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STRONGLY ORTHODOX CONGRUENCES ON AN E-INVERSIVE SEMIGROUP

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Abstract

In this paper we investigate some subclasses of strongly regular congruences on an E-inversive semigroup S. We describe the minimum and the maximum strongly orthodox congruences on S whose characteristic trace coincides with the characteristic trace of given congruences and, in each case, we present an alternative characterization for them. A description of all strongly orthodox congruences on S with characteristic trace τ is given. Further, we investigate the kernel relation of strongly orthodox congruences on an E-inversive semigroup and give the least and the greatest element in the class of the same kernel with a given congruence.

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1. Introduction and preliminaries

A semigroup S is called E-inversive if for any $a \in S$ there exists $x \in S$ such that $ax \in E(S)$, the set of idempotents of S. This class of semigroups was introduced by Thierrin [14], and it contains both the class of all eventually regular semigroups (in which every element has a power that is regular; see [1]) and the class of all Bruck semigroups over a monoid (and also includes all periodic semigroups, all group bound semigroups and all semigroups with zero). The strategy for studying E-inversive semigroups was to generalize known results for regular semigroups and for periodic semigroups to E-inversive semigroups. Mitsch [10] studied the subdirect product of E-inversive semigroups, and Zheng [17] characterized the group congruences on an E-inversive semigroup. Some basic properties of E-inversive

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semigroups were given by Mitsch and Petrich [11]. Weipoltshammer [16] described certain special congruences on E-inversive E-semigroups.

Hayes [3] investigated E^*-dense semigroups and gave a characterization theorem for E^*-dense semigroups whose idempotents form a *-rectangular band. Recently, Luo *et al.* [7] described regular congruences on an E-inversive semigroup S by means of their kernels and traces and proved that each regular congruence on S is uniquely determined by its kernel and trace.

The lattices of congruences on regular semigroups have been explored extensively. Gomes [2] gave descriptions for the lattice of $\mathcal{R}$-unipotent congruences on a regular semigroup, and LaTorre [6] described the θ-classes in $\mathcal{L}$-unipotent semigroups. Pastijn and Petrich [12] considered three different subdirect decompositions of the congruence lattice. The lattice of idempotent-separating congruences on a $\mathcal{P}$-regular semigroup was studied by Sen and Seth in [13].

The aim of this paper is to describe some subclasses of strongly regular congruences on an E-inversive semigroup. After introducing some definitions and results in this section, in Section 2 we describe the minimum strongly orthodox congruence determined by its characteristic trace on an E-inversive semigroup, and we give an alternative characterization for it. A description of all strongly orthodox congruences on an E-inversive semigroup with characteristic trace τ is given in Section 3. In the last section, we investigate strongly orthodox congruences determined by their kernel and give the least and the greatest element of $\kappa(\rho)$.

In this paper S denotes an E-inversive semigroup, unless otherwise stated. We shall use the standard terminology and notation of semigroup theory, and the reader is referred to Higgins [4] and Howie [5]. As usual, $E(S)$ is the set of idempotents of a semigroup S, $\mathrm{Reg}(S)$ is the set of regular elements of S and $V(a)$ is the set of all inverses of a in S. An element x of S is called a weak inverse of a if $xax = x$. We denote by $W(a)$ the set of all weak inverses of a in S. From [11, Lemma 3.1], a semigroup S is E-inversive if and only if $W(a) \neq \emptyset$ for any $a \in S$. Luo and Li [8, 9] described $\mathcal{R}$-unipotent congruences and orthodox congruences on eventually regular semigroups by means of the notion of 'weak inverses', which also play an important role in this paper.

Recall from [15] that the core $C(S) = \langle E(S) \rangle$ of S is its idempotent generated subsemigroup. Define

$$C_c(S) = \left\langle \bigcup \{aC(S)a' \cup a'C(S)a : a' \in W(a), a \in S\} \right\rangle, \quad C_\infty(S) = C_{cc\cdots}(S).$$

Then $C_\infty(S)$ (or just C_∞ if the context is clear) is the self-conjugate core of S. It is easy to show that C_∞ is the least self-conjugate full subsemigroup of S having the property of including all weak inverses of its elements. Let ρ be a congruence on a semigroup S. The subset $\{a \in S : a\rho \in E(S/\rho)\}$ of S is called the kernel of ρ and is denoted by $\ker\rho$. The restriction of ρ to the subset C_∞ of S is called the characteristic trace of ρ and is denoted by $\mathrm{ctr}\,\rho$.

Let S be a semigroup and $e, f \in E(S)$. Define

$$M(e, f) = \{g \in E(S) : ge = g = fg\}$$

and

$$S(e, f) = \{g \in E(S) : ge = g = fg, egf = ef\}.$$

$S(e, f)$ is called the sandwich set of e and f. It is known that $M(e, f) \neq \emptyset$ (respectively, $S(e, f) \neq \emptyset$) for all $e, f \in E(S)$ in an E-inversive (respectively, a regular) semigroup S (see [4]).

The following definition provides a central concept of this paper.

Definition 1.1. A congruence ρ on S is called a strongly regular congruence, if for each $a \in S$ there exists $a' \in W(a)$ such that $a \,\rho\, aa'a$.

Recall that a congruence ρ on a semigroup S is called regular if S/ρ is regular. In [7] strongly regular congruences on an E-inversive semigroup are called regular congruences. An example in [7] illustrates that there exists a regular congruence on an E-inversive semigroup S which does not satisfy the following property:

$$(\forall a \in S) \quad (\exists a' \in W(a)) \quad a \,\rho\, aa'a.$$

For a class C, a C-congruence ρ on an E-inversive semigroup S is a *strongly C-congruence* on S if ρ is a strongly regular. For example, an orthodox congruence ρ on an E-inversive semigroup S is said to be *strongly orthodox* if ρ is strongly regular. It is clear that C_∞/ρ is a band if ρ is a strongly orthodox congruence on an E-inversive semigroup S. We have seen that a congruence on an eventually regular semigroup is regular if and only if it is strongly regular (see [16, Lemma 5.4]). Therefore a congruence on an eventually regular semigroup S is a C-congruence on S if and only if it is a strongly C-congruence on S. Notice the fact that all elements have a weak inverse in an E-inversive semigroup. It follows from that the class of E-inversive semigroups is the largest possible class on which strongly regular congruences exist.

We now list some known results for later use.

Lemma 1.2 [7]. *Let $a, b \in S$, $a' \in W(a)$, $b' \in W(b)$. If $g \in M(a'a, bb')$, then $b'ga' \in W(ab) \cap V(agb)$.*

Lemma 1.3 [7]. *If ρ is a strongly regular congruence on S and $a\rho$ is an idempotent of S/ρ, then an idempotent e can be found in $a\rho$ such that $H_e \leqslant H_a$.*

If ρ is a strongly regular congruence on an E-inversive semigroup S then, according to Lemma 1.3,

$$\ker \rho = \{a \in S : (\exists e \in E(S))\; a \,\rho\, e\}.$$

Lemma 1.4 [7]. *Let ρ be a strongly regular congruence on S. If $x, y \in S$ such that $yxy \,\rho\, y$, then there exists $z \in y\rho$ such that $z \in W(x)$ and $H_z \leqslant H_y$.*

Lemma 1.5 [7]. *Let ρ be a strongly regular congruence on S. If $e, f \in E(S)$ such that $e \,\rho\, f$, then there exists $g \in E(S)$ such that $e \,\rho\, g \,\rho\, f$ and $g \in M(e, f)$.*

Definition 1.6. A congruence τ on the least self-conjugate full subsemigroup C_∞ of S is said to be regular normal if:

(i) $(\forall a \in S)(\exists a^+ \in W(a))(\forall a' \in W(a))\ aa' \ \tau\ aa^+aa', a'a \ \tau\ a'aa^+a$;

(ii) $(\forall x, y \in C_\infty)\ x \ \tau\ y \Rightarrow (\forall a \in S, \forall a' \in W(a))\ axa' \ \tau\ aya', a'xa \ \tau\ a'ya.$

Lemma 1.7. *Let ρ be a strongly orthodox congruence on S with $\tau = \operatorname{ctr} \rho$. Then τ is a regular normal band congruence on C_∞.*

Proof. Since ρ is a strongly regular congruence, for any $a \in S$, there exists $a^+ \in W(a)$ such that $a \ \rho\ aa^+a$. Thus $aa^+aa' \ \tau\ aa'$ and $a'aa^+a \ \tau\ a'a$ for any $a' \in W(a)$. Let $x, y \in C_\infty$ be such that $x \ \tau\ y$. Then for any $a \in S$ and $a' \in W(a)$, we have $axa' \ \tau\ aya'$ and $a'xa \ \tau\ a'ya$. Hence τ is a regular normal congruence on C_∞. It follows from ρ being a strongly orthodox congruence on S that C_∞/τ is a band. Thus τ is a regular normal band congruence on C_∞. □

2. The minimum strongly orthodox congruences determined by characteristic traces

Let τ be an equivalence relation on C_∞. Define the following relation $\tau_{\min}$ on S for $a, b \in S$:

$$a\tau_{\min}b \Leftrightarrow \begin{array}{l}(\forall a' \in W(a))(\exists b' \in W(b))(\exists x, y \in C_\infty)(xa = by, x\tau aa'\tau bb', y\tau a'a\tau b'b)\ \&\\ (\forall b' \in W(b))(\exists a' \in W(a))(\exists x, y \in C_\infty)(xb = ay, x\tau aa'\tau bb', y\tau a'a\tau b'b).\end{array}$$

Theorem 2.1. *Let ρ be a strongly orthodox congruence on S with $\tau = \operatorname{ctr} \rho$. Then $\tau_{\min}$ is the minimum strongly orthodox congruence on S with characteristic trace τ.*

Proof. We first show that $\tau_{\min}$ is an equivalence relation. It is clear that $\tau_{\min}$ is reflexive and symmetric. To show that $\tau_{\min}$ is transitive, let $(a, b), (b, c) \in \tau_{\min}$. Then for any $a' \in W(a)$ there exist $b' \in W(b)$, $x, y \in C_\infty$ such that

$$xa = by, \quad x \ \tau\ aa' \ \tau\ bb' \quad \text{and} \quad y \ \tau\ a'a \ \tau\ b'b,$$

and so for $b' \in W(b)$, there exist $c' \in W(c)$, $z, v \in C_\infty$ such that

$$zb = cv, \quad z \ \tau\ bb' \ \tau\ cc' \quad \text{and} \quad v \ \tau\ b'b \ \tau\ c'c.$$

Let $x_1 = zx$, $y_1 = vy$. Then $x_1, y_1 \in C_\infty$ and $x_1 \cdot a = zxa = zby = cvy = c \cdot y_1$. Notice $x \ \tau\ z$ and $y \ \tau\ v$; we have that

$$x_1 = zx \ \tau\ aa' \ \tau\ cc', \quad y_1 = vy \ \tau\ a'a \ \tau\ c'c.$$

Dually we may show that for any $c' \in W(c)$, there exist $a' \in W(a)$, $p, q \in C_\infty$ such that $pc = aq$, $p \ \tau\ aa' \ \tau\ cc'$ and $q \ \tau\ a'a \ \tau\ c'c$. Therefore $(a, c) \in \tau_{\min}$, as required.

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To show that $\tau_{\min}$ is a congruence, suppose that $(a, b) \in \tau_{\min}$. For any $c \in S$, $(ac)' \in W(ac)$, we have that $a' = c(ac)' \in W(a)$, $c' = (ac)'a \in W(c)$ and $(ac)' = c'a'$, $a'a = cc'$. By the definition of $\tau_{\min}$, there exist $b' \in W(b)$, $x, y \in C_\infty$ such that $xa = by$, $x \,\tau\, aa' \,\tau\, bb'$ and $y \,\tau\, a'a \,\tau\, b'b$. Now

$$bcc'b'x \cdot ac = bc \cdot c'b'byc.$$

Let $s = bcc'b'x$, $t = c'b'byc$. Then $s, t \in C_\infty$ and $s \cdot ac = bc \cdot t$. It follows that

$$s = bcc'b'x = ba'ab'x \,\tau\, bb'x \,\tau\, aa' = (ac)c'a' = (ac)(ac)'$$

and

$$t = c'b'byc \,\tau\, c'a'ac = (ac)'(ac).$$

On the other hand, by Lemma 1.5, there exists $g \in M(b'b, a'a) = M(b'b, cc')$ such that $b'b \,\tau\, g \,\tau\, a'a$. Let $(bc)' = c'gb'$. Then $(bc)' \in W(bc)$. It now follows that

$$s \,\tau\, (ac)(ac)' = aa' \,\tau\, bb' = bb'bb' \,\tau\, bgb' = bcc'gb' = (bc)(bc)'$$

and

$$t \,\tau\, (ac)'(ac) = c'a'ac \,\tau\, c'gc = (c'gb')(bc) = (bc)'(bc).$$

A similar argument will show that for any $(bc)' \in W(bc)$, there exist $(ac)' \in W(ac)$, $p, q \in C_\infty$ such that $p(bc) = (ac)q$ and $p \,\tau\, (ac)(ac)' \,\tau\, (bc)(bc)'$, $q \,\tau\, (ac)'(ac) \,\tau\, (bc)'(bc)$. Hence $ac \,\tau_{\min}\, bc$, and so that $\tau_{\min}$ is a right congruence on S. Similarly, we can show that $\tau_{\min}$ is a left congruence on S. Consequently, $\tau_{\min}$ is a congruence on S.

We now verify that $\operatorname{ctr} \tau_{\min} = \tau$. Suppose first that $(x, y) \in \tau \cap C_\infty$. Then for any $x' \in W(x)$ and $x' \,\rho\, x'yx'$, by Lemma 1.4, there exists $y' \in W(y)$ such that $x' \,\rho\, y'$ and so $xx' \,\tau\, yy'$ and $x'x \,\tau\, y'y$. Since $x'x \,\tau\, y'y$, by Lemma 1.5 there exists $g \in M(x'x, y'y)$ such that $x'x \,\tau\, g \,\tau\, y'y$. Put $m = ygx'$ and $n = g$; then $m, n \in C_\infty$. It follows that $m \cdot x = ygx'x = yg = y \cdot n$. Then

$$m = ygx' \,\tau\, yy'yy' = yy' \,\tau\, xx' \quad \text{and} \quad n = g \,\tau\, x'x \,\tau\, y'y.$$

A similar argument will show that for any $y' \in W(y)$, there exist $x' \in W(x)$, $p, q \in C_\infty$ such that $p \cdot y = x \cdot q$ and $p \,\tau\, xx' \,\tau\, yy'$, $q \,\tau\, x'x \,\tau\, y'y$. Thus $(x, y) \in \tau_{\min}$.

Conversely, let $(x, y) \in \operatorname{ctr} \tau_{\min}$. Since ρ is a strongly regular congruence on S, there exist $x'' \in W(x)$ and $y'' \in W(y)$ such that $x \,\rho\, xx''x$ and $y \,\rho\, yy''y$. By the definition of $\tau_{\min}$, there exist $y' \in W(y)$, $p_1, q_1 \in C_\infty$ such that

$$p_1x = yq_1, \; p_1 \,\tau\, xx'' \,\tau\, yy' \quad \text{and} \quad q_1 \,\tau\, x''x \,\tau\, y'y,$$

and there exist $x' \in W(x)$, $m_1, n_1 \in C_\infty$ such that

$$m_1y = xn_1, \quad m_1 \,\tau\, xx' \,\tau\, yy'' \quad \text{and} \quad n_1 \,\tau\, x'x \,\tau\, y''y.$$

It follows that

$$x \,\tau\, xx''x \,\tau\, p_1x = yq_1 \,\tau\, yy'y \,\tau\, xx''y$$

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and

$$y\ \tau\ yy''y\ \tau\ xx'y\ \tau\ xx''xx'y\ \tau\ xx''y.$$

Hence $x\ \tau\ y$, as required.

To show that $\tau_{\min}$ is a strongly orthodox congruence, we first show that $\tau_{\min}$ is a strongly regular congruence on S. By Lemma 1.7, τ is a regular normal congruence on C_∞; then for each $a \in S$, there exists $a^+ \in W(a)$ such that $aa'\ \tau\ aa^+aa'$ and $a'a\ \tau\ a'aa^+a$ for any $a' \in W(a)$. Clearly, $aa^+a \in \mathrm{Reg}(S)$. Now we show that $a\ \tau_{\min}\ aa^+a$. Notice that $a'a = a'aa'a\ \tau\ a'aa^+aa'a$, so by Lemma 1.4 there exists $(a^+a)' \in W(a^+a) \cap C_\infty$ such that $(a^+a)'\ \tau\ a'a$ and $H_{(a^+a)'} \leqslant H_{a'a}$. Notice that $(a^+a)'a'a = (a^+a)'$, hence

$$(a^+a)'a' \cdot aa^+a \cdot (a^+a)'a' = (a^+a)'a' \quad \text{that is, } (a^+a)'a' \in W(aa^+a).$$

Put $s = aa^+aa'$, $t = a'a$; then $s, t \in C_\infty$ and $s \cdot a = aa^+aa'a = aa^+a \cdot t$. It follows that

$$s = aa^+aa'\ \tau\ aa'\ \tau\ (aa^+a)(a^+a)'a'$$

and

$$t = a'a\ \tau\ a'aa^+a\ \tau\ (a^+a)'a^+a = (a^+a)'a'(aa^+a).$$

On the other hand, for any $u \in W(aa^+a)$, we have $a^+au \cdot a \cdot a^+au = a^+au$ and $au \cdot aa^+ \cdot au = au$. It follows that $a^+au \in W(a)$ and $au \in W(aa^+) \cap C_\infty$. Thus

$$aa^+au \cdot au \cdot aa^+au = a \cdot a^+au \cdot au\ \tau\ aa^+auaa^+au = aa^+au.$$

So by Lemma 1.4 there exists $v \in W(au)$ such that $v\ \tau\ aa^+au$ where $v, aa^+au \in C_\infty$. Let $a^* = uv$. Then $uvauv = uv$ implies $a^* \in W(a)$. Put $l = au$, $h = uaa^+a$. Then $l, h \in C_\infty$ and $l \cdot aa^+a = a \cdot h$. It follows that

$$l = au = auaa^+au\ \tau\ auv = a \cdot a^*\ \tau\ aa^+auv\ \tau\ aa^+au \cdot aa^+au = aa^+a \cdot u$$

and

$$h = u \cdot aa^+a = uaa^+auaa^+a\ \tau\ uvaa^+a = a^*aa^+a\ \tau\ a^* \cdot a.$$

Therefore $a\ \tau_{\min}\ aa^+a$, as required.

Next let $a\tau_{\min}, b\tau_{\min} \in E(S/\tau_{\min})$. Then by Lemma 1.3 there exist $e, f \in E(S)$ such that $a\ \tau_{\min}\ e$, $b\ \tau_{\min}\ f$. It follows from the fact that τ is a strongly orthodox congruence on C_∞ that

$$(ab)^2\ \tau_{\min}\ (ef)^2\ \tau\ (ef)\ \tau_{\min}\ (ab).$$

Then $a\tau_{\min}b\tau_{\min} \cdot a\tau_{\min}b\tau_{\min} = a\tau_{\min}b\tau_{\min}$. Hence $S/\tau_{\min}$ is an orthodox semigroup and $\tau_{\min}$ is a strongly orthodox congruence on S.

Finally, we show that $\tau_{\min}$ is the minimum strongly orthodox congruence on S with characteristic trace τ. Let θ be any strongly orthodox congruence on S with characteristic trace τ, and $(a, b) \in \tau_{\min}$. Since θ is a strongly regular congruence on S, for any $a, b \in S$, there exist $a'' \in W(a)$ and $b'' \in W(b)$ such that $a\ \theta\ aa''a$ and $b\ \theta\ bb''b$.

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By the definition of $\tau_{\min}$, there exist $a' \in W(a)$, $b' \in W(b)$, $x, y, l, h \in C_\infty$ such that

$$xb = ay, \quad x \, \tau \, aa' \, \tau \, bb'', \quad y \, \tau \, a'a \, \tau \, b''b$$

and

$$la = bh, \quad l \, \tau \, aa'' \, \tau \, bb', \quad h \, \tau \, a''a \, \tau \, b'b,$$

so that

$$x \, \theta \, aa' \, \theta \, bb'', \quad y \, \theta \, a'a \, \theta \, b''b$$

and

$$l \, \theta \, aa'' \, \theta \, bb', \quad h \, \theta \, a''a \, \theta \, b'b.$$

It follows that $(b'a) \, \theta \, b'aa''a \, \theta \, b'la = b'bh \, \theta \, b'b$, and so $(b'a)\theta \in E(S/\theta)$. Now

$$a \, \theta \, aa''a \, \theta \, bb'a \, \theta \, b(b'a)(b'a) \, \theta \, ab'a.$$

On the other hand,

$$b \, \theta \, bb''b \, \theta \, aa'b \, \theta \, aa''aa'b \, \theta \, (aa'')(bb''b) \, \theta \, aa''b.$$

It follows that

$$a \, \theta \, aa''a \, \theta \, ab'b \, \theta \, ab' \cdot aa''b = ab'a \cdot a''b \, \theta \, aa''b \, \theta \, b.$$

Therefore $\tau_{\min} \subseteq \theta$, as required. □

We now present an alternate characterization of $\tau_{\min}$.

Theorem 2.2. *Let ρ be a strongly orthodox congruence on S with $\tau = \mathrm{ctr}\,\rho$. Define a binary relation $\delta_{\min}$ on S as follows. For $a, b \in S$, let*

$$a \, \delta_{\min} \, b \Leftrightarrow \begin{array}{l} (\forall a' \in W(a))(\exists b' \in W(b)) \, (aa' \, \tau \, bb', \; a'a \, \tau \, b'b, \; a'b \in \ker \tau_{\min}) \; \& \\ (\forall b' \in W(b))(\exists a' \in W(a)) \, (aa' \, \tau \, bb', \; a'a \, \tau \, b'b, \; b'a \in \ker \tau_{\min}). \end{array}$$

Then $\delta_{\min} = \tau_{\min}$.

Proof. Let $(a, b) \in \tau_{\min}$. Then for any $a' \in W(a)$, there exists $b' \in W(b)$ such that $aa' \, \tau \, bb', a'a \, \tau \, b'b$. Also $(a'b, a'a) \in \tau_{\min}$, and so $a'b \in \ker \tau_{\min}$. A similar argument will show the dual case. Hence $(a, b) \in \delta_{\min}$, as required.

Conversely, let $(a, b) \in \delta_{\min}$. For any $a' \in W(a)$, then there exists $b' \in W(b)$ such that $aa' \, \tau \, bb', a'a \, \tau \, b'b$ and $a'b \in \ker \tau_{\min}$. Notice that $\tau = \mathrm{ctr}\,\tau_{\min}$. Thus $aa' \; \mathrm{ctr}\,\tau_{\min} \; bb'$, $a'a \; \mathrm{ctr}\,\tau_{\min} \; b'b$ and $a'b \in \ker \tau_{\min}$. Similarly, for any $b' \in W(b)$, there exists $a' \in W(a)$ such that $aa' \; \mathrm{ctr}\,\tau_{\min} \; bb'$, $a'a \; \mathrm{ctr}\,\tau_{\min} \; b'b$ and $b'a \in \ker \tau_{\min}$. Since $\tau_{\min}$ is a strongly orthodox congruence on S, it is easy to prove that $a \, \tau_{\min} \, b$ by imitating the corresponding part of Theorem 2.1. Therefore $\delta_{\min} = \tau_{\min}$. □

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3. Strongly orthodox congruences determined by characteristic trace

Definition 3.1. Let S be a semigroup and τ be an equivalence relation on C_∞. Define a binary relation $\tau_{\max}$ on S for $a, b \in S$ by

$$a\, \tau_{\max}\, b \Leftrightarrow \begin{array}{l} (\forall a' \in W(a))\, (\exists b' \in W(b))\, (aa'\ \tau\ bb',\ a'a\ \tau\ b'b)\ \&\\ (\forall b' \in W(b))\, (\exists a' \in W(a))\, (aa'\ \tau\ bb',\ a'a\ \tau\ b'b). \end{array}$$

If τ is an equivalence relation on $E(S)$ then $\tau_{\max}$ is equivalent to the relation $\mathcal{H}_\tau$ given in [7, Definition 2.1]. It is clear that if τ is a congruence on $E(S)$ and $e\ \tau_{\max}\ f$ for any $e, f \in E(S)$, then $e\tau = f\tau$.

Theorem 3.2. *Let ρ be a strongly orthodox congruence on S with $\tau = \operatorname{ctr} \rho$. Then $\tau_{\max}$ is the maximum strongly orthodox congruence on S with characteristic trace τ.*

Proof. It follows from the fact that ρ is a strongly orthodox congruence on S that $\tau = \operatorname{ctr} \rho$ is a band congruence on C_∞.

To show that $\operatorname{ctr} \tau_{\max} = \tau$, let $x, y \in C_\infty$ be such that $x\ \tau_{\max}\ y$. Since ρ is a strongly regular congruence, there exist $x' \in W(x) \cap C_\infty$ such that $x\ \rho\ xx'x$. So by the definition of $\tau_{\max}$, there exist $y' \in W(y) \cap C_\infty$ such that $xx'\ \tau\ yy'$, $x'x\ \tau\ y'y$. It follows from $\tau = \operatorname{ctr} \rho$ being a band congruence on C_∞ that $xy'y\ \tau\ x\ \tau\ xx'x\ \tau\ yy'x$. And so $xy\ \tau\ x\ \tau\ yx$. Dually, we have that $yx\ \tau\ y\ \tau\ xy$. Hence $x\ \tau\ y$, and so $\operatorname{ctr} \tau_{\max} \subseteq \tau$. Conversely, let $x, y \in C_\infty$ be such that $x\ \tau\ y$. For any $x' \in W(x)$, then $x'yx'\ \rho\ x'$. By Lemma 1.4, there exists $y' \in W(y)$ such that $x'\ \rho\ y'$. Hence $xx'\ \tau\ yy'$ and $x'x\ \tau\ y'y$. Dually, for any $y' \in W(y)$, there exists $x' \in W(x)$ such that $xx'\ \tau\ yy'$, $x'x\ \tau\ y'y$. Hence $x\ \tau_{\max}\ y$ and so $\tau \subseteq \tau_{\max}$. Therefore $\tau = \operatorname{ctr} \tau_{\max}$.

As in [7, Theorem 2.3] we may deduce that $\tau_{\max}$ is the maximum strongly regular congruence on S. Then $\tau_{\max}$ is the maximum strongly orthodox congruence on S with characteristic trace τ. □

Proposition 3.3. *Let ρ be any strongly orthodox congruence on S with $\tau = \operatorname{ctr} \rho$. Then for all $e \in E$,*

$$e\rho = e\tau_{\max} \cap \ker \rho.$$

Proof. Let $a \in e\tau_{\max} \cap \ker \rho$, that is, $(a, e) \in \tau_{\max}$ and $a \in \ker \rho$. Then there exists $f \in E$ such that $(a, f) \in \rho$. It follows that $faf\ \rho\ f$. By Lemma 1.4, there exists $a' \in W(a)$ such that $a'\ \rho\ f$. Hence $(aa', f) \in \rho$, and $a\ \rho\ aa'$. It is easy to show that $\rho \subseteq \tau_{\max}$, and so $a\ \tau_{\max}\ aa'$. By the definition of $\tau_{\max}$, there exists $a^* \in W(a)$ such that $aa'\ \rho\ aa^*\ \rho\ a^*a$. Thus $a\ \rho\ aa'\ \rho\ aa^*$. Since $(a, e) \in \tau_{\max}$, there exists $e^* \in W(e)$ such that $aa^*\ \rho\ ee^*$. Therefore

$$a\ \rho\ aa^*\ \rho\ ee^* = e \cdot ee^*\ \rho\ e \cdot a\ \rho\ ef.$$

On the other hand,

$$a\ \rho\ aa^*\ \rho\ fa^* = f \cdot fa^*\ \rho\ fa.$$

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Since $(a, e) \in \tau_{\max}$ again, there exists $a'' \in W(a)$ such that $e \;\rho\; aa'' \;\rho\; a''a$. Then $e \,\rho\, fa'' \,\rho\, a''f$, and so $e \,\rho\, fef \,\rho\, fa \,\rho\, a$. Therefore $a \in e\rho$, as required.

Conversely, let $a \in e\rho$ for some $e \in E$. Thus $a \in \ker \rho$. For any $a' \in W(a)$, then $a' \;\rho\; a'ea'$ and, by Lemma 1.4, there exists $e' \in W(e)$ such that $e' \;\rho\; a'$. Therefore $aa' \,\rho\, ee'$ and $a'a \,\rho\, e'e$, that is, $aa' \;\tau\; ee'$ and $a'a \;\tau\; e'e$. A similar argument will show that for any $e' \in W(e)$ there exists $a' \in W(a)$ such that $aa' \;\tau\; ee'$ and $a'a \;\tau\; e'e$. Thus $a \in e\tau_{\max}$, and so $a \in e\tau_{\max} \cap \ker \rho$. Consequently, $e\rho = e\tau_{\max} \cap \ker \rho$. □

We now present an alternate characterization of $\tau_{\max}$. The following theorem is very easily proved by imitating the style of Theorem 2.2. We omit the details.

Theorem 3.4. *Let ρ be a strongly orthodox congruence on S with $\tau = \operatorname{ctr} \rho$. Define the following relation $\delta_{\max}$ on S for $a, b \in S$ by*

$$a \;\delta_{\max}\; b \Leftrightarrow \begin{array}{l} (\forall a' \in W(a))\, (\exists b' \in W(b))\, (aa' \;\tau\; bb',\;\; a'a \;\tau\; b'b,\;\; a'b \in \ker \tau_{\max})\ \& \\ (\forall b' \in W(b))\, (\exists a' \in W(a))\, (aa' \;\tau\; bb',\; a'a \;\tau\; b'b,\; b'a \in \ker \tau_{\max}). \end{array}$$

Then $\delta_{\max} = \tau_{\max}$.

Definition 3.5. A subset K of S is called complete if, for $a, b \in S$ and $x \in C_\infty$:

(i) $E(S) \subseteq K$, that is, K is full;
(ii) $xa \in K$ implies $xaa^+a \in K$ for each $a^+ \in W(a)$;
(iii) $b \in K$ implies $(ab^2 \in K \Leftrightarrow ab \in K)$.

Definition 3.6. Let τ be a regular normal congruence on C_∞. A subset K of S is called τ-normal if, for any $a, b \in S$, $x \in C_\infty$,

(C_1) for $y, z \in S$, $a' \in W(a)$ and $b' \in W(b)$, $aa' \;\tau\; bb'$, $a'a \;\tau\; b'b$, $a'b \in K$ and $yb'z \in K \Rightarrow ya'z \in K$.

(C_2) for any $a' \in W(a)$, $a'b \in K$ and $(x, aa') \in \tau \Rightarrow a'xb \in K$.

(C_3) for $a^+ \in W(a)$, $xa \in K$ and $x \;\tau\; aa^+ \Rightarrow a \in K$.

Proposition 3.7. *Let ρ be a strongly orthodox congruence on S with $\tau = \operatorname{ctr} \rho$. If $K = \ker \rho$, then K is a complete and τ-normal subset.*

Proof. We first show that K is a complete subset. It is clear that K is full. Let $a \in S$ and $x \in C_\infty$ be such that $xa \in K = \ker \rho$. Then there exists $e \in E(S)$ such that $xa \,\rho\, e$. Since ρ is a strongly regular congruence on S, there exists $a'' \in W(a)$ such that $a \,\rho\, aa''a$. By Lemma 1.7, $\tau = \operatorname{ctr} \rho$ is a regular normal band congruence on C_∞. Then $aa''\ \operatorname{ctr} \rho\ aa^+aa''$ for each $a^+ \in W(a)$, that is, $aa'' \;\rho\; aa^+aa''$. It follows that

$$a\rho = (aa''a)\rho = (aa^+aa''a)\rho = (aa^+a)\rho.$$

It follows that $xaa^+a \;\rho\; xa \;\rho\; e$, and so $xaa^+a \in K$. Now let $b \in K$. Then there exists $e \in E(S)$ such that $b \,\rho\, e$. Thus $ab \,\rho\, ae \,\rho\, ae^2 \,\rho\, ab^2$, and so $ab \in K \Leftrightarrow ab^2 \in K$. Therefore K is a complete subset.

Consider $a, b, y, z \in S$ and $a' \in W(a)$, $b' \in W(b)$ with $aa' \;\tau\; bb'$, $a'a \;\tau\; b'b$, $a'b \in K$ and $yb'z \in K$. Then $(a'b)\rho, (yb'z)\rho \in E(S/\rho)$. It follows that

$$a' = a'aa' \;\rho\; a'bb' \;\rho\; a'ba'bb' \;\rho\; a'ba'$$

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and

$$b' = b'bb' \;\rho\; b'aa' \;\rho\; b'aa'ba' \;\rho\; b'ba' \;\rho\; a'aa' = a'.$$

Therefore $(ya'z)\rho = (yb'z)\rho \in E(S/\rho)$. By Lemma 1.3, $ya'z \in \ker\rho$, and so (C_1) holds.

For any $a \in S$ and $x \in C_\infty$, $a' \in W(a)$, if $a'b \in K$ and $(x, aa') \in \tau$, then there exists $f \in E(S)$ such that $a'b \;\rho\; f$. It follows that $a'xb \;\rho\; a'aa'b = a'b \;\rho\; f$, that is, $a'xb \in K$. And so (C_2) holds.

Let $x \in C_\infty$ be such that $xa \in \ker\rho$ and $x \;\operatorname{ctr}\rho\; aa^+$ for $a^+ \in W(a)$. Then $(xa)\rho \in E(S/\rho)$. As ρ is a strongly regular congruence on S, one can deduce that $a \;\rho\; aa^+a$ as in the proof above. Hence $a\rho = (aa^+a)\rho = (xa)\rho \in E(S/\rho)$. Thus $a \in \ker\rho$ by Lemma 1.3, and so (C_3) holds. Therefore K is a complete and τ-normal subset. □

The following theorem gives a description of all strongly orthodox congruences with characteristic trace τ on S. Denote

$$N = \{K : \ker\tau_{\min} \subseteq K \subseteq \ker\tau_{\max} \text{ where } K \text{ is a complete and } \tau\text{-normal subset of } S\}.$$

Notice that $\ker\tau_{\min}$ and $\ker\tau_{\max}$ are both the kernels of strongly orthodox congruences. It follows from Proposition 3.7 that $\ker\tau_{\min}$ and $\ker\tau_{\max}$ belong to N.

Theorem 3.8. *Let ρ be a strongly orthodox congruence on S with $\tau = \operatorname{ctr}\rho$. Define a binary relation ρ_K on S as follows. For $a, b \in S$, let*

$$a \,\rho_K\, b \Leftrightarrow \begin{array}{l} (\forall a' \in W(a))\,(\exists b' \in W(b))\,(aa' \;\tau\; bb',\; a'a \;\tau\; b'b,\; a'b \in K)\ \& \\ (\forall b' \in W(b))\,(\exists a' \in W(a))\,(aa' \;\tau\; bb',\; a'a \;\tau\; b'b,\; b'a \in K). \end{array}$$

Then the map $K \to \rho_K$ is a one-to-one order-preserving map of N onto the set of all strongly orthodox congruences on S with characteristic trace τ.

Proof. First we shall show that ρ_K is an equivalence relation on S. It is clear that ρ_K is symmetric and reflexive. To prove that ρ_K is transitive, let $(a, b) \in \rho_K$, $(b, c) \in \rho_K$. For any $a' \in W(a)$, then there exists $b' \in W(b)$ such that $aa' \;\tau\; bb'$, $a'a \;\tau\; b'b$ and $a'b \in K$, and for $b' \in W(b)$ there exists $c' \in W(c)$ such that $bb' \;\tau\; cc'$, $b'b \;\tau\; c'c$ and $b'c \in K$. Since τ is transitive, we have that $aa' \;\tau\; cc'$ and $a'a \;\tau\; c'c$. Since $b'c \in K$ and $ab'ba' \;\tau\; aa' \;\tau\; bb'$, by (C_2), $b' \cdot ab'ba' \cdot c \in K$. Since $aa' \;\tau\; bb'$, $a'a \;\tau\; b'b$ and $a'b \in K$, by (C_1) we have $b'aa'ba'c \in K$. Thus $b'aa'b \cdot (a'c)(a'c)^+(a'c) \in K$ as K is a complete subset. Notice that $c(a'c)^+ \in W(a')$, hence we have $b'aa'b \cdot a'c(a'c)^+ \;\tau\; a'c(a'c)^+$. By (C_3), $a'c \in K$. Dually, we may show that for any $c' \in W(c)$, there exists $a' \in W(a)$ such that $aa' \;\tau\; cc'$, $a'a \;\tau\; c'c$ and $c'a \in K$. Hence $a \,\rho_K\, c$ and ρ_K is transitive. Consequently, ρ_K is an equivalence relation on S.

To show that ρ_K is a congruence, let $a, b, c \in S$ be such that $(a, b) \in \rho_K$. For any $(ca)' \in W(ca)$, we have that $a' = (ca)'c \in W(a)$, $c' = a(ca)' \in W(c)$ and $(ca)' = a'c'$, $aa' = c'c$. By the definition of ρ_K, there exists $b' \in W(b)$ such that $aa' \;\tau\; bb'$, $a'a \;\tau\; b'b$ and $a'b \in K$. Hence $(ca)'(cb) = a'b \in K$. On the other hand, by Lemma 1.5, there exists $g \in M(aa', bb') = M(c'c, bb')$ such that $c'c = aa' \;\tau\; g \;\tau\; bb'$. Let $(cb)' = b'gc'$. Then $(cb)' = b'gc' \in W(cb)$. It now follows that

$$(ca)(ca)' = caa'c' \;\tau\; cbb'gc' = (cb)(cb)'$$

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and

$$(cb)'(cb) = b'gc'cb = b'gb \;\tau\; b'b \;\tau\; a'a = a'aa'a = a'c'ca = (ca)'(ca).$$

A similar argument will show that for any $(cb)' \in W(cb)$, there exists $(ca)' \in W(ca)$ such that $(ca)(ca)' \;\tau\; (cb)(cb)'$, $(ca)'(ca) \;\tau\; (cb)'(cb)$ and $(cb)'ca \in K$. Hence $(ca, cb) \in \rho_K$, and so ρ_K is a left congruence on S. Similarly, we can show that ρ_K is a right congruence on S. Consequently, ρ_K is a congruence.

It is easy to show that ρ_K is a strongly orthodox congruence by following exactly the same argument of the corresponding part of Theorem 2.1.

Next we show that $\operatorname{ctr} \rho_K = \tau$. Let $x, y \in C_\infty$ be such that $x \;\rho_K\; y$. Since ρ_K is a strongly regular congruence, there exist $x' \in W(x) \cap C_\infty$ such that $x \;\rho\; xx'x$. So by the definition of ρ_K, there exist $y' \in W(y) \cap C_\infty$ such that $xx' \;\tau\; yy'$, $x'x \;\tau\; y'y$. It follows from the fact that τ is a band congruence on C_∞ that $xy'y \;\tau\; x \;\tau\; xx'x \;\tau\; yy'x$. And so $xy \;\tau\; x \;\tau\; yx$. Dually, we have that $yx \;\tau\; y \;\tau\; xy$. Hence $x \;\tau\; y$, and so $\operatorname{ctr} \rho_K \subseteq \tau$.

Conversely, let $x, y \in C_\infty$ be such that $x \;\tau\; y$. Take any $x' \in W(x)$; then $x'yx' \;\rho\; x'$. By Lemma 1.4, there exists $y' \in W(y)$ such that $x' \;\rho\; y'$. Hence $xx' \;\tau\; yy'$ and $x'x \;\tau\; y'y$. Now $y'y \in K$ and $xx' \;\tau\; yy'$, then by (C_2) we have $y'xx'y \in K$, and so $y'xx'y(x'y)^+(x'y) \in K$ as K is a complete subset. It follows that

$$y'xx'y(x'y)^+ \;\tau\; x'x(x'y)^+ \;\tau\; x'y(x'y)^+.$$

By (C_3), we have $x'y \in K$. Dually, for any $y' \in W(y)$, there exists $x' \in W(x)$ such that $xx' \;\tau\; yy'$, $x'x \;\tau\; y'y$ and $y'x \in K$. Hence $x \;\rho_K\; y$ and so $\tau \subseteq \rho_K$. Therefore $\tau = \operatorname{ctr} \rho_K$.

Next, to prove that the given map is onto, let μ be a strongly orthodox congruence with $\tau = \operatorname{ctr} \mu$ on S and let $(a, b) \in \rho_{\ker \mu}$. Then for any $a' \in W(a)$, there exists $b' \in W(b)$ such that $aa' \;\operatorname{ctr} \mu\; bb'$, $a'a \;\operatorname{ctr} \mu\; b'b$ and $a'b \in \ker \mu$. Dually for any $b' \in W(b)$, there exists $a' \in W(a)$ such that $aa' \;\operatorname{ctr} \mu\; bb'$, $a'a \;\operatorname{ctr} \mu\; b'b$ and $b'a \in \ker \mu$. Recall that μ is a strongly orthodox congruence. Then it is easy to show that $a \;\mu\; b$. Conversely, let $a \;\mu\; b$. Then for any $a' \in W(a)$, $a' \;\mu\; a'ba'$. By Lemma 1.4, there exists $b' \in W(b)$ such that $a' \;\mu\; b'$. Thus $aa' \;\mu\; bb'$ and $a'a \;\mu\; b'b$, that is, $aa' \;\tau\; bb'$ and $a'a \;\tau\; b'b$. Also $(a'b, a'a) \in \mu$, and so $a'b \in \ker \mu$. A similar argument will show that for any $b' \in W(b)$, there exists $a' \in W(a)$ such that $aa' \;\tau\; bb'$, $a'a \;\tau\; b'b$ and $b'a \in \ker \mu$. Hence $(a, b) \in \rho_{\ker \mu}$, as required.

The given map is clearly order-preserving. We shall now show that the given map is one-to-one. To this end, let $K, L \in N$ with $\rho_K = \rho_L$ and let $a \in K$. Since $K \subseteq \ker \tau_{\max}$, $a \in \ker \tau_{\max}$. Therefore $(a, e) \in \tau_{\max}$ for some $e \in E(S)$. Then $(a, a^2) \in \tau_{\max}$. By the definition of $\tau_{\max}$, for any $a' \in W(a)$, there exists $c \in W(a^2)$ such that $aa' \;\tau\; a^2c$ and $a'a \;\tau\; ca^2$. Since $a \in K$ and $a'a \in K$, $a'aa \in K$ as K is a complete subset. On the other hand, for any $c \in W(a^2)$, there exists $a' \in W(a)$ such that $aa' \;\tau\; a^2c$ and $a'a \;\tau\; ca^2$. Since $ca^2 \in K$ and $a \in K$, $ca \in K$ as K is complete subset. Thus $(a, a^2) \in \rho_K = \rho_L$. Then there exists $f \in E(S)$ such that $a \;\rho_L\; f$. For $a^+ \in W(a)$, by the definition of ρ_L, there exists $f' \in W(f)$ such that $aa^+ \;\tau\; ff'$, $a^+a \;\tau\; f'f$ and $a^+f \in L$. On the other hand, for $f \in W(f)$, there exists $a' \in W(a)$ such that $aa' \;\tau\; f \;\tau\; a'a$ and $fa \in L$. Since $fa \in L$, $faa^+a \in L$ as L is a complete subset. Now $faa^+ \;\tau\; f \cdot ff' = ff' \;\tau\; aa^+$. By (C_3), we have $a \in L$.

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Thus $K \subseteq L$. Similarly, we can prove that $L \subseteq K$. Therefore $K = L$. We conclude that the given map is one-to-one. □

4. Strongly orthodox congruences determined by kernel

In this section we investigate the κ-relation of strongly orthodox congruences on an E-inversive semigroup and give the least and the greatest element of $\kappa(\rho)$.

Definition 4.1 [5]. If θ and ρ are congruences on S such that $\theta \subseteq \rho$, then the relation ρ/θ on S/θ is defined by

$$(x\theta, y\theta) \in \rho/\theta \quad \text{if and only if } (x, y) \in \rho.$$

This relation ρ/θ is in fact a congruence on S/θ.

The set of all strongly regular (orthodox) congruences on S is denoted by $\mathrm{SRC}(S)$ ($\mathrm{SOC}(S)$).

Definition 4.2. Define

$$\kappa = \{(\rho, \theta) \in \mathrm{SOC}(S) \times \mathrm{SOC}(S) : \ker\rho = \ker\theta\}$$

and denote the κ-class containing $\rho \in \mathrm{SOC}(S)$ by $\kappa(\rho)$.

The following is a direct analogue of [7, Lemma 2.7] and the proof carries across with minimal change. We omit the details.

Proposition 4.3. *For any $\rho \in \mathrm{SOC}(S)$, the relation*

$$\rho^{\max} = \{(a, b) \in S \times S : (\forall x, y \in S^1)\ xay \in \ker\rho \Leftrightarrow xby \in \ker\rho\}$$

is the greatest element of $\kappa(\rho)$.

Let

$$\tau_S = \{(a, b) \in S \times S : (\forall x, y \in S^1)\ xay \in E(S) \Leftrightarrow xby \in E(S)\}.$$

Theorem 4.4. *Let $\rho, \theta \in \mathrm{SRC}(S)$. Then the following statements are equivalent:*

(1) $\rho\ \kappa\ \theta$;
(2) $\rho \subseteq \theta^{\max}$ *and* $\theta^{\max}/\rho = \tau_{S/\rho}$;
(3) $a\rho\ \tau_{S/\rho}\ b\rho \Leftrightarrow a\theta\ \tau_{S/\theta}\ b\theta$.

Proof. (1)⇒(2). Let $\rho\ \kappa\ \theta$; then $\ker\rho = \ker\theta$. Thus $\rho \subseteq \rho^{\max} = \theta^{\max}$. For any $a, b \in S$, from the definition of $\tau_{S/\rho}$,

$$\begin{aligned} a\rho\ \theta^{\max}/\rho\ b\rho &\Leftrightarrow a\ \theta^{\max}\ b \\ &\Leftrightarrow (\forall x, y \in S^1)\ (xay \in \ker\theta \Leftrightarrow xby \in \ker\theta) \\ &\Leftrightarrow (\forall x, y \in S^1)\ (xay \in \ker\rho \Leftrightarrow xby \in \ker\rho) \\ &\Leftrightarrow (\forall x, y \in S^1)\ ((xay)\rho \in E(S/\rho) \Leftrightarrow (xby)\rho \in E(S/\rho)) \\ &\Leftrightarrow (\forall x\rho, y\rho \in (S/\rho)^1)\ ((x\rho)(a\rho)(y\rho) \in E(S/\rho) \Leftrightarrow (x\rho)(b\rho)(y\rho) \in E(S/\rho)) \\ &\Leftrightarrow a\rho\ \tau_{S/\rho}\ b\rho. \end{aligned}$$

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(2)⇒(1). By $\rho \subseteq \theta^{\max}$, we have $\ker\rho \subseteq \ker\theta^{\max} = \ker\theta$. Conversely, let $a \in \ker\theta = \ker\theta^{\max}$; then there exists $e \in E(S)$ such that $(a, e) \in \theta^{\max}$. So we have

$$\begin{aligned} a\,\theta^{\max}\,e \Leftrightarrow a\rho\ \theta^{\max}/\rho\ e\rho \Leftrightarrow&\ a\rho\ \tau_{S/\rho}\ e\rho \\ &\Leftrightarrow (\forall x\rho, y\rho \in (S/\rho)^1)\ ((x\rho)(a\rho)(y\rho) \in E(S/\rho) \Leftrightarrow (x\rho)(e\rho)(y\rho) \in E(S/\rho)) \\ &\Leftrightarrow (\forall x, y \in S^1)\ ((xay)\rho \in E(S/\rho) \Leftrightarrow (xey)\rho \in E(S/\rho)) \\ &\Leftrightarrow (\forall x, y \in S^1)\ (xay \in \ker\rho \Leftrightarrow xey \in \ker\rho) \\ &\Leftrightarrow (a, e) \in \rho^{\max}. \end{aligned}$$

Thus $a \in \ker\rho^{\max} = \ker\rho$. That is, $\ker\theta \subseteq \ker\rho$. Hence $\rho\ \kappa\ \theta$.

(1)⇒(3). For any $a, b \in S$,

$$\begin{aligned} a\rho\ \tau_{S/\rho}\ b\rho &\Leftrightarrow (\forall x\rho, y\rho \in (S/\rho)^1)\ ((xay)\rho \in E(S/\rho) \Leftrightarrow (xby)\rho \in E(S/\rho)) \\ &\Leftrightarrow (\forall x, y \in S^1)\ (xay \in \ker\rho \Leftrightarrow xby \in \ker\rho) \\ &\Leftrightarrow (\forall x, y \in S^1)\ (xay \in \ker\theta \Leftrightarrow xby \in \ker\theta) \\ &\Leftrightarrow (\forall x, y \in S^1)\ ((xay)\theta \in E(S/\theta) \Leftrightarrow (xey)\theta \in E(S/\theta)) \\ &\Leftrightarrow a\theta\ \tau_{S/\theta}\ b\theta. \end{aligned}$$

(3)⇒(1). Let $a \in \ker\rho$; then there exists $e \in E(S)$ such that $(a, e) \in \rho$. Thus

$$\begin{aligned} (a, e) \in \rho \Rightarrow&\ (a\rho, e\rho) \in 1_{S/\rho} \subseteq \tau_{S/\rho} \\ &\Rightarrow (a\rho\tau_{S/\rho}e\rho \Leftrightarrow a\theta\tau_{S/\theta}e\theta) \\ &\Leftrightarrow (\forall x\theta, y\theta \in (S/\theta)^1)\ ((x\theta)(a\theta)(y\theta) \in E(S/\theta) \Leftrightarrow (x\theta)(e\theta)(y\theta) \in E(S/\theta)) \\ &\Leftrightarrow (\forall x, y \in S^1)\ ((xay)\theta \in E(S/\theta) \Leftrightarrow (xey)\theta \in E(S/\theta)) \\ &\Leftrightarrow (\forall x, y \in S^1)\ (xay \in \ker\theta \Leftrightarrow xey \in \ker\theta). \end{aligned}$$

Since $e = 1 \cdot e \cdot 1 \in \ker\theta$, $a = 1 \cdot a \cdot 1 \in \ker\theta$, that is, $\ker\rho \subseteq \ker\theta$. By symmetry, $\ker\theta \subseteq \ker\rho$. Thus $\rho\ \kappa\ \theta$. □

Definition 4.5. A subset K of S is called strongly orthodox normal, if K is the kernel of a strongly orthodox congruence on S.

Proposition 4.6. *Let K be a strongly orthodox normal subset of S. Define a relation R on S by*

$$R = \{(a, aa'a), (a, a^2) : a \in K, \text{ for some } a' \in W(a)\}.$$

Then R^*, the congruence generated by R, is the least strongly orthodox congruence on S with kernel equal to K, and we denote it by $\rho^{\min}$.

Proof. It suffices to prove $K = \ker R^*$. If K is a strongly orthodox normal subset of S, clearly $K \subseteq \ker R^*$. Conversely, let $a \in K$; then there exist $e \in E(S)$ and $\rho \in \mathrm{SOC}(S)$ such that $a\ \rho\ e$, and so $a\ \rho\ a^2$. Since ρ is a strongly orthodox congruence, there exists $a' \in W(a)$ such that $a\ \rho\ aa'a$. Then $R \subseteq \rho$, and thus $R^* \subseteq \rho$. Therefore $\ker R^* \subseteq \ker\rho = K$. Thus $K = \ker R^*$. □

Proposition 4.7. *Let $\rho, \theta \in \mathrm{SRC}(S)$. Then:*

(1) *if $\rho \subseteq \theta$ and $\mathrm{ctr}\,\rho = \mathrm{ctr}\,\theta$, then $(a, b) \in \rho$ if and only if $(a, b) \in \theta$ and $a'b \in \ker\rho$ for any $a' \in W(a)$;*

(2) *$(a, b) \in \rho$ if and only if $(a, b) \in (\mathrm{ctr}\,\rho)_{\max}$ and $a'b \in \ker\rho$ for any $a' \in W(a)$.*

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PROOF. (1) Assume that $\rho \subseteq \theta$ and $\operatorname{ctr}\rho = \operatorname{ctr}\theta$. Let $(a, b) \in \rho$; then $(a, b) \in \theta$. It is clear that $a'b \in \ker\rho$ for all $a' \in W(a)$. Conversely, let $(a, b) \in \theta$ and $a'b \in \ker\rho$ for any $a' \in W(a)$. Since ρ is a strongly regular congruence on S, for any $a, b \in S$, there exist $a'' \in W(a)$ and $b'' \in W(b)$ such that $a \;\rho\; aa''a$ and $b \;\rho\; bb''b$. Since $(a, b) \in \theta$, then $a'' \;\theta\; a''ba''$. By Lemma 1.4, there exists $b' \in W(b)$ such that $a'' \;\theta\; b'$. Thus $aa'' \;\theta\; bb'$ and $a''a \;\theta\; b'b$. Notice that $\operatorname{ctr}\rho = \operatorname{ctr}\theta$, then we have $aa'' \;\rho\; bb'$ and $a''a \;\rho\; b'b$. Similarly, there exists $a' \in W(a)$ such that $aa' \;\rho\; bb''$ and $a'a \;\rho\; b''b$. Now $a'b \in \ker\rho$. It follows that

$$b \;\rho\; bb''b \;\rho\; aa'b \;\rho\; aa'ba'b \;\rho\; ba'b.$$

On the other hand,

$$a \;\rho\; aa''a \;\rho\; bb'a \;\rho\; bb''bb'a \;\rho\; bb''a.$$

Therefore,

$$b \;\rho\; bb''b \;\rho\; ba'a \;\rho\; ba'bb''a \;\rho\; bb''a \;\rho\; a.$$

(2) This is easy to show, so we omit the details. □

PROPOSITION 4.8. *For any $\rho \in \mathrm{SOC}(S)$, we have $\rho = \rho_{\min} \vee \rho^{\min} = \rho_{\max} \cap \rho^{\max}$.*

PROOF. Clearly, $\rho_{\min} \vee \rho^{\min} \subseteq \rho \subseteq \rho_{\max} \cap \rho^{\max}$. Then

$$\operatorname{ctr}(\rho_{\min} \vee \rho^{\min}) \subseteq \operatorname{ctr}\rho \subseteq \operatorname{ctr}(\rho_{\max} \cap \rho^{\max}),$$
$$\ker(\rho_{\min} \vee \rho^{\min}) \subseteq \ker\rho \subseteq \ker(\rho_{\max} \cap \rho^{\max}).$$

Furthermore,

$$\operatorname{ctr}(\rho_{\max} \cap \rho^{\max}) \subseteq \operatorname{ctr}\rho_{\max} = \operatorname{ctr}\rho = \operatorname{ctr}\rho_{\min} \subseteq \operatorname{ctr}(\rho_{\min} \vee \rho^{\min}),$$
$$\ker(\rho_{\max} \cap \rho^{\max}) \subseteq \ker\rho_{\max} = \ker\rho = \ker\rho_{\min} \subseteq \ker(\rho_{\min} \vee \rho^{\min}).$$

Therefore

$$\operatorname{ctr}(\rho_{\min} \vee \rho^{\min}) = \operatorname{ctr}\rho = \operatorname{ctr}(\rho_{\max} \cap \rho^{\max}),$$
$$\ker(\rho_{\min} \vee \rho^{\min}) = \ker\rho = \ker(\rho_{\max} \cap \rho^{\max}).$$

Thus $\rho = \rho_{\min} \vee \rho^{\min} = \rho_{\max} \cap \rho^{\max}$. □

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