t)} \delta_t * f \right\|_{\omega\eta} \omega(t)\, d|\mu|(t) \\ &\leq \sup\{D(s) \colon s \geq a\} \|\mu\|_\omega. \end{aligned}$$

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This gives the reverse inequality, and completes the proof of the lemma.

Proof of Theorem 2.1 First suppose that T_f is a compact operator. For each h in $C_0(1/\omega)$, we have $\langle \frac{1}{\omega(s)}\delta_s, h\rangle = h(s)/\omega(s)$, which has a limit 0 as $s \to \infty$, by the definition of $C_0(1/\omega)$. Thus $\frac{1}{\omega(s)}\delta_s \to 0$ weak* in $M(\omega)$. Hence the compactness of T_f implies that $D(s) = \|T_f(\frac{1}{\omega(s)}\delta_s)\| \to 0$ as $s \to \infty$.

Conversely, suppose that $\lim_{s\to\infty} D(s) = 0$. For each $a > 0$, let Q_a be the natural projection from $M(\omega)$ onto $M(\omega)_a$ given by $(Q_a\lambda)(E) = \lambda\big(E\cap[a,\infty)\big)$. Let P_a be the complementary projection, so that the range of P_a is the space of measures in $M(\omega)$ which are concentrated on $[0, a)$. Since Q_a is a projection of norm 1, the norm of $T_f Q_a$ is the same as the norm of the restriction of T_f to $M(\omega)_a$. Thus, by Lemma 2.2, we have that

$$\lim_{a\to\infty}(T_f - T_f P_a) = \lim_{a\to\infty} T_f Q_a = 0.$$

Thus to prove the compactness of T_f, it will be enough to show that all $T_f P_a$ are compact. This can be done without any hypothesis on $D(s)$, so we separate this out as a lemma, which we will use elsewhere in this paper.

Lemma 2.3 *Suppose that ω is an algebra weight and that P_a is the natural projection from $M(\omega)$ onto the measures concentrated in $[0, a)$. Then $T_f P_a$ is a compact operator from $M(\omega)$ to $L^1(\omega)$, for every f in $L^1(\omega)$.*

Proof Suppose that $\lambda_n \to 0$ weak* in $M(\omega)$. The operator P_a is weak*-continuous since it is the adjoint of the analogous restriction operator on the pre-dual $C_0(1/\omega)$. Hence $P_a\lambda_n \to 0$ weak* as $n \to \infty$. We need to show that $T_f P_a(\lambda_n) = f * (P_a\lambda_n)$ also has limit 0 in norm as $n \to \infty$. Since the functions with compact support are dense in $L^1(\omega)$, we may assume that f has bounded support, say support $(f) \subseteq [0, b]$. Then the support of each $f * P_a\lambda_n$ is contained in $[0, a + b]$. Thus

$$\|T_f(P_a\lambda_n)\| = \int_0^{a+b} |f * (P_a\lambda_n)(t)|\omega(t)\,dt.$$

Since $\omega(t)$ is locally bounded and $P_a\lambda_n \to 0$ weak* it follows from Theorem 1.1 that $\lim_{n\to\infty} \|T_f(P_a\lambda_n)\| = 0$, as required. This completes the proof of the lemma, and of Theorem 2.1.

In order to efficiently use limit statements like $D(s) \to 0$, we will need to go back and forth between limits of integrals and pointwise limits of functions. We collect the needed results in the following lemma. The analogue for L^1 limits is standard measure theory. The extension to $L^1_{\mathrm{loc}}(\mathbf{R}^+)$ limits is routine in (b) and uses a diagonalization argument in (a).

Lemma 2.4 *Suppose that $\{g_n\}_{n=1}^{\infty}$ and g belong to $L^1_{\mathrm{loc}}(\mathbf{R}^+)$.*

(a) *If $g_n \to g$ in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$, then $\{g_n\}$ has a subsequence which converges almost everywhere on $\mathbf{R}^+$ to g.*

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(b) *If $\{g_n\}$ is dominated by a locally integrable function, then $g_n \to g$ in $L^1_{\text{loc}}(\mathbf{R}^+)$ if and only if every subsequence of $\{g_n\}$ has a subsequence which converges almost everywhere to g. If the dominating function is integrable, then $g_n \to g$ in $L^1_{\text{loc}}(\mathbf{R}^+)$ as well.*

In the above lemma, we can replace $\mathbf{R}^+$ by any $[a, \infty)$, in which case convergence in $L^1_{\text{loc}}(\mathbf{R}^+)$ is replaced by convergence on $L^1_{\text{loc}}[a, \infty)$. That is $g_n \to g$ means

$$\lim_{n\to\infty} \int_a^b |g_n(t) - g(t)|\, dt = 0,$$

for all $b \geq a$. We can also allow $b = \infty$ if there is a dominating function in $L^1[a, \infty)$.

3 Convergence Factors for a Fixed ω

In this section we determine when η is a convergence factor for ω at a (see Definition 2.2) and prove some related results. The following theorem is the basic result.

Theorem 3.1 *Suppose that ω is an algebra weight, that $a \geq 0$ and that η is a bounded weight on $\mathbf{R}^+$. Then the following are equivalent:*

(a) *There is an f in $L^1(\omega)$ with $\alpha(f) = a$ for which T_f is a compact operator from $M(\omega)$ to $L^1(\omega\eta)$;*
(b) *η is a convergence factor for ω at a;*
(c) *for every sequence $\{s_n\}$ of positive numbers increasing to ∞, there is a subsequence $\{s_n'\}$ for which*

$$\lim_{n\to\infty} \frac{\omega(t + s_n')}{\omega(s_n')}\eta(t + s_n') = 0,$$

for almost every $t > a$;
(d) *for all $b > a$,*

$$\lim_{s\to\infty} \int_a^b \frac{\omega(t + s)}{\omega(s)}\eta(t + s)\, dt = 0.$$

If $\omega(t)$ is integrable, we can also allow $b = \infty$.

Proof The equivalence of (c) and (d) is just Lemma 2.4 and the remark following it. Also (b) is a stronger statement than (a) since it asserts that T_f is compact for all f in $L^1(\omega)$ with $\alpha(f) \geq a$. So we need only prove (c) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c).

Suppose that (c) holds and let f belong to $L^1(\omega)$ with $\alpha(f) \geq a$. The subsequence limit condition in (c) also holds for $|f(t)|\big(\omega(t+s)/\omega(s)\big)\,\eta(t+s)$, which is dominated by the integrable function $|f(t)|\omega(t)\|\eta\|_\infty$. Since $\alpha(f) \geq a$, the integral defining $D(s)$ in formula (2.1) can be written with a in place of 0 as the lower limit. It therefore follows from Lemma 2.4 that $\lim_{s\to\infty} D(s) = 0$, which by Theorem 2.1 implies that T_f is compact. This proves (b).

Suppose that (a) holds. Since the integral formula for $D(s)$ involves $|f(t)|$ and not $f(t)$, it follows that convolution by $|f|$ is also compact from $M(\omega)$ to $L^1(\omega\eta)$. Choose some $e^{-\lambda t}$ in $L^1(\omega)$; then convolution by $e^{-\lambda x} * |f|(x) = e^{-\lambda x} \int_a^x |f(t)|e^{\lambda t}\, dt$ is also compact. Since $\alpha(f) = a$, the function $e^{-\lambda x} * |f|$ is never 0 on (a, ∞). Thus there is no loss of generality in assuming that T_f is compact for some f which vanishes on $[0, a]$ but is strictly positive on (a, ∞). By Theorem 2.1 $\lim_{s\to\infty} D(s, f) = 0$. Thus by Lemma 2.4 again we have that for every unbounded increasing sequence, there is a subsequence $\{s'_n\}$ for which

$$\lim_{t\to 0} f(t)\bigl(\omega(t + s'_n)/\omega(s'_n)\bigr)\,\eta(t + s'_n) = 0,$$

for almost every $t > a$. Since $f(t)$ is never 0 on (a, ∞), this gives (c) and completes the proof of the theorem.

The condition in (c) is complicated because for an arbitrary η we know nothing about the structure of $L^1(\omega\eta)$, except that it contains the algebra $L^1(\omega)$ as a continuously embedded subspace. This condition can be simplified when $M(\omega\eta)$ is an algebra, or even when it is just *translation invariant* in the sense that $\delta_b * M(\omega\eta) \subseteq M(\omega\eta)$ for all $b > 0$. It would then follow from the closed graph theorem that convolution with δ_b was a continous operator on $M(\omega\eta)$. Thus $\delta_b * M(\omega\eta) \subseteq M(\omega\eta)$ if and only if $\bigl(\omega(t + b)/\omega(t)\bigr)\,\bigl(\eta(t + b)/\eta(t)\bigr)$ is bounded on $\mathbf{R}^+$.

The case $\eta \equiv 1$ in the following theorem is the standard characterization of compactness on $L^1(\omega)$ in terms of regulated weights [BD1, Theorem 2.10, p. 91].

Theorem 3.2 *Suppose that ω is an algebra weight and that $M(\omega\eta)$ is translation invariant. Then the following are equivalent:*

(a) *η is a convergence factor for ω at a;*
(b) $\lim_{s\to\infty} \frac{\omega(t+s)}{\omega(s)}\eta(t + s) = 0$*, for all $t > a$.*

Proof Condition (b) is clearly stronger than Theorem 3.1 (c), so (b) $\Rightarrow$ (a) holds even if $M(\omega\eta)$ is not translation invariant. We complete the proof by assuming translation invariance and the condition in Theorem 3.1 (c), and proving condition (b).

Suppose therefore that (b) were not true. Then there would be a $t_0 > a$ for which the limit were not 0. We could then find a sequence $\{s_n\}$ increasing to infinity for which $\bigl(\omega(t_0 + s_n)/\omega(s_n)\bigr)\,\eta(t_0 + s_n)$ is bounded away from 0; so that no subsequence has limit 0. We let $\{s'_n\}$ be a subsequence given by Theorem 3.1 (c) and reach a contradiction. Since 3.1 (c) guarantees convergence for almost every $t > a$, there is a t_1 between a and t_0 for which

$$r_n = \frac{\omega(t_1 + s'_n)}{\omega(s'_n)}\eta(t_1 + s'_n)$$

has limit 0. Also, since $M(\omega\eta)$ is translation invariant, the sequence

$$\frac{\omega(t_0 + s'_n)\eta(t_0 + s'_n)}{\omega(t_1 + s'_n)\eta(t_1 + s'_n)}$$

is bounded above by some number K. We now have

$$\frac{\omega(t_0 + s'_n)}{\omega(s'_n)}\eta(t_0 + s'_n) \le \frac{\omega(t_0 + s'_n)\eta(t_0 + s'_n)}{\omega(t_1 + s'_n)\eta(t_0 + s'_n)} r_n \le K r_n.$$

Since $\lim_{n\to\infty} Kr_n = 0$, this contradicts our assumption on $\{s_n\}$ and proves the theorem.

In the next result we show how to construct a convergence factor at 0 from a convergence factor at $a > 0$. Following [Gr1, pp. 540–541], for any subspace L of $L^1_{\text{loc}}(\mathbf{R}^+)$ we let

$$\mathcal{S}^{-a}(L) = \{f \in L^1_{\text{loc}}(\mathbf{R}^+) : \delta_a * f \in L\},$$

and we notice that for any weight ω on $\mathbf{R}^+$, we have $\mathcal{S}^{-a}\big(L^1(\omega)\big) = L^1\big(\omega(t+a)\big)$. Also the *translation map* $T_a g = \delta_a * g$ is an isometry from $L^1\big(\omega(t+a)\big)$ onto $L^1(\omega)_a$. When $M(\omega)$ is translation invariant, then $\mathcal{S}^{-a}\big(L^1(\omega)\big) \supseteq L^1(\omega)$, and when ω is a radical algebra weight the containment is proper [Gr1, p. 541].

Theorem 3.3 *Suppose that ω is an algebra weight, $a > 0$, and η is a bounded weight. Then the following are equivalent:*

(a) *η is a convergence factor for ω at a;*
(b) *there is an f in $\mathcal{S}^{-a}\big(L^1(\omega)\big)$, with $\alpha(f) = 0$, for which T_f is a compact operator from $M(\omega)$ to $\mathcal{S}^{-a}\big(L^1(\omega\eta)\big) = L^1\big(\omega(t+a)\eta(t+a)\big)$;*
(c) *for all f in $\mathcal{S}^{-a}\big(L^1(\omega)\big)$, T_f is a compact operator from $M(\omega)$ to $L^1\big(\omega(t+a)\eta(t+a)\big)$;*
(d) *$\eta_a(t) = \frac{\omega(t+a)}{\omega(t)}\eta(t+a)$ is a convergence factor for ω at 0.*

Proof Whenever f belongs to $\mathcal{S}^{-a}\big(L^1(\omega)\big)$, we have $g = \delta_a * f$ in $L^1(\omega)$. Hence $g * M(\omega) = \delta_a * f * M(\omega) \subseteq L^1(\omega\eta)$, so that T_f maps $M(\omega)$ into $\mathcal{S}^{-a}\big(L^1(\omega\eta)\big)$. Also if $g = \delta_a * f$ belongs to $L^1(\omega)$, then f belongs to $\mathcal{S}^{-a}\big(L^1(\omega)\big)$. Notice that $\eta_a(t)$ is defined so that $\omega(t)\eta_a(t) = \omega(t+a)\eta(t+a)$. Thus applying Theorem 3.1 to both η and η_a, we see that it is enough to prove that whenever f beongs to $\mathcal{S}^{-a}\big(L^1(\omega)\big)$ and $g = \delta_a * f$, then $T_f \colon M(\omega) \to \mathcal{S}^{-a}\big(L^1(\omega\eta)\big)$ is compact if and only if $T_g \colon M(\omega) \to L^1(\omega\eta)$ is compact. But the translation operator $T_a \colon \mathcal{S}^{-a}\big(L^1(\omega\eta)\big) \to L^1(\omega\eta)_a$ is an isometry, and $T_g = T_a T_f$. Hence T_f is compact if and only if T_g is compact. This completes the proof.

4 Universal Convergence Factors

In this section we give several conditions for η to be a universal factor (see Definition 1.2).

The following is the basic result.

Theorem 4.1 *For a bounded weight η on $\mathbf{R}^+$, the following are equivalent:*

(a) *η is a universal convergence factor;*
(b) *there is a non-zero f in $L^1(\mathbf{R}^+)$ for which T_f is a compact operator from $M(\mathbf{R}^+) = M(1)$ to $L^1(\eta) = L^1(1\eta)$;*
(c) *$\lim_{s\to\infty} \int_s^{s+1} \eta(t)\,dt = 0$;*
(d) *$\lim_{k\to\infty} \int_0^1 \eta(k+t)\,dt = 0$.*

Proof Since $\eta \geq 0$, it makes no difference whether the limit is taken for real numbers s or integers k; so the equivalence of (c) and (d) is clear. Also, by definition, (a) $\Rightarrow$ (b). We will prove (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

Suppose that (b) holds and let $\alpha(f) = a$. If we write $f = \delta_a * g = g * \delta_a$, then g also belongs to $L^1(\mathbf{R}^+)$ and $\alpha(g) = 0$. The translation operator $T_a(\mu) = \delta_a * \mu$ is an isometry from $M(\mathbf{R}^+)$ to $M(\mathbf{R}^+)_a$. Let $S_a\colon M(\mathbf{R}^+)_a \to M(\mathbf{R}^+)$ be the inverse of this isometry and let T'_g be the restriction of T_g to a map from $M(\mathbf{R}^+)_a$ to $L^1(\eta)$. As in the proofs of Theorem 2.1 and Lemma 2.3, let Q_a be the natural projection of $M(\mathbf{R}^+)$ onto $M(\mathbf{R}^+)_a$ and let P_a be the complementary projection. Then $T'_g T_a = T_f$, so that $T'_g = T_f S_a$ is compact. Hence $T_g Q_a$ is a compact operator from $M(\mathbf{R}^+)$ to $L^1(\eta)$. Since $T_g P_a$ is always compact, by Lemma 2.3, we have that $T_g = T_g Q_a + T_g P_a$ is compact. Since $\alpha(g) = 0$, it follows from Theorem 3.1 that η is a convergence factor for $\omega(t) \equiv 1$ at 0.

Hence if we let h be the characteristic function of the unit interval $[0, 1]$, the convolution operator $T_h\colon \mu \mapsto \mu * h$ is compact from $M(\mathbf{R}^+)$ to $L^1(\eta)$. Hence, by Theorem 2.1

$$D(s) = D(s, h) = \int_0^1 \eta(t + s)\, dt = \int_s^{s+1} \eta(t)\, dt$$

has limit 0 as $s \to \infty$. This proves (c).

Now we suppose that η satisfies (c) and we fix an algebra weight ω. We need to show that η is a convergence factor at 0 for this ω. By Theorems 2.1 and 3.1, we need only find some f in $L^1(\omega)$ with $\alpha(f) = 0$, for which $\lim_{s\to\infty} D(s, f) = 0$. Let f be $1/\omega$ times the characteristic function of $[0, 1]$. Then

$$D(s, f) = \int_0^1 \frac{\omega(t + s)}{\omega(t)\omega(s)} \eta(t + s)\, dt \leq \int_0^1 \eta(t + s)\, dt,$$

which has limit 0 by (c). This completes the proof of the theorem.

There are a variety of simple conditions equivalent to (c) and (d) above. We have already remarked that we could let $s \to \infty$ for real s, or for s restricted to a fixed increasing unbounded sequence. Also one could replace 1 in these formulas by any fixed $b > 0$. As in Lemma 2.4 one could rephrase (d) in terms of almost-everywhere convergence of subsequences. Since the integrals in (c) and (d) are all bounded by $\|\eta\|_\infty$, we can replace η by any η^p in (c) and (d). Hence we have:

Corollary 4.2 *Let $0 < p < \infty$ and let η be a non-negative bounded measurable function. Then η is a universal convergence factor if and only if η^p is.*

There are a variety of simple easily verified sufficent conditions for the bounded weight η to be a universal convergence factor. In [GG2, Theorem 3.2, p. 512], we showed that η belonging to $L^1(\mathbf{R}^+)$ was sufficient. This is strictly stronger than (c) and (d) in Theorem 4.1; because $\eta > 0$ is integrable precisely when the infinite series $\sum_{k=0}^{\infty} \int_k^{k+1} \eta(t)\, dt$ converges. Other useful conditions are in the next result.

Corollary 4.3 *The bounded weight η is a universal convergence factor if either of the following conditions holds:*

(i) *There is a $0 < p < \infty$ for which η belongs to $L^p(\mathbf{R}^+)$;*

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(ii) $\lim_{t\to\infty} \eta(t) = 0$.

Proof Suppose that (i) holds. We have already observed that the condition is sufficient for $p = 1$. The general case then follows from Corollary 4.2. If (ii) holds, then $\lim_{k\to\infty} \eta(k+t) = 0$, so that Theorem 3.1 (d) follows from the bounded convergence theorem.

Neither (i) or (ii), or both together, is necessary for η to be a universal convergence factor. But Theorem 3.1 (d) does imply that when η is a universal convergence factor, then there is a sequence $\{k_n\}$ increasing to infinity for which $\lim_{n\to\infty} \eta(k_n + t) = 0$, for almost every t in $[0, 1]$.

5 Applications to the Structure of $L^1(\omega)$

In this section we give applications of the techniques we have been developing to studying closed ideals in $L^1(\omega)$ and, using our results together with information about the ideals, to other questions. The ideals $L^1(\omega)_a$ for $0 \le a < \infty$ and $\{0\}$ are called *standard ideals* of the convolution algebra $L^1(\omega)_a$. In [BD2, Proposition 1.9, p. 72] Bade and Dales show that if the algebra weight ω is regulated at 0, that is, $\eta(t) \equiv 1$ is a convergence factor for ω at 0, then every closed ideal whose weak*-closure is standard is already weak*-closed. The following result extends the Bade-Dales result in two ways; we obtain their conclusion when ω is regulated at any a, and we obtain a stronger result when ω is regulated at 0.

Theorem 5.1 *Suppose that ω is an algebra weight and that I is a closed ideal in $L^1(\omega)$.*

(a) *If ω is regulated at 0, then I is weak*-closed.*

(b) *If ω is regulated at any $a > 0$ and if the weak*-closure of I is standard, then I is weak*-closed.*

The conclusion in (a) is strictly stronger than that in (b), because there are examples of regulated weights for which $L^1(\omega)$ has closed non-standard ideals [DM]. Our results will be in terms of an arbitrary convergence factor η, so that Theorem 5.1 will be the special case when $\eta \equiv 1$.

Recall [DS, Definition V.5.3, p. 427], [DY, Definition 2, p. 48] that the bw* topology on a dual space, such as $M(\omega)$, is defined by saying that a set E is bw*-closed if and only if the intersection of E with every multiple of the closed unit ball in X is weak*-closed. Thus the bw* topology is the strongest topology in which every norm-bounded weak*-convergent net converges. Although we will not need an explicit description of bw*-convergent nets [DY, Lemma 2, p. 49], we should point out that there are unbounded nets which converge in the bw*-topology [DY, p. 48].

For a subspace Y, like $L^1(\omega)$, which is weak*-dense in a dual space X, like $M(\omega)$, we will use the terms bw* and weak* topologies on Y for the relative topologies inherited from X.

We will need the following lemma, which observes that the Krein-Smulian Theorem [DS, Theorem V.5.7, p. 429], remains true for weak*-dense subspaces.

Lemma 5.2 *Suppose that Y is a weak*-dense subspace of a dual space X. A convex subset K of Y is weak*-closed in Y if its intersection with each closed ball about the origin in Y is weak*-closed. That is, K is weak*-closed if and only if K is* bw**-closed.*

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Proof It follows from the usual Krein-Smulian Theorem [DS, Theorem V.5.7, p. 429], that Y is bw*-dense in X. Hence X and Y have the same weak*-continuous linear functionals and the same bw*-continuous linear functionals. Hence we can extend from the dual space X to its weak*-dense subspace Y, the result [DS, Theorem V, 5.6, p. 428] that a linear functional is weak*-continuous if and only if it is bw*-continuous. The lemma follows immediately from this result for Y just as the Krein-Smulian Theorem followed immediately [DS, p. 429] from the result for X.

We can now prove the $a = 0$ case of Theorem 5.1, for arbitrary convergence factors.

Theorem 5.3 *Suppose that η is a convergence factor at 0 for the algebra weight ω. If I is an ideal in $L^1(\omega)$ which is closed in the (relative) $L^1(\omega\eta)$ norm, then I is weak*-closed in $L^1(\omega)$.*

Proof By the extension of the Krein-Smulian Theorem given in Lemma 5.2 above, we just need to show that $I \cap B$ is weak*-closed when B is a closed ball in $L^1(\omega)$. Since $M(\omega)$ is the dual space of a separable space, the relative weak*-topology on B is metrizable. So we only need to show that if $\{g_n\}$ is a sequence in $I \cap B$ which converges weak* to g, then g is in I. Since η is a convergence factor, it follows (see Definition 1.2) that $f * g_n \to f * g$ in the norm of $L^1(\omega\eta)$ for all f in $L^1(\omega)$. Since I is an $L^1(\omega\eta)$ closed ideal this implies that $L^1(\omega) * g \subseteq I$. Since $L^1(\omega)$ is continuously embedded in $L^1(\omega\eta)$, I is a closed ideal in $L^1(\omega)$. But $L^1(\omega)$ has a bounded approximate identity so we have $g \in c\ell\big(L^1(\omega) * g\big) \subseteq I$, as required.

Combining the previous theorem with Theorem 3.3 gives the following corollary:

Corollary 5.4 *Suppose that η is a convergence factor at a for the algebra weight ω. If I is an ideal in $L^1(\omega)$ which is closed in $L^1(\omega)$ in the norm of $\mathcal{S}^{-a}\big(L^1(\omega\eta)\big) = L^1\big(\omega(t+a)\eta(t+a)\big)$, then I is weak*-closed in $L^1(\omega)$.*

The following result is the version of Theorem 5.1 (b) which applies to arbitrary convergence factors.

Theorem 5.5 *Suppose that η is a convergence factor at a for the algebra weight ω. Suppose also that I is an ideal of $L^1(\omega)$ which is closed in the $L^1(\omega\eta)$ norm. If the weak*-closure of I is standard, then I is already weak*-closed.*

The above theorem will follow easily from the following result, which does not assume that the weak*-closure is standard.

Theorem 5.6 *Suppose that η is a convergence factor at $a \geq 0$ for ω and that I is an ideal of $L^1(\omega)$ which is closed in the $L^1(\omega\eta)$ norm on $L^1(\omega)$. If J is the weak*-closure of I, then $\delta_a * J$ and $M(\omega)_a * J$ are both subsets of I.*

For regulated weights, that is, for $\eta = 1$, the above result says:

Corollary 5.7 *Suppose that ω is an algebra weight regulated at $a \geq 0$, and that I is a closed ideal of $L^1(\omega)$. If J is the weak*-closure of I, then $\delta_a * J \subseteq I$ and $M(\omega)_a * J \subseteq I$.*

Notice that Theorem 5.3 is just the case $a = 0$ of Theorem 5.6. It is also easy to deduce Theorem 5.5 from Theorem 5.6.

Proof of Theorem 5.5 (Assuming Theorem 5.6) Let $J = L^1(\omega)_b$. It follows from Theorem 5.6 and the fact that I is closed in $L^1(\omega)$ that $c\ell\big(\delta_a * L^1(\omega)_b\big) \subseteq I$. Since $\delta_a * L^1(\omega)_b$ contains all locally integrable functions whose support is a bounded subset of $[a + b, \infty)$ we have that $c\ell\big(I_a * L^1(\omega)_b\big) = L^1(\omega)_{a+b}$. Thus I contains a standard ideal and is therefore standard, by [Gr1, Lemma 6.2, p. 548].

In our proof of Theorem 5.6, we will need the following two relatively simple results. The second of the results augments Theorem 1.3.

Lemma 5.8 *Suppose that ω is an algebra weight. If I is a closed ideal in $L^1(\omega)$, then I is also a closed ideal in $M(\omega)$.*

Proof Suppose that $f \in I$ and $\mu \in M(\omega)$. We must show that $\mu * f$ belongs to I. Let $\{e_n\}$ be a bounded approximate identity for $L^1(\omega)$. Then $(\mu * e_n) * g \in I$, and hence $\mu * g = \lim_{n\to\infty} \mu * (e_n * g) = \lim_{n\to\infty}(\mu * e_n) * g$ belongs to I, as required.

Theorem 5.9 *For an algebra weight ω, a bounded weight η, and an element f of $L^1(\omega)$, the following are equivalent:*

(a) *The convolution operator T_f is a compact operator from $M(\omega)$ to $L^1(\omega\eta)$;*
(b) *T_f is continuous from the* bw*-topology on $M(\omega)$ to the norm topology on $L^1(\omega\eta)$;*
(c) *whenever the net $\{\lambda_n\}$ in $M(\omega)$ converges in the* bw*-topology on $M(\omega)$ to λ, then $\lambda_n * f \to \lambda * f$ in the norm of $L^1(\omega\eta)$.*

Proof The equivalence of (b) and (c) is just the standard characterization of continuity for any topology in terms of nets. Since any weak*-convergent net which is bounded is *b*w*-convergent, it is clear that (c) $\Rightarrow$ (a), (*cf.* Theorem 1.3 (a)).

We complete the proof by assuming that T_f is compact and proving (b). Let E be a norm-closed subset of $L^1(\omega\eta)$. We need to show that $T_f^{-1}(E)$ is *b*w*-closed in $M(\omega)$. That is, if B is a closed ball in $M(\omega)$, then we must show that $F = T_f^{-1}(E) \cap B$ is weak*-closed in $M(\omega)$. Suppose that $\{\lambda_n\}$ is a sequence in F which converges weak* in F to λ. It follows from Theorem 1.3 that $\lambda_n * f \to \lambda * f$ in the norm topology of $L^1(\omega\eta)$. Thus $T_f(\lambda)$ belongs to E, and hence λ belongs to F. Since the relative weak*-topology on bounded sets like B is metrizable, this proves that F is closed, as required.

The following lemma is the heart of the proof of Theorem 5.6.

Lemma 5.10 *Under the hypotheses of Theorem 5.6, we have $f * J \subseteq I$, for all f in $L^1(\omega)$ with $\alpha(f) \ge a$.*

Proof Suppose that f belongs to $L^1(\omega)_a$ and that g belongs to J. We need to prove that $f * g \in I$. Since J is the weak*-closure of I, it follows from Lemma 5.2 that J is also the *b*w*-closure of I. Hence there is a net $\{g_n\}$ in I whose *b*w*-limit is g. Since $T_f \colon M(\omega) \to L^1(\omega\eta)$ is compact, it follows from Theorem 5.9 that

$$\lim(f * g_n) = f * g$$

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in $L^1(\omega\eta)$ norm. Since I is an $L^1(\omega\eta)$ norm-closed ideal this implies that $f * g$ belongs to I, as required.

Proof of Theorem 5.6 Suppose that $g \in J$ and let $\{e_n\}$ be a bounded approximate identity in $L^1(\omega)$. Since $L^1(\omega)$ is continuously embedded in $L^1(\omega\eta)$, I is a closed ideal of $L^1(\omega)$. Hence $\delta_a * g = \lim\big((\delta_a * e_n) * g\big)$ belongs to I. Thus $\delta_a * J \subseteq I$, as asserted. Then Lemma 5.8 shows that

$$\big(\delta_a * M(\omega)\big) * J \subseteq M(\omega) * J \subseteq I.$$

The measures in $M_{\mathrm{loc}}(\mathbf{R}^+)$ with compact support in $[a, \infty)$ all belong to $M(\omega)_a$ and form a dense subspace of $M(\omega)_a$. Thus $M(\omega)_a * J$ is a subset of the closed ideal J, as required. This completes the proof of Theorem 5.6 and therefore completes the proofs of Theorems 5.5 and 5.1.

Our next two applications use the results in this paper to improve results in [GG1]. The following result extends [GG1, Theorem 2.5, p. 511] and answers a variant of Question 2 in the introduction to this paper (which is the same as [GG1, Question 3, p. 507]).

Theorem 5.11 *Suppose that ω is an algebra weight, that η is a convergence factor for ω at some $a \geq 0$, and that $L^1(\omega\eta)$ is translation invariant. If $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$ for which there is some g in $L^1(\omega)$ with $\alpha(g) = 0$ and $\lambda_n * g \to \lambda * g$ in $L^1(\omega\eta)$ norm, then $\lim_{n\to\infty}(\lambda_n * f) = \lambda * f$ in the $L^1(\omega\eta)$ norm for all f in $L^1(\omega)$.*

Proof Let

$$I = \{f \in L^1(\omega) \colon \lambda_n * f \to \lambda * f \text{ in } L^1(\omega\eta) \text{ norm}\}.$$

Since $\{\lambda_n\}$ is bounded in $M(\omega)$, the maps $f \mapsto \lambda_n * f$ are uniformly bounded in the operator norm from $L^1(\omega)$ to $L^1(\omega\eta)$. Hence I is closed in $L^1(\omega\eta)$ and is therefore a closed subspace of $L^1(\omega)$. Since $L^1(\omega\eta)$ is translation invariant, I is a translation invariant closed subspace, and hence a closed ideal, in $L^1(\omega)$. It follows from Theorem 1.1 that $\lambda_n \to \lambda$ weak $*$ in $M(\omega)$. Hence our hypotheses on η imply that I contains the standard ideal $L^1(\omega)_a$ and must therefore be a standard ideal [Gr1, Lemma 6.2, p. 548]. Since I contains a function g with $\alpha(g) = 0$, this forces I to be all of $L^1(\omega)$, and proves the theorem.

The next result is a small improvement of [GG1, Corollary 3.4, p. 514].

Theorem 5.12 *Suppose that ω_1 and ω_2 are algebra weights and that $\phi \colon L^1(\omega_1) \to L^1(\omega_2)$ is a continuous non-zero homomorphism. If ω_3 is an algebra weight for which $\eta = \omega_3/\omega_2$ is a convergence factor for ω_2 at some $a \geq 0$, then as a map from $L^1(\omega_1)$ to $L^1(\omega_3) \supseteq L^1(\omega_2)$ the homomorphism is standard.*

Proof Although stronger conditions on η are stated in [GG1, Corollary 3.4, p. 514], the proof given works as long as η is a convergence factor at 0. If we use the fact noted just before Question 3 in [GG1, p. 507] that for the semigroup $\mu_t = \phi(\delta_t)$ in $M(\omega_2)$ there is a g in $L^1(\omega_2)$ for which $\alpha(g) = 0$ and $\lim_{t\to 0} \mu_t * g = g$ in $L^1(\omega_2)$, then the proof of [GG1, p. 507] will still work if we use Theorem 5.11 above in place of the weaker result [GG1, Theorem 3.2, p. 512] used in the original proof.

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6 Dual Spaces and L^p Results

In this section we characterize compactness in terms of properties of the adjoints of the convolution operator T_f, and we prove the L^p analogues of our L^1 results on compactness and convergence factors. The results on dual spaces are the natural generalizations to arbitrary convergence factors, but with different proofs, of results of Bade and Dales [BD2] for regulated weights. The results for $L^p(\omega)$ in a sense extend our results for the regulated case, that is $\eta \equiv 1$, [GG2] to arbitrary η, though some of the new results only apply for η different from 1.

We start by defining the adjoint of convolution, as we did in [GG1, p. 516], by

(6.1) $$f \mathbin{\hat{*}} h(x) = \int_0^\infty f(t)h(x+t)\,dt \quad \text{for } x \geq 0,$$

so that

(6.2) $$\langle f * g, h\rangle = \langle g, f \mathbin{\hat{*}} h\rangle,$$

under hypotheses adequate to use the Fubini Theorem.

In particular, if ω is an algebra weight, the map $h \mapsto f \mathbin{\hat{*}} h$ is the adjoint on $L^\infty(1/\omega)$ of the convolution operator $T_f \colon g \mapsto f * g$ on $L^1(\omega)$. When $T_f \colon L^1(\omega) \to L^1(\omega\eta)$, then the adjoint is just the restriction of $h \mapsto f \mathbin{\hat{*}} h$ to $L^\infty\big(1/(\omega\eta)\big) \subseteq L^\infty(1/\omega)$.

It is easy to see that $L^1(\omega) \mathbin{\hat{*}} L^\infty(1/\omega)$ is composed of continuous functions, and in fact these functions satisfy a uniform continuity property with respect to $1/\omega$ [GG2, Lemma 4.3]. There are formulas for $\mu \mathbin{\hat{*}} h$ analogous to formulas (6.1) and (6.4) [G2, p. 595], but we will not need them here. We will only need

(6.3) $$\langle f * \mu, h\rangle = \langle \mu, f \mathbin{\hat{*}} h\rangle$$

for $f \in L^1(\omega)$, $\mu \in M(\omega)$ and $h \in L^\infty(1/\omega)$.

It turns out that the compactness of T_f is related to $f \mathbin{\hat{*}} h$ belonging to the space $C_0(1/\omega)$. The next result extends to $L^1(\omega) \mathbin{\hat{*}} L^\infty\big(1/(\omega\eta)\big)$, the results of Bade and Dales [BD2, Theorem 1.5, p. 71] for $\eta = 1$. Although they only consider the case $a = 0$, their proof extends readily to $a > 0$.

Theorem 6.1 *Suppose that ω is an algebra weight and that $a \geq 0$. For a bounded weight η, the following are equivalent:*

(a) *η is a convergence factor for ω at a;*
(b) $L^1(\omega)_a \mathbin{\hat{*}} L^\infty\big(1/(\omega\eta)\big) \subseteq C_0(1/\omega)$;
(c) *there is an f in $L^1(\omega)$ with $\alpha(f) = a$ for which $|f| \mathbin{\hat{*}} (\omega\eta) \in C_0(1/\omega)$;*
(d) *there is an f in $L^1(\omega)$ with $\alpha(f) = 0$ for which $f \mathbin{\hat{*}} L^\infty\big(1/(\omega\eta)\big) \subseteq C_0(1/\omega)$.*

Proof Fix an f in $L^1(\omega)$ with $\alpha(f) = a$. We know from Theorem 3.1 that η is a convergence factor at a if and only if T_f is a compact map from $M(\omega)$ to $L^1(\omega\eta)$ (or equivalently by Theorem 1.3, from $L^1(\omega)$ to $L^1(\omega\eta)$). Thus we need to show, for this fixed f, that (c), (d), and the compactness of T_f are equivalent. Formula (2.1) can be expressed as

$$D(s) = \frac{\big(|f|\hat{*}(\omega\eta)\big)(s)}{\omega(s)},$$

so it follows from Theorem 2.1 that T_f is compact if and only if $|f| * (\omega\eta) \in C_0(1/\omega)$. If h in $L^\infty(1/\omega\eta)$ has norm no more than one, it is clear that

$$|f\hat{*}h(s)| \le |f|\hat{*}(\omega\eta)(s).$$

Thus, for each fixed f, we have (c) $\Rightarrow$ (d).

Hence we just need to assume (d) for a fixed f and use this assumption to show that $\lim_{s\to\infty} D(s) = 0$. By formula (2.1), this means that we need to show that $\frac{1}{\omega(s)}\delta_s * f \to 0$ in the norm of $L^1(\omega\eta)$. Since $\frac{1}{\omega(s)}\delta_s \to 0$ weak* in $M(\omega)$, it follows, exactly as in the proof of (d) $\Rightarrow$ (c) in Theorem 1.3, that it is enough to show that $\frac{\delta_s}{\omega(s)} * f \to 0$ weakly in $L^1(\omega\eta)$. Suppose therefore that h belongs to $L^\infty(1/\omega\eta)$. Then by our asumption, $f\hat{*}h$ belongs to $C_0(1/\omega)$. Hence

$$\lim_{s\to\infty} \left\langle \frac{1}{\omega(s)}\delta_s * f, h \right\rangle = \lim_{s\to\infty} \left\langle \frac{1}{\omega(s)}\delta_s, f\hat{*}h \right\rangle = 0,$$

as required. This completes the proof of the theorem.

We can use the above theorem to give an alternate proof of Lemma 5.10, which would then give an alternate proof to all our applications to weak*-closed ideals in the previous section. Since the ideal I is closed in the $L^1(\omega\eta)$ norm on $L^1(\omega)$, to prove that $f * J \subseteq I$, we need only show that if h is in $L^1(\omega\eta)^* = L^\infty\big(1/(\omega\eta)\big)$ and satisfies $\langle J, h\rangle = 0$, then we also have $\langle f * J, h\rangle = 0$. But

$$\langle f * I, h\rangle = \langle I, f\hat{*}h\rangle = 0.$$

By Theorem 6.1, $f\hat{*}h \in C_0(1/\omega)$ and therefore it defines a weak*-continuous linear functional on $L^1(\omega)$. Since J is the weak*-closure of I, the fact that $\langle I, f\hat{*}h\rangle = 0$ implies that $\langle J, f\hat{*}h\rangle = 0$ as well. But

$$\langle J, f\hat{*}h\rangle = \langle f * J, h\rangle,$$

from which Lemma 5.10 follows as noted above.

We now give the p-analogues of our main results on compactness and convergence factors, and relate the results for general $1 < p < \infty$, with those for $p = 1$.

For a weight ω, the Banach space $L^p(\omega)$ is composed of all f in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$ for which $f\omega$ belongs to $L^p(\mathbf{R}^+)$. We give $L^p(\omega)$ the inherited norm

$$\|f\| = \|f\|_{\omega,p} = \left[\int_0^\infty |f(t)|^p \omega(t)^p \, dt\right]^{1/p}.$$

All f in $L^p(\omega)$ belong to $L^p_{\mathrm{loc}}(\mathbf{R}^+)$ which is a proper subspace continuously embedded in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$ when given the usual Fréchet topology determined by the seminorms

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$[\int_0^b |f(t)|^p \, dt]^{1/p}$ for $b > 0$. When ω is an algebra weight, we have $M(\omega) * L^p(\omega) \subseteq L^p(\omega)$, so that for each f in $L^p(\omega)$ the convolution map $T_f \colon \mu \mapsto f * \mu$ maps $M(\omega)$ and $L^1(\omega)$ into $L^p(\omega)$ and therefore into $L^p(\omega\eta)$ for any bounded weight η. We define p-convergence factors analogously to the case $p = 1$ given in Definition 1.2.

Definition 6.2 If ω is an algebra weight and η is a bounded weight, then η is a *p-convergence factor for ω at $a \geq 0$* if $\lambda_n \to \lambda$ weak* in $M(\omega)$ and $f \in L^p(\omega)$ with $\alpha(f) \geq a$ together imply that $\lambda_n * f \to \lambda * f$ in norm in $L^p(\omega\eta)$. If this holds for all ω and all f in $L^p(\omega)$, then η is a *universal p-convergence factor*.

Our goal is to give p-analogues of each of our major results on compactness and convergence factors and to relate the properties for $p > 1$ to those for $p = 1$.

In those cases where the proofs are very similar to the $p = 1$ case, we give only quick sketches of the proofs.

In Lemma 6.3 we add p-characterization to the characterizations of "very weak convergence" given in Theorem 1.1. In Theorem 6.4, we relate compactness and convergence factors giving a p-analogue of Theorem 1.3. In Theorem 6.5, we prove the analogue of Theorem 2.1, giving a formula characterizing the compactness of T_f for a fixed f and we also show that $T_f \colon M(\omega) \to L^p(\omega\eta)$ is compact if and only if $T_{|f|^p} \colon M(\omega^p) \to L^1(\omega^p\eta^p)$ is compact. From this it is easy for us to characterize in Theorem 6.6, p-convergence factors in terms of convergence factors, and to show, in Theorem 6.7, that η is a universal p-convergence factor if and only if η is a universal convergence factor.

We start with the extension of Theorem 1.1.

Lemma 6.3 *Suppose that ω is an algebra weight and that $\{\lambda_n\}$ is a bounded seqence in $M(\omega)$. Then the following are equivalent:*

(a) *$\lambda_n \to \lambda$ weak* in $M(\omega)$;*
(b) *for some (equivalently all) $1 < p < \infty$, and some (equivalently all) non-zero f in $L^1(\omega)$, we have $\lambda_n * f \to \lambda * f$ weakly in $L^p(\omega)$,*
(c) *for some (equivalently all) $1 \leq p < \infty$ and some (equivalently all) non-zero f in $L^p_{\mathrm{loc}}(\mathbf{R}^+)$, we have $\lambda_n * f \to \lambda * f$ in the usual Fréchet topology on $L^p_{\mathrm{loc}}(\mathbf{R}^+)$.*

Proof The equivalence of (a) and (b) is [GG2, Lemma 2.1, p. 51]. Since all the $L^p_{\mathrm{loc}}(\mathbf{R}^+)$ are continuously embedded in $L^1_{\mathrm{loc}}(\mathbf{R}^+)$ it follows, from Theorem 1.1 (a), that (a) $\Rightarrow$ (c) for all p and f. The proof that (c) implies (a) is exactly the same as the proof (a) $\Rightarrow$ (b) in Thoerem 1.1.

We now relate compactness and convergence factors when $p > 1$. The main difference from Theorem 1.3 is that $L^p(\omega)$ is reflexive when $1 < p < \infty$, so that weak-compactness is equivalent to continuity and not to compactness.

Theorem 6.4 *Suppose that ω is an algebra weight, that η is a bounded weight, and that $1 < p < \infty$. For f in $L^p(\omega)$, the following are equivalent:*

(a) *Whenever $\lambda_n \to \lambda$ weak* in $M(\omega)$, then $\lambda_n * f \to \lambda * f$ in norm in $L^p(\omega\eta)$;*
(b) *T_f is a compact operator from $M(\omega)$ to $L^p(\omega\eta)$;*
(c) *T_f is a compact operator from $L^1(\omega)$ to $L^p(\omega\eta)$.*

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Proof We will only prove (b) $\Rightarrow$ (a), since the rest of the proof is essentially the same as the proof of Theorem 1.3. Suppose therefore that $\lambda_n \to \lambda$ weak* in $M(\omega)$ and that $T_f \colon M(\omega) \to L^p(\omega\eta)$ is compact. Since $\{\lambda_\eta * f\}$ belongs to a sequentially compact subset of $L^p(\omega\eta)$, we need only show that whenever a subsequence $\{\lambda'_n * f\}$ converges to some g in $L^p(\omega\eta)$, then $g = \lambda * f$. From Lemma 6.3 we have that $\lambda'_n * f$ converges weakly to $\lambda * f$ in $L^p(\omega)$, and therefore in the larger space $L^p(\omega\eta)$. Hence $\lambda * f = g$ as required, and the proof is complete.

To characterize the compactness of T_f for a fixed f, we need the p-analogue of $D(s)$ defined in formula (2.1). Fix an algebra weight ω and a bounded weight η. For f in $L^p(\omega)$ we consider the maps $T_f \colon M(\omega) \to L^p(\omega\eta)$ and $T_f \colon M(\omega) \to L^1(\omega^p\eta^p)$. We then define

$$D_p(s) = D_p(s,f) = \left\| \frac{1}{\omega(s)} \delta_s * f \right\|_{\omega\eta,p} = D(s, |f|^p), \tag{6.4}$$

where $D(s,|f|^p)$ is defined with respect to the weights ω^p and η^p. For an integral formula for $D_p(s) = D(s,|f|^p)$ see formula (2.1). We then obtain the following characterization of compactness.

Theorem 6.5 *For a fixed f in $L^p(\omega)$, the following are equivalent:*

(a) *$T_f \colon \mu \mapsto f * \mu$ is a compact operator from $M(\omega)$ to $L^p(\omega\eta)$;*
(b) *$T_{|f|^p}$ is a compact operator from $M(\omega^p)$ to $L^1(\omega^p\eta^p)$;*
(c) *$\lim_{s\to\infty} D_p(s) = 0$.*

Proof Since $D_p(s) = D(s,|f|^p)$, the equivalence of (b) and (c) is just Theorem 2.1 for the map $T_{|f|^p}$. As noted in the proof of Theorem 2.1, we have $\frac{1}{\omega(s)}\delta_s \to 0$ weak* in $M(\omega)$. Hence if T_f is compact, it follows from Lemma 6.3 that $\frac{1}{\omega(s)}\delta_s * f \to 0$ in the norm of $L^p(\omega\eta)$. This proves that (a) implies (c).

The proof that (c) implies (a) is very similar to the proof of Theorem 2.1. Using the same integral formula as in the proof of Lemma 2.2, but with the integrand having values in $L^p(\omega\eta)$, we obtain, as in Lemma 2.2, that $\sup\{D_p(s) \colon s \geq a\}$ is the norm of the restriction of T_f to $M(\omega)_a$. As in the proof of Theorem 2.1, we let P_a and Q_a be the natural projections of $M(\omega)$ onto the measures concentrated on $[0,a]$ and onto $M(\omega)_a$, respectively. As before $T_f Q_a$ has the same norm as the restriction of T_f to $M(\omega)_a$. Thus we need only show that $T_f P_a$ is compact. This follows as in the proof of Lemma 2.3 by using Lemma 6.3 (c) in place of Theorem 1.1 (a). This completes the proof of the theorem.

The following characterization of p-convergence factors follows immediately from Theorems 3.1 and 6.5.

Theorem 6.6 *Suppose that ω is an algebra weight and that η is a bounded weight. For each $a \geq 0$, the following are equivalent:*

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(a) *There is an f in $L^p(\omega)$ with $\alpha(f) = a$ for which T_f is a compact operator from $M(\omega)$ to $L^p(\omega\eta)$;*
(b) *η is a p-convergence factor for ω at a;*
(c) *η^p is a convergence factor ω^p at a.*

For the formulas that characterize p-convergence factors, we can just use Theorem 3.1 (c) and (d) for ω^p and η^p. When $M(\omega\eta)$ is translation invariant, we can use the simpler formula of Theorem 3.2 (b), again for ω^p and η^p. It is now easy to characterize universal p-convergence factors.

Theorem 6.7 *For a bounded weight η on $\mathbf{R}^+$, the following are equivalent:*

(a) *η is a universal p-convergence factor for all $1 \le p < \infty$;*
(b) *η is a universal convergence factor;*
(c) *η is a universal convergence factor for some $1 \le p < \infty$;*
(d) *there is some $1 \le p < \infty$ and some non-zero f in $L^p(\mathbf{R}^+)$ for which $T_f \colon L^1(\mathbf{R}^+) \to L^p(\eta)$ is compact.*

Proof We know from Corollary 4.2 that η is a universal convergence factor if and only if η^p is. The equivalence of (a), (b), and (c) now follows from Theorem 6.6. Since (a) $\Rightarrow$ (d) follows by definition, we need only prove that (d) implies (b). Since $L^p(\mathbf{R}^+) = L^p(1)$ and $1^p = 1$, it follows from Theorem 6.6 that $T_{|f|^p}$ is a compact operator from $L^1(\mathbf{R}^+)$ to $L^1(\eta^p)$. It then follows from Theorem 4.1 that η^p is a universal convergence factor. Corollary 4.2 then gives us that η is a universal convergence factor, as required.

For formulas that characterize universal convergence factors, see Theorem 4.1 (c) and (d) and the discussion after Theorem 4.1. For sufficient conditions, see Corollary 4.3.

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