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HYPERBOLIC MANIFOLDS ADMITTING HOLOMORPHIC FIBERINGS

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We give a simple proof of the result that if the total space of a holomorphic fiber bundle is (complete) hyperbolic then both the fiber and the base manifold must be (complete) hyperbolic. Shoshichi Kobayashi tried to set up examples where the total space is hyperbolic but the base is not; our theorem shows that any such example is bound to fail.

Kobayashi defined an invariant pseudo-distance on any complex manifold (see [2]) and called a manifold (complete) hyperbolic if this pseudo-distance is a (complete) distance on the manifold.

In his book [2], Kobayashi proves a theorem due to Kiernan [1]: if $\pi : E \to M$ is a holomorphic fiber bundle with base M and fiber F then the total space E is (complete) hyperbolic if M and F are (complete) hyperbolic. Kobayashi then remarks ([2], p. 64) that if E is (complete) hyperbolic then F is also (complete) hyperbolic but according to him the base M need not be hyperbolic even if E and F are. Theorem 1 below shows that M must be hyperbolic if E is. In Remark 1 we will explain why Kobayashi's examples fail.

THEOREM 1. *Let* E *be a holomorphic fiber bundle over* M *with fiber* F *and projection* $\pi : E \to M$ *. Then if* E *is (complete) hyperbolic the*

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fiber F *and base space* M *must both be (complete) hyperbolic.* *(*E, M *, and* F *are complex manifolds.)*

Proof. Let us recall some facts about hyperbolic manifolds.

(A) If $p : \tilde{M} \to M$ is a holomorphic covering space between complex manifolds then $\tilde{M}$ is (complete) hyperbolic if and only if M is (complete) hyperbolic (see Kobayashi [2], p. 58).

(B) A (closed) complex submanifold X' of a (complete) hyperbolic manifold X is (complete) hyperbolic (see Kobayashi [2], Proposition 4.2, p. 57).

(C) A holomorphic fiber bundle with simply connected base and hyperbolic fiber is holomorphically trivial (see Royden [3], Corollary 1).

Now if E is (complete) hyperbolic then F is (complete) hyperbolic by (B) above. Let $p : \tilde{M} \to M$ be the holomorphic universal covering of M . Pull back the fiber bundle $\pi : E \to M$ to a fiber bundle $\tilde{\pi} : \tilde{E} \to \tilde{M}$ getting a commutative diagram

$$\begin{array}{ccc} E & \overset{\tilde{p}}{\longleftarrow} & \tilde{E} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \underset{p}{\longleftarrow} & \tilde{M} \end{array} .$$

Since p is a covering space clearly the induced $\tilde{p} : \tilde{E} \to E$ is a covering space also. Now, by (C), $\tilde{E} = \tilde{M} \times F$ and, by (A), $\tilde{E}$ is (complete) hyperbolic as E is assumed (complete) hyperbolic. But then $\tilde{M}$ embeds as a closed submanifold in the product $\tilde{M} \times F = \tilde{E}$; therefore (by (B)) $\tilde{M}$, and consequently (by (A)) also M , is (complete) hyperbolic.

REMARK 1. Kobayashi considers the following example on p. 64 of [2]: let

$$B^* = \{(z, w) \in \mathbb{C}^2 : 0 < |z|^2+|w|^2 < 1\}$$

and

$$D^* = \{z \in \mathbb{C} : 0 < |z| < 1\} .$$

Then B^* and D^* are hyperbolic and the natural projection $\pi : B^* \to \mathbb{P}^1(\mathbb{C})$ given by $\pi((z, w)) = \langle z, w \rangle$ (homogeneous coordinates in $\mathbb{P}^1$) is a holomorphic fiber space onto a non-hyperbolic base manifold $\mathbb{P}^1$.

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However, although this fiber space is a topological (C^∞) fiber bundle with D^* as fiber it is not a holomorphic fiber bundle. This can be seen as follows.

The map $h : U \times D^* \to \pi^{-1}(U)$ given by

$$h((x, t)) = \left(t/\left(|x|^2+1\right)^{\frac{1}{2}}, tx/\left(|x|^2+1\right)^{\frac{1}{2}}\right)$$

is a C^∞ local trivialization of π over the neighbourhood $U \subset \mathbb{P}^1$ given by $x \in \mathbb{C}$ where x is identified with $\langle 1, x\rangle \in \mathbb{P}^1$. Since the automorphisms of D^* are simply rotations about the origin it is easy to see that one can find a holomorphic local trivialization if and only if one can find a holomorphic function f of x in a neighbourhood of $x = 0$ in $\mathbb{C}$ such that

$$|f(x)|^2 = |x|^2 + 1$$

(because then one can replace h by $(x, t) \mapsto \left(t/f(x), tx/f(x)\right)$). But since $\log\left(1+|x|^2\right)$ is not a harmonic function in the x-plane no such holomorphic f exists - thus directly proving that this is not a holomorphic fiber bundle.

Alternatively, one may use fact (C) above to say that $\pi : B^* \to \mathbb{P}^1$ must be holomorphically trivial if the example were correct. But a little topology shows that the fiber space π does not even admit a continuous global cross-section. Indeed if $f : \mathbb{P}^1 \to B^*$ is a global cross-section then $\pi \circ f = 1$ and on second homology groups one has induced maps $H_2(f)$ and $H_2(\pi)$ whose composition is the identity. But $H_2\left(\mathbb{P}^1\right) = \mathbb{Z}$, whereas B^* is topologically $S^3 \times \mathbb{R}$, so $H_2(B^*) = 0$. Thus $H_2(f)$ is the 0 map, and hence $H_2(\pi) \circ H_2(f)$ cannot be the identity.

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