Can. J. Math., Vol. XLI, No. 3, 1989, pp. 462–479

TOPOLOGICAL PROPERTIES OF CYCLIC COVERINGS BRANCHED ALONG AN AMPLE DIVISOR

ANTONIO LANTERI AND DANIELE C. STRUPPA

0. Introduction. Let $\pi: X' \to X$ be a finite morphism between two complex connected projective k-folds. Since π is surjective, the Betti numbers of X and X' are related as follows

(0.1) $\quad b_i(X) \leqq b_i(X').$

In particular, if π is a cyclic covering and the branch locus Δ is an ample divisor, (0.1) is in fact an equality for $i \leqq k - 1$ (see 1.10 or, more generally, [**5**]). It seems natural to look for such coverings satisfying

(0.2) $\quad b_k(X) = b_k(X').$

Let us see what happens for $k = 2$. In this case (0.2) can be rephrased as

(0.3) $\quad 2\chi(O_X) + h^{1,1}(X) + g(\Delta) = 2,$

where g stands for the arithmetic genus (see 1.13). This implies that X is ruled; if $g(\Delta) \leqq 1$ this follows from [**9**, Section 2] while if $g(\Delta) \geqq 2$, (0.3) yields $\chi(O_X) < 0$ and then X is ruled by the Castelnuovo De Franchis theorem [**2**, p. 154]. The ruledness of X implies $\chi(O_X) = 1 - q(X)$ and $h^{1,1}(X) \geqq 2$ unless $X \simeq \mathbf{P}^2$. Assume that $(X, [\Delta])$ is not a scroll: then one can show that $g(\Delta) \geqq 2q(X)$ (see 3.2), which contradicts (0.3). When $X \simeq \mathbf{P}^2$ or $(X, [\Delta])$ is a scroll, a direct computation shows that (0.2) cannot hold. This proves the following.

0.4 *Fact.* In the case of surfaces no such covering exists satisfying (0.2).

This simple result can be thought of as a variation of Buium's result [**4**] stating that for a double covering $\pi: X' \to X$ of surfaces with branch locus $\Delta \in |L^2|$, L very ample, X and X' have the same Picard numbers if and only if $h^\circ(K_X \otimes L) \neq 0$. In our approach the Picard number never plays any role, but 0.4 immediately shows that $h^{1,1}(X) = h^{1,1}(X')$ implies $h^\circ(K_X \otimes L) \neq 0$. Notice that we do not make any assumption on the degree of π and that L may not be very ample. On the other hand, in our

Received March 17, 1988. This work was partially supported by M.P.I. of the Italian Government.

more general context the converse does not hold, i.e., $h^{\circ}(K_X \otimes L) \neq 0$ does not imply $h^{1,1}(X) = h^{1,1}(X')$; this is immediately shown by looking e.g. at case (ii) in Proposition 2.11.

As a consequence of 0.4 one is naturally led to: (i) determine the smallest possible value of $\delta = b_2(X') - b_2(X)$ and (ii) classify cyclic coverings for which δ is small.

As to (i) we show (see Sections 2 and 3) that apart from the obvious double covering $\mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^2 = \mathrm{Sym}^2\, \mathbf{P}^1$, we have $\delta \geqq 2$, the equality implying that π is a double covering and X is ruled (for an explicit example with $\delta = 2$ see 1.19). Moreover $\delta \geqq 6$ if X is not ruled and $\delta \geqq 13$ if X is of general type. As to (ii) we supply a detailed classification up to $\delta = 10$. In particular we show that with a single exception (a triple cyclic covering of $\mathbf{P}^2$ branched along a cubic) all such coverings are double. The results are summarized in the table at the end of the introduction.

The example of odd dimensional hyperquadrics which doubly cover the projective spaces shows that 0.4 does not immediately extend to dimension $k \geqq 3$. In Section 4, however, we provide a partial generalization of it when $\deg \pi \geqq k + 2$, under some extra assumptions.

This result cannot completely cover the case $\deg \pi = k + 1$, due to the existence of Fano k-folds with the same integral cohomology as $\mathbf{P}^k$, not isomorphic to $\mathbf{P}^k$ itself.

1. Background material. Let X, X' be two complex connected projective k-folds and let $\pi:X' \rightarrow X$ be a cyclic covering of order n (shortly an n-cyclic covering) branched along a divisor Δ. As is known there exists a line bundle $L \in \mathrm{Pic}(X)$ such that $\Delta \in |L^n|$.

1.1 *Remark.* The branch locus Δ is smooth. Actually in the total space of the bundle $L \rightarrow X$, X' is locally defined by an equation of the form

$$z^n - \phi(x_1, \dots, x_k) = \Phi(x_1, \dots, x_k, z) = 0,$$

where $\phi = 0$ is a local equation of Δ. Then a singular point p of Δ would define a singular point of X' since

$$(\mathrm{grad}\, \Phi)_p = [\, (\mathrm{grad}\, \phi)_p, 0].$$

We recall, from [**1**, pp. 42-43], the following basic facts:

(1.2) $\quad K_{X'} = \pi^*(K_X \otimes L^{n-1})$;

(1.3) let Δ' be the reduced divisor $\pi^{-1}(\Delta)$ on X', then

$$\Delta' \approx \Delta \quad \text{and} \quad \pi^*\Delta = n\Delta';$$

(1.4) $\quad \pi_* O_{X'} = \bigoplus_{j=0}^{n-1} L^{-j}.$

$\pi : X' \to X$ n-cyclic covering branched along $\Delta \in |L^n|$, L ample, satisfying $\delta \leqq 10$

δ	n	(L, L)	$g(L)$	(X, L)	X'
1	2	1	0	$(\mathbf{P}^2, O_{\mathbf{P}^2}(1))$	$\mathbf{P}^1 \times \mathbf{P}^1$
2	2	1	$q > 0$	scroll, $e \leqq -1$, odd; $[\xi] \otimes [f]^{(e+1)/2}$	ruled
4	2	2	$q \geqq 0$	scroll, $e \leqq 0$, even; $[\xi] \otimes [f]^{(e+2)/2}$	ruled
6	2	3	$q \geqq 0$	scroll, $e \leqq 1$, odd; $[\xi] \otimes [f]^{(e+3)/2}$	ruled
6	3	1	0	$(\mathbf{P}^2, O_{\mathbf{P}^2}(1))$	rational
6	2	1	2	X = minimal elliptic surface over $\mathbf{P}^1$, $\chi(O_X) = 0$	of general type
7	2	4	0	$(\mathbf{P}^2, O_{\mathbf{P}^2}(2))$	rational
7	2	1	2	X = Blow-up at one point of an abelian or hyperelliptic surface	of general type
7	2	2	2	X = Blow-up at one point of an elliptic $\mathbf{P}^1$-bundle, $e = 0, -1$, deg $L_F = 3$	of general type
8	2	4	$q \geqq 0$	scroll, $e \leqq 2$, even; $[\xi] \otimes [f]^{(e+4)/2}$	ruled
8	2	2	2	X = abelian or hyperelliptic surface	of general type
8	2	1	2	X = Blow-up at 2 points of an elliptic $\mathbf{P}^1$-bundle, $e = 0, -1$, deg $L_F = 3$	of general type
9	2	1	2	X = Blow-up at one point of an elliptic $\mathbf{P}^1$-bundle, $e = 0, -1$, deg $L_F = 5$ X = Blow-up at 3 points of an elliptic $\mathbf{P}^1$-bundle, $e = 0, -1$, deg $L_F = 2$	of general type elliptic
10	2	5	$q \geqq 0$	scroll, $e \leqq 3$, odd; $[\xi] \otimes [f]^{(e+5)/2}$	ruled
10	2	2	2	X = Blow-up at 2 points of an elliptic $\mathbf{P}^1$-bundle, $e = 0, -1$, deg $L_F = 2$	elliptic
10	2	1	3	X = minimal elliptic surface over a base of genus 0 or 1, $\chi(O_X) = 0$	of general type

1.5 *Remark*. We have

$$h^\circ(K_{X'}) = h^\circ(K_X) + h^\circ(K_X \otimes L) + \ldots + h^\circ(K_X \otimes L^{n-1}).$$

Actually, due to (1.2) and (1.4)

$$H^\circ(X', K_{X'}) \simeq H^\circ\left(X, K_X \otimes L^{n-1} \otimes \left[\bigoplus_{j=0}^{n-1} L^{-j}\right]\right).$$

1.6 PROPOSITION. *The following equality holds*:

$$\chi(X') = n\chi(X) - (n-1)\chi(\Delta),$$

where χ *is the topological Euler-Poincaré characteristic.*

Proof. Let V be a tubular neighborhood of Δ in X, and let $V' = \pi^{-1}(V)$; since V, V' retract on Δ, Δ' respectively, recalling (1.3), we have

$$\chi(V) = \chi(V') = \chi(\Delta). \tag{1.7}$$

Applying the Mayer-Vietoris theorem to the triads $(X'\setminus V', V', \partial V')$, $(X\setminus V, V, \partial V)$ one gets

$$\chi(X') = \chi(X\setminus V') + \chi(V'), \tag{1.8}$$

$$\chi(X) = \chi(X\setminus V) + \chi(V); \tag{1.9}$$

actually $\chi(\partial V) = \chi(\partial V') = 0$ since ∂V is an S^1-bundle over Δ and $\partial V'$ an unbranched covering of it.

The assertion now follows from (1.7), (1.8), (1.9), noticing that $\chi(X'\setminus V') = n\chi(X\setminus V)$.

1.10 *Remark*. For $k = 2$ the above formula agrees with the main result in [**8**], under the assumption that the branch locus is smooth.

Now assume that the branch locus Δ is ample. Then Δ' is ample too, hence $X\setminus\Delta$ and $X'\setminus\Delta'$ are both Stein and the Lefschetz duality theorem provides

$$H_q(X, \Delta) = H_q(X', \Delta') = 0 \quad \text{for } q \leqq k - 1.$$

Then from the exact sequences of the pairs (X, Δ) and (X', Δ') one immediately sees that π induces an isomorphism between $H_q(X', \mathbf{Q})$ and $H_q(X, \mathbf{Q})$ up to $q = k - 2$. In fact a more refined Lefschetz type theorem has been proved by Cornalba for the morphism

$$\pi_{*q}:\Pi_q(X', x') \to \Pi_q(X, \pi(x'))$$

induced by π on the homotopy groups [**5**]. In particular Cornalba's result implies the following

1.11 PROPOSITION. *If* Δ *is ample, then the morphism*

$$\pi_{*q}:H_q(X') \to H_q(X)$$

induced by π on the integral q^{th} homology groups is an isomorphism for $q \leqq k - 1$ and a surjection for $q = k$. In particular

$$b_q(X') = b_q(X) \quad \textit{for } q \leqq k - 1 \textit{ and } b_k(X') \geqq b_k(X).$$

1.12 *Remark*. Since $\pi:X' \to X$ is holomorphic, it follows from the elliptic operator theory that it induces a morphism between the Hodge structures of X and X'. Then from 1.11 we deduce

$$h^{p,q}(X') = h^{p,q}(X) \quad \text{for } p + q \leqq k - 1,$$

$$h^{p,q}(X') \geqq h^{p,q}(X) \quad \text{for } p + q = k.$$

1.13 COROLLARY. *If Δ is ample, then π_{*k} is an isomorphism if and only if*

$$2b_{k-1}(X) = b_k(X) + b_{k-1}(\Delta).$$

Proof. By 1.11, if π_{*k} is an isomorphism, then $\chi(X) = \chi(X')$ due to Poincaré duality; hence (1.6) becomes $\chi(X) = \chi(\Delta)$. On the other hand, since Δ is ample, $b_i(X) = b_i(\Delta)$ for $i \leqq k - 2$ by the Lefschetz theorem [**12**], and this immediately gives the assertion. Conversely, by the Lefschetz theorem and Poincaré duality, one sees that $b_k(X') = b_k(X)$ and this concludes the proof.

Now assume that X, X' are surfaces, i.e. $k = 2$. We let

$$\delta = b_2(X') - b_2(X).$$

Due to 1.11 we have also $\delta = \chi(X') - \chi(X) \geqq 0$. Letting

$$d = (L, L) \quad \text{and} \quad g = g(L) = 1 + \frac{1}{2}(L, K_X \otimes L),$$

1.6 gives the following expression for δ:

(1.14) $\quad \chi + 2n(g - 1) + n(n - 1)d = \delta/(n - 1),$

where $\chi = \chi(X)$. In particular, (1.14) implies that

(1.15) $\quad dn^2 + (2g - 2 - d)n + \chi - \delta \leqq 0.$

Now consider the pair (X, L). By the Riemann-Roch and the Kodaira vanishing theorems we have

(1.16) $\quad h^\circ(K_X \otimes L) = h^\circ(K_X) + g - q$

where $q = h^1(O_X)$ is the irregularity of X. The pair (X, L) is said to be a *scroll* if X is a $\mathbf{P}^1$-bundle and $L_f = O_{\mathbf{P}^1}(1)$ for every fibre f of X. We recall the following fact.

1.17 PROPOSITION ([**9**, Theorem 3.2]). *$h^\circ(K_X \otimes L) = 0$ if and only if (X, L) is either a scroll or $(\mathbf{P}^2, O_{\mathbf{P}^2}(e))$, $e = 1, 2$.*

In the next two sections we will classify cyclic coverings of surfaces for which $\delta \leqq 10$. As it will be apparent (2.6, 3.8 and 3.9), if (X, L) is a scroll, condition $\delta \leqq 10$ implies $n = 2$. So the following Proposition will be of interest in the sequel.

1.18 PROPOSITION. *Let (X, L) be any scroll, $n = 2$ and $\delta \leqq 10$. Then δ is even, $\delta > 0$ and the possible cases are listed below where ξ is a fundamental section, f a fibre and e the invariant of X.*

δ	e	numerical class of L
2	$e \leqq -1$, odd	$[\xi] \otimes [f]^{(e+1)/2}$
4	$e \leqq 0$, even	$[\xi] \otimes [f]^{(e+2)/2}$
6	$e \leqq 1$, odd	$[\xi] \otimes [f]^{(e+3)/2}$
8	$e \leqq 2$, even	$[\xi] \otimes [f]^{(e+4)/2}$
10	$e \leqq 3$, odd	$[\xi] \otimes [f]^{(e+5)/2}$

Proof. From 1.14 we get $\delta = 2d$. Let $[\xi] \otimes [f]^b$ be the numerical equivalence class of L: then we have $\delta = 2(2b - e)$, hence δ is even. Moreover the ampleness condition for L [7, p. 382] shows that $\delta > 0$. Then the above table follows from a close check using the assumption $\delta \leqq 10$ and the ampleness condition.

We produce an explicit example with $\delta = 2$.

1.19 *Example.* Let X be the $\mathbf{P}^1$-bundle of invariant $e = -1$ over an elliptic curve and let ξ be a fundamental section of it. Then $L = [\xi]$ is ample and $d = (\xi, \xi) = -e = 1$. Moreover, L^2 is spanned by global sections (e.g. see [7, p. 385]). Then, due to Bertini's theorem, $|L^2|$ contains a smooth divisor Δ and the double covering $\pi:X' \rightarrow X$ branched along Δ satisfies $\delta = 2d = 2$.

Another useful result concerning double coverings is the following congruence.

1.20 PROPOSITION. *If a double covering $\pi:X' \rightarrow X$ exists with invariants g, d, δ, then*

$$(K_X, K_X) + 8(g - 1) + \delta - 2d \equiv 0 \ (12).$$

Proof. By (1.2) and the genus formula, we have

$$(K_{X'}, K_{X'}) = 2((K_X, K_X) + 4g - 4 - d).$$

On the other hand, $\chi(X') = \chi(X) + \delta$ and then Noether's formula both for X' and X gives

$$\begin{aligned} 12\chi(O_{X'}) &= \chi(X') + (K_{X'}, K_{X'}) \\ &= 12\chi(O_X) + (K_X, K_X) + \delta + 8(g - 1) - 2d. \end{aligned}$$

Finally, to justify the assertions about X' in the table at the end of the introduction, we need a last result. First we recall some more terminology. Let X be a surface and L an ample line bundle on X. (X, L) is said to be a *Del Pezzo pair* if $L = K_X^{-1}$. (X, L) is said to be a *conic bundle* if X is a ruled surface and $L_F = O_{\mathbf{P}^1}(2)$ for the general fibre F of the ruling.

1.21 PROPOSITION. *Let X be a surface and L an ample line bundle on it such that L^2 contains a smooth divisor Δ. Then the double covering $\pi:X' \to X$ branched along Δ is classified according to (X, L)* as follows:

X' is a ruled surface if and only if (X, L) is either $(\mathbf{P}^2, O_{\mathbf{P}^2}(e))$, $e = 1, 2$ or a scroll;

X' is a $K3$ surface if and only if (X, L) is a Del Pezzo pair;

X' is a properly elliptic minimal surface if and only if (X, L) is a conic bundle not in the above list;

X' is a surface of general type, otherwise.

Proof. Recalling (1.2) we have

$$(1.22) \quad h^\circ(K_{X'}^n) = h^\circ(K_X^n \otimes L^n) + h^\circ(K_X^n \otimes L^{n-1}) \quad \text{for every } n \geqq 1.$$

If X' is ruled, then 1.17 and (1.22) show that (X, L) is either a scroll or $(\mathbf{P}^2, O_{\mathbf{P}^2}(e))$, $e = 1, 2$. The converse follows again from (1.22) in view of the Enriques ruledness criterion [**2**]. Let (X, L) be a Del Pezzo pair: then $K_{X'}$ is trivial by (1.2). Moreover, as

$$h^1(O_{X'}) = h^1(O_X) + h^1(L^{-1}) = 0,$$

it follows that X' is a $K3$ surface. Conversely, if X' is a $K3$ surface, then (X, L) is a Del Pezzo pair due to (1.2) and the injectivity of

$$\pi^*:\mathrm{Pic}(X) \to \mathrm{Pic}(X').$$

Assume (X, L) is not as before; then X' has Kodaira dimension $k(X') \geqq 1$ and

$$(K_{X'}, K_{X'}) = 2(K_X \otimes L, K_X \otimes L) \geqq 0,$$

with equality if and only if (X, L) is a conic bundle [**9**, Theorem 2.5]. Then X' is of general type if (X, L) is not a conic bundle, while, if (X, L) is a conic bundle with ruling projection $p:X \to B$, then the map

$$X' \xrightarrow{\pi} X \xrightarrow{p} B$$

exhibits X' as a properly elliptic minimal surface with base curve B.

2. Surfaces: $\chi \geqq 0$. From now on, up to Section 3, we assume that $\pi:X' \to X$ is an n-cyclic covering of surfaces. In this section we will establish some facts on δ when $\chi \geqq 0$.

2.1 LEMMA. *If* $g \geqq 2$, *then either* $\chi < 0$ *or*

$$d \leqq \frac{\delta}{n(n-1)} - \frac{2}{n-1}.$$

Proof. Since $n \geqq 2$, (1.14) implies that

$$d \leqq \frac{2(1-g)}{n-1} + \frac{\delta - \chi}{n(n-1)}.$$

Then the assertion follows from the assumption on g.

2.2 LEMMA. *Assume* $\chi \geqq 0$ *and* $n \geqq 3$.
(i) *If* $g \geqq 3$, *then* $\delta \geqq 18$;
(ii) *If* $g \geqq 2$, *then* $\delta \geqq 12$.

Proof. (i) Let $g \geqq 3$; by (1.15) we have

$$dn^2 + (4 - d)n + \chi - \delta \leqq 0.$$

Therefore

$$n \leqq (d - 4 + \sqrt{(d-4)^2 + 4d(\delta - \chi)}/2d,$$

and due to the assumption $n \geqq 3$, we conclude that

$$d \leqq (\delta - \chi - 12)/6.$$

Since $d \geqq 1$, this gives $\delta \geqq 18$. Case (ii) is dealt with in the same way.

2.3 LEMMA. *Let* $\chi \geqq 0$. *If* $n = 2$, *then* $\delta \geqq 4g - 2$.

Proof. Letting $n = 2$ in (1.14) one gets

$$\delta \geqq 4(g-1) + 2 + \chi \geqq 4g - 2.$$

Lemmas 2.2, 2.3 immediately imply the following.

2.4 PROPOSITION. *Let* $\chi \geqq 0$. *If* $g \geqq 3$, *then either* $\delta \geqq 18$ *or* $n = 2$ *and* $\delta \geqq 10$, *equality implying that* $\chi = 0$, $d = 1$, $g = 3$.

So, in order to study what happens when $\delta \leqq 9$, we can assume $g \leqq 2$. Moreover, in case $g = 2$, we have $n = 2$, by 2.2, and then 2.1 implies $d \leqq 2$. So, for $\delta \leqq 10$ we have only to consider the following cases.
Case (a) $g \leqq 1$;
Case (b) $g = 2$ with $n = 2$, $d = 1$ or 2;
Case (c) $g = 3$ with $n = 2$, $\chi = 0$, $d = 1$.
As to case (a) we recall the following facts.

2.5 PROPOSITION ([**9**, Corollaries 2.3 and 2.4]). *If* $g = 0$, *then* (X, L) *is either* $(\mathbf{P}^2, O_{\mathbf{P}^2}(e))$, $e = 1, 2$ *or a rational scroll*; *if* $g = 1$, *then* (X, L) *is either an elliptic scroll or a Del Pezzo pair.*

2.6 *Remark*. Let (X, L) be a scroll, $g \leqq 1$ and $n \geqq 3$, then (1.14) implies

$$\delta \geqq 16 \quad \text{if } g = 0,$$

$$\delta \geqq 12 \quad \text{if } g = 1.$$

Since we are interested in the case $\delta \leqq 10$, then $n = 2$ when (X, L) is either a rational or an elliptic scroll, and so 1.18 applies. These cases will be cumulated in the study of scrolls in the next section.

As to Del Pezzo pairs, we have the following.

2.7 PROPOSITION. *If (X, L) is a Del Pezzo pair, then $\delta \geqq 13$.*

Proof. Actually, recalling the structure of Del Pezzo surfaces, (1.14) gives

$$\delta = (n - 1)(n(n - 1)d + 12 - d) \geqq 13,$$

since $n \geqq 2$, $d \geqq 1$.

Now we consider the case $X \simeq \mathbf{P}^2$. We have

2.8 PROPOSITION. *Let $X \simeq \mathbf{P}^2$ and assume $\delta \leqq 10$. Then only the following cases can occur.*

$$n = 2 \quad \text{with } L = O_{\mathbf{P}^2}(e), \quad e = 1, 2,$$

$$n = 3 \quad \text{with } L = O_{\mathbf{P}^2}(1).$$

Proof. Let $L = O_{\mathbf{P}^2}(e)$; then (1.15) and $\delta \leqq 10$ imply

$$e^2n^2 - 3en - 7 \leqq 0;$$

hence $en \leqq 4$: since $n \geqq 2$ it can only be $(n, e) = (2, 1), (2, 2), (3, 1), (4, 1)$. However for $(n, e) = (4, 1)$, we get from (1.14), $\delta = 21$.

Now we come to case (b). The polarized surfaces (X, L) with $g = 2$ are classified in [3]. First of all we show the following general fact:

2.9 PROPOSITION. *If X is of general type, then $\delta \geqq 13$. Equality implies that $n = 2$, $\chi = 3$, $d = 1$, $g = 3$ and that K_X is numerically equivalent to L^3.*

Proof. Let

$$f(n, g, d, \chi) = (n - 1)\{\chi + 2n(g - 1) + n(n - 1)d\}.$$

By (1.14) $\delta = f(n, g, d, \chi)$. Since χ is of general type, Noether's formula and Miyaoka's inequality imply $\chi \geqq 3$. Moreover, due to 2.5, we can assume $g \geqq 2$. We have

$$f(2, 2, 1, \chi) = \chi + 6;$$

however from [**3**, Theorem 1.4] it follows that if $g = 2$, then $(K_X, K_X) = 1$ and $\chi \geqq 11$. Hence $\delta \geqq 17$ (which in turn, implies $\delta \geqq 19$, in view of 1.20). On the other hand, if $g \geqq 3$ we have

$$\delta \geqq f(2, 3, 1, 3) = 13.$$

Note that if $g = 3$, then $(L, K_X) \leqq 3$ and the Hodge index theorem implies

$$(K_X, K_X) \leqq 9/d \leqq 9.$$

On the other hand $\chi(O_X) \geqq 1$, hence $(K_X, K_X) \geqq 9$. So, if $\delta = 13$, it has to be $n = 2$, $\chi = 3$, $d = 1$, $g = 3$ and $(K_X, K_X) = 9$; this implies that K_X is numerically equivalent to L^3.

In view of 2.9, to deal with case (b) we can assume that X has Kodaira dimension $k(X) \leqq 1$. Since $n = g = 2$, (1.14) reads

(2.10) $\quad \delta = \chi + 4 + 2d.$

First we consider the non-ruled case.

2.11 PROPOSITION. *Assume that X is non-ruled and that $n = g = 2$. If $\delta \leqq 17$, then either*

(i) $\delta = 8$, $d = 2$ *and X is either an abelian or a hyperelliptic surface.*

(ii) $\delta = 7$, $d = 1$ *and X is the blow-up at a single point of a surface as in* (i), *or*

(iii) $\delta = 6$, $d = 1$ *and X is a minimal elliptic surface with base* $\mathbf{P}^1$ *and* $h^\circ(K_X) = 0$, $q = 1$.

Proof. If $k(X) = 0$, then either: (j) $d = 2$ and X is a minimal surface or (jj) $d = 1$ and X is the blow-up at a single point of a surface X_0 as in (j) [**3**, Proposition 2.1]. Note that in case (j) $\chi = 0$ if X is abelian or hyperelliptic and $\chi \geqq 12$ otherwise; in case (jj), $\chi = \chi(X_0) + 1$. Then (i) and (ii) follow from (2.10): otherwise $\delta \geqq 19$. Now assume $k(X) = 1$; then $d = 1$ and X is a minimal elliptic surface, the base of the elliptic fibration is $\mathbf{P}^1$, $h^\circ(K_X) = 0$ and $q = 0$ or 1 ([**3**, Theorem 1.5]). Thus, by Noether's formula, (2.10) implies $\delta = 6$ or 18 according to whether $q = 1$ or 0 and this gives (iii).

Note that in all cases above, if X' exists, it is a surface of general type in view of 1.21.

In case (ii) the pair (X, L) has a reduction (X_0, L_0) (in the sense of [**13**]) as in (i). Some restrictions on the position of the point of X_0 one has to blow up for obtaining X are needed in order to insure the ampleness of L (see [**3**]).

In case (i), if X is an abelian surface, (X, L) can only be one of the following pairs [**3**, Theorem 2.7]:

(α) X is the jacobian of a smooth curve C of genus two and $L = [C]$ is the line bundle corresponding to C embedded in X;

https://doi.org/10.4153/CJM-1989-021-3 Published online by Cambridge University Press

(β) $X = E \times F$ is the product of two elliptic curves and $L = [E + F]$.

In both cases, by using Reider's method [**11**] one can see that L^2 is spanned by global sections: hence $|L^2|$ contains a smooth divisor and therefore a smooth surface X' giving a double covering as in (i) does exist.

Where X is hyperelliptic, the possible numerical classes of L are described in [**3**]. Reider's method can be used again to show that in many cases L^2 is spanned by global sections; hence double coverings with X hyperelliptic and $\delta = 8$ exist.

Finally we have

2.12 PROPOSITION. *Let X be ruled, $\chi \geqq 0$, $n = g = 2$, $d \leqq 2$ and assume $\delta \leqq 10$. Then $q = 1$, X is gotten by blowing up a $\mathbf{P}^1$-bundle of invariant $e = 0$ or -1 at s $(1 \leqq s \leqq 3)$ points (possibly infinitely near if $s = 2$). The possible values of δ and the corresponding invariants are listed below, where t is the degree of the restriction of L to the general fibre of X.*

δ	$s = \chi$	$p_1, \ldots, p_s$	d	t
7	1		2	3
8	2	distinct or infinitely near*	1	3
9	1		1	5
9	3	on distinct fibres*	1	2
10	2	on distinct fibres*	2	2

For further restrictions on the position of $p_1, \ldots, p_s$ on X_0 see* [3**].

Proof. We have $q \leqq g = 2$. However it cannot be $q = 2$ since otherwise (X, L) would be a scroll and then $\chi = 4(1 - q) < 0$, a contradiction. On the other hand, if $q = 0$, then X is rational, hence $\chi \geqq 8$ and (2.10) implies $\delta \geqq 14$. It remains to consider case $q = 1$. In this case, the allowable pairs (X, L) are to be found in [**3**, Theorem 3.3] using the assumption $d \leqq 2$. A close check using 1.20 immediately proves the assertion.

Note that, by 1.21, if X' exists in the cases above, it is an elliptic surface with $h^{\circ}(K_X) = 0$, $q = 1$ when $t = 2$, while it is of general type in the remaining cases.

As a last thing let us deal with case (c). Since $\chi = 0$, in view of the Enriques-Kodaira classification [**1**, **2**] X can be either

(i) a ruled surface with $q > 0$,

(ii) an abelian or a hyperelliptic surface, or

(iii) a properly elliptic minimal surface with $\chi(O_X) = 0$.

If (i) holds, we have $q \leqq g = 3$ and if $q = 3$, then (X, L) is a scroll [**9**, Theorem 3.2]. But in this case $\chi = 4(1 - q) = -8 < 0$, a contradiction. It thus follows that $q = 1$ (see 3.2). This implies that X is a $\mathbf{P}^1$-bundle over an elliptic curve; since $g \neq q$, (X, L) is not a scroll and this contradicts the equality $d = 1$ (see 3.5). So, case (i) cannot happen. Case (ii) cannot happen as well; actually, the assumptions $g = 3$, $d = 1$ imply $(L, K_X) = 3$ by the genus formula, but this contradicts the numerical triviality of K_X. It remains to consider case (iii). Since X is minimal we have $(K_X, K_X) = 0$ and then $\chi(O_X) = 0$ by Noether's formula. Let $\psi: X \rightarrow B$ be the elliptic fibration of X and let $b = g(B)$ be the genus of the base curve B. In view of the canonical bundle formula, K_X is numerically equivalent to

$$\left[(2b - 2)F + \sum_i (m_i - 1)f_i\right],$$

where F is the general fibre of ψ, f_i is the reduced component of a fibre of multiplicity m_i and the sum involves all multiple fibres. Since $(L, K_X) = 3$ and $F \underset{n}{\sim} m_i f_i$, we get

$$3 = \left\{2(b - 1) + \sum_i (m_i - 1)/m_i\right\}([F], L),$$

and therefore, since $(L, [F]) \geqq \max\{m_i\}$, we see that $b \leqq 1$.

Taking into account 1.21, what we said proves the following

2.13 PROPOSITION. *Let $n = 2$, $\chi = 0$, $g = 3$, $d = 1$ and assume $\delta \leqq 10$. Then $\delta = 10$ and X is a minimal elliptic surface with base $\mathbf{P}^1$ or an elliptic curve; moreover X', if it exists, is a surface of general type.*

3. Surfaces: $\chi < 0$. In this section we study the case $\chi < 0$. First of all we recall the following fact:

3.1 PROPOSITION. *If $\chi < 0$, then X is a ruled surface of irregularity $q \geqq 2$.*

Proof. X is ruled by the Castelnuovo-De Franchis theorem [**2**, p. 154]. Let X_0 be a minimal model of X; since

$$\chi(X) \geqq \chi(X_0) = \begin{cases} 3 & \text{if } X_0 \simeq \mathbf{P}^2 \\ 4(1 - q) & \text{otherwise} \end{cases}$$

the assumption $\chi < 0$ implies $q \geqq 2$.

3.2 LEMMA. *Assume that X is ruled, but $X \not\simeq \mathbf{P}^2$ and (X, L) is not a scroll. Then*

(3.3) $\quad q \leqq h^{\circ}(K_X \otimes L) + 1 - d/4,$

with equality if and only if X is a $\mathbf{P}^1$-bundle and (X, L) a conic bundle. In particular, $g \geqq 2q$.

Proof. Due to the assumptions, we have [**9**, Theorem 2.5]

$$(K_X \otimes L, K_X \otimes L) \geqq 0,$$

with equality if and only if (X, L) is a conic bundle. Moreover, since $X \not\cong \mathbf{P}^2$, we have

$$(K_X, K_X) \leqq 8(1 - q)$$

with equality if and only if X is a $\mathbf{P}^1$-bundle. Putting together these inequalities and recalling that

$$h^\circ(K_X \otimes L) = g - q$$

by (1.16), we get (3.3). The last assertion is obvious.

3.4 PROPOSITION. *Assume that $\chi < 0$ and that (X, L) is not a scroll. Then $\delta \geqq 10$, equality implying that $n = 2$, $d = 1$, $g = 4$ and that X is a $\mathbf{P}^1$-bundle of irregularity $q = 2$.*

Proof. By 3.1 we have $q \geqq 2$ and $\chi \geqq 4(1 - q)$ the last being an equality if and only if X is a $\mathbf{P}^1$-bundle.

Since $n \geqq 2$, (1.14) gives

$$4(1 - q) + 2n(g - 1) + n(n - 1)d \leqq \delta,$$

and by using 3.2 we get

$$\delta \geqq n(n - 1)d + 4q(n - 1) + 4 - 2n.$$

Then the assertion follows from the inequalities $q \geqq 2$, $n \geqq 2$, $d \geqq 1$ and from 3.2.

However we have the following fact.

3.5 REMARK. *Let X be a $\mathbf{P}^1$-bundle. If (X, L) is not a scroll, it cannot happen that $d = 1$.*

Actually, let ξ and f be a fundamental section and a fibre of X respectively and let e be the invariant of X. Then L is numerically equivalent to $[\xi]^a \otimes [f]^b$ and in view of the ampleness conditions [**7**, p. 382] it cannot happen that

$$1 = (L, L) = a(2b - ae)$$

unless $a = 1$. This means however that (X, L) is a scroll.

Since we are looking for surfaces with $\delta \leqq 10$, it follows from 3.4, 3.5 that in case $\chi < 0$ we have only to consider scrolls.

3.6 LEMMA. *Let (X, L) be a scroll. Then*

$$h^\circ(K_X \otimes L^r) = (r - 1)(q - 1) + \binom{r}{2}d.$$

Proof. By the Riemann-Roch and the Kodaira vanishing theorems we have

$$h^\circ(K_X \otimes L^r) = g(L^r) - q.$$

On the other hand, the genus formula gives

$$g(L^r) - g = (r - 1)(q - 1) + \binom{r}{2}d,$$

and then the assertion follows recalling that $g = q$ as (X, L) is a scroll.

3.7 Lemma. *Let* (X, L) *be a scroll. Then*

$$\delta \geqq (n - 1)(n - 2)(q - 1 + d).$$

Proof. We have, in view of the Hodge decomposition

$$\delta = h^{1,1}(X') - h^{1,1}(X) + 2(h^\circ(K_{X'}) - h^\circ(K_X)).$$

Then, by 1.12, 1.5 and 3.6 we get

$$\begin{aligned}\delta &\geqq 2 \sum_{r=0}^{n-1} h^\circ(K_X \otimes L^r) \\ &= 2\left\{ (q - 1)(1 + 2 + \dots + n - 2) \right. \\ &\qquad\qquad \left. + d\left(1 + \binom{3}{2} + \dots + \binom{n-1}{2}\right)\right\} \\ &\geqq 2\left\{\binom{n-1}{2}(q - 1 + d)\right\}.\end{aligned}$$

An immediate consequence of Lemmas 3.1, 3.7 is

3.8 Corollary. *Let* (X, L) *be a scroll with* $\chi < 0$: *if* $n \geqq 4$, *then* $\delta \geqq 12$.

The above inequality is probably rough. In fact, as to the case $n = 3$, we have the following

3.9 Proposition. *Let* (X, L) *be a scroll with* $\chi < 0$; *if* $n = 3$, *then* $\delta \geqq 16$.

Proof. Since (X, L) is a scroll, $K_X \otimes L^2$ is numerically equivalent to some power of $[f]$, where f is a fibre of X. Recalling (1.2) this implies

$$(K_{X'}, K_{X'}) = 0.$$

In addition, by 1.5 and by 3.6 we have

$$\begin{aligned}h^\circ(K_{X'}) &= h^\circ(K_X) + h^\circ(K_X \otimes L) + h^\circ(K_X \otimes L^2) \\ &= h^\circ(K_X \otimes L^2) = q - 1 + d,\end{aligned}$$

and since $q(X') = q(X)$ we get $\chi(O_{X'}) = d$.

Now, Noether's formula for X' gives

$$\delta = \chi(X') - \chi(X) = 12\chi(O_{X'}) - (K_{X'}, K_{X'}) - 4(1 - q)$$
$$= 12d + 4(q - 1) \geqq 16,$$

in view of 3.1.

The study of cyclic coverings for which $\delta \leqq 10$ is therefore completed in view of 1.18. The complete list of such coverings can be found in the table at the end of the introduction.

4. Higher dimension. In this section X, X' are projective k-folds, $k \geqq 3$. Throughout this section we shall also assume that

(4.1) L is ample and spanned by its global sections.

We recall the following result by Sommese [**13**].

4.2 PROPOSITION. *Assume that* (4.1) *holds. If*

$$h^\circ(K_X \otimes L^t) = 0 \quad \textit{for some } t \geqq k - 1,$$

then either

(a) $t = k, k - 1$ and $(X, L) = (\mathbf{P}^k, O_{\mathbf{P}^k}(1))$,

(b) $t = k - 1$ and $(X, L) = (Q^k, O_{Q^k}(1))$,

where $Q^k \subset \mathbf{P}^{k+1}$ *is a smooth quadric hypersurface, or*

(c) $t = k - 1$ *and* X *is a* $\mathbf{P}^{k-1}$*-bundle over a smooth curve.* $L_f = O_{\mathbf{P}^{k-1}}(1)$ *for every fibre* f *of* X.

Proof. In view of (4.1) the cohomology sequence of

$$0 \to K_X \otimes L^{t-1} \to K_X \otimes L^t \to K_Y \otimes L_Y^{t-1} \to 0$$

($Y \in |L|$ a general element) shows that $h^\circ(K_X \otimes L^t) = 0$ implies

$$h^\circ(K_X \otimes L^{k-1}) = 0.$$

Then the assertion follows from [**13**, Corollary 3.6.1].

Now, if our n-cyclic cover $\pi:X' \to X$ would satisfy the condition

(4.3) $\quad b_k(X') = b_k(X),$

then by Remarks 1.5 and 1.12 it would follow that

$$h^\circ(K_X \otimes L^{n-1}) = 0.$$

Hence Proposition 4.2 implies the following

4.4 THEOREM. *Let* $\pi:X' \to X$ *be an* n*-cyclic covering of projective* k*-folds,* $n \geqq k \geqq 3$, *branched along* $\Delta \in |L^n|$, *where* L *is ample and spanned by* *global sections. Then*

$$b_k(X') > b_k(X),$$

unless, possibly, either

(i) $n = k$ *and* (X, L) *is as in* (a), (b), (c) *of* 4.2, *or*

(ii) $n = k + 1$ *and* $(X, L) = (\mathbf{P}^k, O_{\mathbf{P}^k}(1))$.

A classification of pairs (X, L) for which $h^{\circ}(K_X \otimes L^t) = 0$ for small values of t would allow to extend Theorem 4.4 to cyclic coverings of degree $n < k$. Such coverings satisfying (4.3) do really exist, as the following example shows.

4.5 *Example.* The k-dimensional hyperquadric X' doubly covers $X = \mathbf{P}^k$ with branch locus $\Delta \in |O_{\mathbf{P}^k}(2)|$. When k is odd this covering satisfies (4.3).

An obvious corollary of Theorem 4.4 is the following partial extension of 0.4.

4.6 COROLLARY. *For* $n \geqq k + 2$ *no* n*-cyclic covering with* L *ample and spanned satisfies* (4.3).

A close inspection of cases (i) and (ii) in 4.4 provides more information on X'.

4.7 PROPOSITION. *Let* $n = k + 1$. *If* (4.3) *holds, then* X' *is a Fano* k*-fold of first kind of index* $r = 1$ *and degree* $d = k + 1$ *and with the same integral cohomology as* $\mathbf{P}^k$.

Proof. We get from (1.2)

$$K_{X'} = \pi^* O_{\mathbf{P}^k}(-1).$$

This shows that X' is Fano. Moreover 1.10 and (4.3) say that X' is a homological $\mathbf{P}^k$. In particular, since $b_2(X') = b_2(\mathbf{P}^k) = 1$, X' is of the first kind. Finally, from

$$(-)^k(k + 1) = (\dot{K_{X'}})^k = (-)^k r^k d,$$

we deduce $r = 1$, $d = k + 1$.

On the other hand, recalling that

$$H^q(X') \cong [H_q(X')]^* \oplus \text{Tors } H_{q-1}(X')$$

and Poincaré duality, from 1.11 we see that

$$H^q(X') \cong H^q(\mathbf{P}^k) \quad \text{for } q \leqq k - 1 \text{ and } q \geqq k + 1.$$

Moreover, since $H_{k-1}(X')$ is torsion free, from 4.3 we also get

$$H^k(X') \cong H^k(\mathbf{P}^k).$$

4.8 *Remark*. Note that $H^*(\mathbf{P}^k)$ and $H^*(X')$ are isomorphic as graded groups and not (at least "a priori") as rings. Note also that this isomorphism is not induced by π. Actually, by Poincaré duality and 1.11, the Gysin map

$$\pi_*: H^q(X') \to H^q(\mathbf{P}^k)$$

is an isomorphism for $q \geqq k + 1$; however $\pi_*\pi^*$ is the multiplication by n and so π^* cannot be an isomorphism in this range.

The same argument used for proving 4.7 shows

4.9 PROPOSITION. *Let* $n = k$ *and assume that* (4.3) *holds.*

(α) *If* (X, L) *is as in* (a), *then* X' *is a Fano k-fold of first kind of index* $r = 2$ *and degree* $d = k$ *and with the same integral cohomology as* $\mathbf{P}^k$.

(β) *If* (X, L) *is as in* (b), *then* X' *is a Fano k-fold of first kind, with* $r = 1$, $d = 2k$ *and with the same integral cohomology as* Q^k.

(γ) *If* (X, L) *is as in case* (c) *then* X' *is a bundle* (*over the same base curve as* X) *of Fano* $(k - 1)$*-folds of first kind, with* $r = 2$, $d = k - 1$, *having the same integral cohomology as* $\mathbf{P}^{k-1}$.

Proof. (α) follows in the same way as 4.7. As

$$K_{X'} = \pi^* O_{\mathbf{P}^k}(-2).$$

We get $r \geqq 2$ and $k2^k = r^k d$. Since $k \geqq 3$ one easily sees that the only solution is $r = 2$, $d = k$. Similarly, in case (β) one gets $2k = r^k d$. In case (γ) note that π acts fibrewise and then apply the argument to prove (α) to each fibre of X'.

Fujita [**6**, Theorem 1] showed that a Fano k-fold with the same integral cohomology ring as $\mathbf{P}^k$ is $\mathbf{P}^k$ itself if $k \leqq 5$. However, even for $k \leqq 4$, Fano k-folds with the same integral cohomology groups as $\mathbf{P}^k$, but different from $\mathbf{P}^k$ exist (see [**10**] for $k = 3$ and [**14**] for $k = 4$). So, even for low values of $k \geqq 3$ we cannot exclude that there exist $(k + 1)$-cyclic coverings with L ample and spanned, satisfying (4.3). "A fortiori", the situation appears more intricate for k-cyclic covers, in view of the above and 4.9.

REFERENCES

1. W. Barth, C. Peters and A. Van, de Ven *Compact complex surfaces* (Springer-Verlag, 1984).

2. A. Beauville, *Surfaces algébriques complexes*, Asterisque *54* (1978).

3. M. Beltrametti, A. Lanteri and M. Palleschi, *Algebraic surfaces containing an ample divisor of arithmetic genus two*, Ark. Mat. *25* (1987).

4. A. Buium, *Sur le nombre de Picard des revètements doubles des surfaces algébriques*, C.R. Acad. Sci. Paris *296* (1983), 361-364.

5. M. Cornalba, *Un'osservazione sulla topologia dei rivestimenti ciclici di varietà algebriche*, Boll. Un. Mat. Ital. *18*-A (1981), 323-328.

6. T. Fujita, *On topological characterizations of complex projective spaces and affine linear spaces*, Proc. Japan Acad. *56* A (1980), 231-234.

7. R. Hartshorne, *Algebraic geometry* (Springer-Verlag, 1977).

8. B. Iversen, *Numerical invariants and multiple planes*, Amer. J. Math. *92* (1970), 968-996.

9. A. Lanteri and M. Palleschi, *About the adjunction process for polarized algebraic surfaces*, J. Reine Angew. Math. *351* (1984), 15-23.

10. A. Lanteri and D. Struppa, *Projective manifolds with the same homology as* $\mathbf{P}^k$, Monatsh. Math. *101* (1986), 53-58.

11. I. Reider, *Vector bundles of rank* 2 *and linear systems on algebraic surfaces*, Ann. of Math. *127* (1988), 309-316.

12. A. J. Sommese, *On manifolds that cannot be ample divisors*, Math. Ann. *221* (1976), 55-72.

13. ——— *On the adjunction theoretic structure of projective varieties*, in *Complex analysis and algebraic geometry*, Proc., Göttingen (1985), 175-213. Lect. Notes Math. *1194* (Springer-Verlag, 1986).

14. P. M. H. Wilson, *On projective manifolds with the same rational cohomology as* $\mathbf{P}^4$, Rend. Sem. Mat. Univ. Politec. Torino (1986), Special Issue, 15-23.

Universita di Milano,
Milano, Italy