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P-SPACES AND THE VOLTERRA PROPERTY

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Abstract

We study the relationship between generalisations of P-spaces and Volterra (weakly Volterra) spaces, that is, spaces where every two dense G_δ have dense (nonempty) intersection. In particular, we prove that every dense and every open, but not every closed subspace of an almost P-space is Volterra and that there are Tychonoff nonweakly Volterra weak P-spaces. These results should be compared with the fact that every P-space is hereditarily Volterra. As a byproduct we obtain an example of a hereditarily Volterra space and a hereditarily Baire space whose product is not weakly Volterra. We also show an example of a Hausdorff space which contains a nonweakly Volterra subspace and is both a weak P-space and an almost P-space.

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1. Introduction

A real-valued function f is called *pointwise discontinuous* if the set of all points where it is continuous is dense. In 1881, eighteen years before René-Louis Baire published the Baire category theorem [1], a 20-year-old student of the Scuola Normale Superiore di Pisa named Vito Volterra proved that there are no two pointwise discontinuous real-valued functions on $\mathbb{R}$ such that the set of all points of continuity of one is equal to the set of all discontinuity points of the other [16] (see also [4]). Volterra's theorem has inspired an interesting generalisation of the Baire property.

Given $f : X \to \mathbb{R}$, let $C(f)$ be the set of all continuity points of f.

DEFINITION 1.1 [6]. A topological space X is called *Volterra* (respectively, *weakly Volterra*) if for every pair of pointwise discontinuous functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ the set $C(f) \cap C(g)$ is dense in X (respectively, nonempty).

Thus Volterra's theorem can be rephrased by stating that the real line is a Volterra space. Gauld and Piotrowski proved the following internal characterisation of Volterra and weakly Volterra spaces. Recall that a set is called a G_δ *set* if it can be represented as a countable intersection of open sets.

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Proposition 1.2 [6]. *A space is Volterra (respectively, weakly Volterra) if and only if for every pair G and H of dense G_δ subsets of X, the set $G \cap H$ is dense (respectively, nonempty).*

Recall that a space is Baire if every countable intersection of dense open sets is dense. From the above characterisation it is clear that every Baire space is Volterra. The problem of when a Volterra space is Baire has been extensively studied (see [2, 7]).

This note was inspired by the simple observation that every P-space (that is, a space where every G_δ set is open) is hereditarily Volterra. Weak P-spaces and almost P-spaces are the two most popular weakenings of P-spaces. We compare these properties with the notions of Volterra and weakly Volterra space. We find that every dense subset and every open subset of an almost P-space is Volterra, while weak P-spaces may fail to be weakly Volterra. Our example of a nonweakly Volterra weak P-space shows that the product of a hereditarily Baire space and a hereditarily Volterra space may fail to be weakly Volterra. Finally, we introduce the class of *pseudo P-spaces*, a natural new weakening of P-spaces, and construct a Hausdorff Baire pseudo P-space with a nonweakly Volterra subspace. The existence of a Tychonoff space with the same properties is left as an open question.

2. P-spaces and generalisations

Definition 2.1.

(1) A space X is called a *P-space* if every countable intersection of open subsets of X is open.
(2) A point $x \in X$ is called a P-point if for every countable family $\{U_n : n < \omega\}$ of neighbourhoods of x we have that $x \in \operatorname{Int}(\bigcap_{n<\omega} U_n)$.
(3) A space X is called an *almost P-space* if every nonempty G_δ subset of X has nonempty interior.
(4) A space X is called a *weak P-space* if every countable subset of X is closed (and discrete).
(5) A point $x \in X$ is called a *weak P-point* if $x \notin \overline{C}$ for every countable $C \subset X \setminus \{x\}$.

Every P-space is an almost P-space and a weak P-space. For the reader's convenience we now recall examples to distinguish the notions of almost P-space and weak P-space.

Example 2.2. There are almost P-spaces which are not weak P-spaces and weak P-spaces which are not almost P-spaces.

Proof. Rudin proved in [14] that ω^*, the remainder of the Čech stone compactification of the integers, is an almost P-space. Hence this is an example of an almost P-space which is not a weak P-space, as weak P-spaces cannot be compact. Watson [17] was even able to construct a compact almost P-space where every point is the limit of a nontrivial convergent sequence.

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We now present a simple example of a weak P-space which is not an almost P-space. Let X be the set of all weak P-points in ω^*. Kunen [9] proved that X is dense in ω^*. Since ω^* is not a P-space we can fix open sets $\{U_n : n < \omega\}$ such that $\bigcap_{n\in\omega} U_n$ is not open, but $U = \mathrm{Int}(\bigcap_{n<\omega} U_n)$ is a nonempty open set. Now $X \setminus U$ is a weak P-space which is not an almost P-space, as $\bigcap\{U_n \cap (X \setminus U) : n < \omega\}$ is a nonempty relative G_δ subset of $X \setminus U$ with empty interior. □

Definition 2.3. Given a property $\mathcal{P}$ of subsets of a topological space X, we say that X is *$\mathcal{P}$-hereditarily Volterra (Baire)* if every subspace of X satisfying $\mathcal{P}$ is Volterra (Baire). A space is *hereditarily Volterra (Baire)* if each one of its subspaces is Volterra (Baire).

Contrast our Definition 2.3 with the common habit of calling a space *hereditarily Baire* if each of its closed subsets is Baire. For example, the real line is not hereditarily Baire according to our definition.

Since every subspace of a P-space is a P-space, the following proposition is clear.

Proposition 2.4. *Every P-space is hereditarily Volterra.*

Proposition 2.5. *Every almost P-space is dense-hereditarily Volterra and open-hereditarily Volterra.*

Proof. Let X be an almost P-space. We claim that X is Volterra. Indeed, let G and H be dense G_δ subspaces of X. We claim that $\mathrm{Int}(G) \cap H$ is a dense set. Since H is dense and $\mathrm{Int}(G)$ is open, $\overline{\mathrm{Int}(G) \cap H} = \overline{\mathrm{Int}(G)}$. So if $\mathrm{Int}(G) \cap H$ were not dense then $X \setminus \overline{\mathrm{Int}(G)}$ would be a nonempty open set, and thus it would have to meet G. Thus, $G \cap (X \setminus \overline{\mathrm{Int}(G)})$ would be a nonempty G_δ set with empty interior. But that contradicts the fact that X is an almost P-space.

To prove the statement of the proposition it now suffices to recall a result of Levy [11] stating that every open set and every dense set of an almost P-space is an almost P-space. □

Almost P-spaces need not be hereditarily Volterra.

Example 2.6. There is a Baire regular almost P-space with a closed nonweakly Volterra subspace.

Proof. Levy [10] constructed a Baire regular almost P-space containing a closed copy of the rational numbers, and the rational numbers are not weakly Volterra. □

On the other hand, weak P-spaces need not even be weakly Volterra. The construction of our counterexample will exploit the density topology on the real line. We recall its definition.

Definition 2.7. A measurable set $A \subset \mathbb{R}$ has density d at x if the limit

$$\lim_{h\to 0} \frac{m(A \cap [x-h, x+h])}{2h}$$

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exists and is equal to d. We denote by $d(x, A)$ the density of A at x and let

$$\phi(A) = \{x \in \mathbb{R} : d(x, A) = 1\}.$$

DEFINITION 2.8. The family of all measurable sets $A \subset \mathbb{R}$ such that $\phi(A) \supset A$ defines a topology on $\mathbb{R}$ called the *density topology* and denoted by $\mathbb{R}_d$.

Since the density topology is finer than the Euclidean topology on the real line, every point is a G_δ set in $\mathbb{R}_d$. Moreover, every measure zero set is easily seen to be closed in $\mathbb{R}_d$. In particular, the density topology is a weak P-space. (See [15] for a comprehensive study of the density topology.)

Recall that a space is *resolvable* if it contains two disjoint dense sets. Dontchev *et al.* [3] proved that the density topology is resolvable. (This was later improved by Luukkainen [12] who proved that $\mathbb{R}_d$ even contains a pairwise disjoint family of dense sets of size continuum.) In the following lemma we review all properties of the density topology that are relevant to us here.

LEMMA 2.9. *The density topology $\mathbb{R}_d$ is a Tychonoff resolvable weak P-space with points G_δ.*

We also need the following lemma of Gruenhage and Lutzer.

LEMMA 2.10 [7]. *Suppose that $\mathcal{U}$ is a point-finite collection of open subsets of a space X and that each $U \in \mathcal{U}$ contains a G_δ set $G(U)$. Then $\bigcup\{G(U) : U \in \mathcal{U}\}$ is a G_δ set.*

EXAMPLE 2.11. There is a nonweakly Volterra Tychonoff weak P-space.

PROOF. Let $X = \{f \in 2^{\omega_1} : |f^{-1}(1)| < \omega\}$ with the topology inherited from the countably supported product topology on 2^{ω_1}. Let

$$U_n = X \setminus \{f \in 2^{\omega_1} : |f^{-1}(1)| \leq n\},$$

and note that U_n is an open dense set in X.

Use Lemma 2.9 to fix disjoint dense sets D_1 and D_2 inside $\mathbb{R}_d$.

Since $\mathbb{R}_d$ is a weak P-space and X is a P-space, $X \times \mathbb{R}_d$ is a weak P-space. Note that the family $\{U_n \times \mathbb{R}_d : n < \omega\}$ is point-finite and $U_n \times \{x\}$ is a G_δ set contained in $U_n \times \mathbb{R}_d$ for every $x \in \mathbb{R}_d$. Thus, by Lemma 2.10,

$$\bigcup_{x \in D_1} U_n \times \{x\} \quad \text{and} \quad \bigcup_{x \in D_2} U_n \times \{x\}$$

are disjoint dense G_δ sets in $X \times \mathbb{R}_d$. □

Since every subspace of $\mathbb{R}_d$ is Baire (see [15]), Example 2.11 shows that the product of a hereditarily Volterra space and a hereditarily Baire space may fail to be weakly Volterra. This suggests the following question.

QUESTION 2.12. Are there hereditarily Baire spaces X and Y such that $X \times Y$ is not weakly Volterra?

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Note that there are metric Baire spaces whose square is not weakly Volterra (see [5, Example 3.9]), but if an example answering Question 2.12 in the positive exists, none of its factors can be metric. Indeed, the product of a Baire space and a closed-hereditary Baire metric space is Baire (see [13]).

3. A new weakening of P-spaces

Definition 3.1. We call a space X a *pseudo P-space* if it is both an almost P-space and a weak P-space.

Example 3.2. There are regular pseudo P-spaces which are not P-spaces.

Proof. For one example, let X be the subspace of all weak P-points of ω^*. Since X is dense in the almost P-space $\mathbb{N}^*$, X is also an almost P-space. Clearly X is a weak P-space. However, since there is a weak P-point which is not a P-point in ω^*, X is not a P-space.

Another example was essentially presented in [8]. Let X be a Lindelöf P-space without isolated points. Van Mill (see [8, Lemma 3.1]) proved that there is a point $p \in \beta X \setminus X$ such that p is not in the closure of any countable subset of X. Then $X \cup \{p\}$ is a weak P-space. But, from the fact that X is a P-space it follows that $X \cup \{p\}$ is an almost P-space. Now, $X \cup \{p\}$ is not a P-space, or otherwise it would be a Lindelöf P-space, and thus each of its Lindelöf subspaces should be closed. But X is a nonclosed Lindelöf subspace of $X \cup \{p\}$. □

Pseudo P-spaces are in some sense very close to P-spaces, closer than almost P-spaces, which suggests the following question.

Question 3.3. Is there a regular pseudo P-space which is not hereditarily weakly Volterra?

The following example provides a partial answer to this question.

Example 3.4. There is a Hausdorff (nonregular) Baire pseudo P-space which is not hereditarily weakly Volterra.

Proof. Let $X = \{f \in 2^{\omega_1} : |f^{-1}(1)| \le \aleph_0\}$. Let C be the set of all functions from a countable subset of ω_1 to 2. For every $\sigma \in C$, let $[\sigma] = \{f \in 2^{\omega_1} : \sigma \subset f\}$. Moreover, for every $n < \omega$, let $X_n = \{f \in 2^{\omega_1} : |f^{-1}(1)| = n\}$. Define a topology on X by declaring $\{[\sigma] \setminus X_n : \sigma \in C, n < \omega\}$ to be a subbase.

Claim 1. X is a pseudo P-space.

Proof of Claim 1. The topology on X is a refinement of the countably supported box product topology on 2^{ω_1} and thus X is a weak P-space. To prove that X is an almost P-space, let $G = \bigcap\{U_n : n < \omega\}$ be a nonempty G_δ set and $x \in G$. For every $n < \omega$, choose α_n and a finite set $\mathcal{F}_n \subset \{X_k : k < \omega\}$ such that $V_n := [x \restriction \alpha_n] \setminus \bigcup \mathcal{F}_n \subset U_n$. Let $h \in \bigcap_{n<\omega} V_n$ be a function with infinite support and $\beta < \omega_1$ be an ordinal such that $\beta \ge \sup_{n<\omega} \alpha_n$. Then $[h \restriction \beta] \subset \bigcap_{n<\omega} V_n \subset \bigcap_{n<\omega} U_n$. □

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Claim 2. The space X is Baire.

Proof of Claim 2. We prove that every meagre set is nowhere dense. Let $\{N_n : n < \omega\}$ be a countable family of nowhere dense subsets of X. Let σ be a countable partial function with domain $\alpha < \omega_1$ and k be an integer. We will prove that the basic open set $[\sigma] \setminus \bigcup\{X_k : k \leq n\}$ is not contained in the closure of $\bigcup_{n<\omega} N_n$. Since N_0 is nowhere dense there must be a countable partial function σ_0 extending σ with domain $\alpha_0 > \alpha$ and an integer $k_0 < \omega$ such that $([\sigma_0] \setminus \bigcup\{X_k : k \leq k_0\}) \cap N_0 = \emptyset$.

Suppose that we have found an increasing sequence of countable partial functions $\{\sigma_i : i < n\}$ and an increasing sequence of integers $\{k_i : i < n\}$. Since N_n is nowhere dense there must be a countable partial function σ_n extending σ_{n-1} and an integer $k_n > k_{n-1}$ such that $[\sigma_n] \cap N_n = \emptyset$. Let $\sigma_\omega = \bigcup_{i<\omega} \sigma_i$. Then

$$([\sigma_\omega] \setminus \cup \{X_k : k < \omega\}) \cap \bigcup_{n<\omega} N_n = \emptyset \quad \text{and} \quad \emptyset \neq [\sigma_\omega] \subset ([\sigma] \setminus \cup \{X_n : n \leq k\}).$$

Thus $[\sigma] \setminus \bigcup\{X_n : n \leq k\}$ is not contained in $\overline{\bigcup_{n<\omega} N_n}$ and since the choice of σ and k was arbitrary, this shows that $\bigcup_{n<\omega} N_n$ is nowhere dense. □

Claim 3. Let $Y = \bigcup_{n<\omega} X_n \subset X$. Then Y is not weakly Volterra.

Proof of Claim 3. Let $G = \bigcap\{X \setminus X_k : k \text{ is even}\}$ and $H = \bigcap\{X \setminus X_k : k \text{ is odd}\}$. Then G and H are dense G_δ subsets of Y with empty intersection. □

This completes the proof of Example 3.4.

As pointed out by Gary Gruenhage in a private communication, Example 3.4 is not regular. For example, the closed set X_1 and the null function cannot be separated by disjoint open sets.

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