17

Quasi-free states

Suppose that we have a state ψ on the polynomial algebra generated by the fields $\phi(y)$ satisfying the CCR or CAR relations. For simplicity, assume that ψ is *even*, that is, vanishes on odd polynomials. Clearly, this state determines a bilinear form on $\mathcal{Y}$ given by the "2-point function"

$$\mathcal{Y} \times \mathcal{Y} \ni (y_1, y_2) \mapsto \psi\big(\phi(y_1)\phi(y_2)\big). \tag{17.1}$$

We say that a state ψ is *quasi-free* if all expectation values

$$\psi\big(\phi(y_1)\cdots\phi(y_m)\big), \quad y_1, \dots, y_m \in \mathcal{Y}, \tag{17.2}$$

can be expressed in terms of (17.1) by the sum over all pairings.

This chapter is devoted to a study of (even) quasi-free states, both bosonic and fermionic. This is an important class of states, often used in physical applications. Fock vacuum states belong to this class. It also includes Gibbs states of quadratic Hamiltonians.

Representations obtained by the GNS construction from quasi-free states will be called *quasi-free representations*. They are usually reducible. Many interesting concepts from the theory of von Neumann algebras can be nicely illustrated in terms of quasi-free representations.

Quasi-free states can be easily realized on Fock spaces, using the so-called *Araki–Woods,* resp. *Araki–Wyss representations* in the bosonic, resp. fermionic case. Under some additional assumptions, in particular in the case of a finite number of degrees of freedom, these representations can be obtained as follows. First we consider a Fock space equipped with a quadratic Hamiltonian. Then we perform the GNS construction with respect to the corresponding Gibbs state. Finally, we apply an appropriate Bogoliubov rotation.

The last section of this chapter is devoted to a lattice of von Neumann algebras generated by fields based on real subspaces of the one-particle space. The most interesting result of this section gives a description of the commutant of such an algebra. The proof of this result uses Araki–Woods, resp. Araki–Wyss representations together with the modular theory of von Neumann algebras.

We will extensively use the terminology of the theory of operator algebras, in particular the modular theory of W^*-algebras; see Chap. 6.

17.1 Bosonic quasi-free states

In this section we discuss bosonic quasi-free states. They can be introduced in two different ways: by demanding that n-point functions can be expressed by the 2-point function, or by demanding that their value on Weyl operators is given by a Gaussian function. We choose the latter approach as the basic definition, since it does not involve unbounded operators.

In the literature, in the bosonic case, the name "quasi-free states" is often used to designate a wider class of states, which do not need to be even. For such states the "1-point function"

$$\mathcal{Y} \ni y \mapsto \psi\big(\phi(y)\big), \quad y \in \mathcal{Y}, \tag{17.3}$$

may be non-zero and fixes a linear functional on $\mathcal{Y}$. Quasi-free states are then determined by both (17.1) and (17.3). Gaussian coherent states considered in Subsects. 9.1.4 and 11.5.1 are examples of non-even quasi-free states. It is easy to see that an appropriate translation of the fields (see Subsect. 8.1.9) reduces a non-even quasi-free state to an even quasi-free state. Therefore, we will not consider non-even quasi-free states.

17.1.1 Definitions of bosonic quasi-free states

Let $(\mathcal{Y}, \omega)$ be a pre-symplectic space, that is, a real vector space $\mathcal{Y}$ equipped with an anti-symmetric form ω. Recall that $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ denotes the Weyl CCR algebra, that is, the C^*-algebra generated by operators $W(y)$ satisfying the (Weyl) CCR commutation relations; see Subsect. 8.3.5.

Definition 17.1 (1) *A state ψ on* $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ *is a* quasi-free state *if there exists $\eta \in L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{\#})$ (a symmetric form on $\mathcal{Y}$) such that*

$$\psi\big(W(y)\big) = \mathrm{e}^{-\frac{1}{2} y\cdot\eta y}, \quad y \in \mathcal{Y}. \tag{17.4}$$

(2) *If $\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H})$ is a CCR representation, a normalized vector $\Psi \in \mathcal{H}$ is called a* quasi-free vector *if*

$$\psi\big(W(y)\big) := \big(\Psi|W^{\pi}(y)\Psi\big), \quad y \in \mathcal{Y},$$

defines a quasi-free state on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$.

(3) *A representation $\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H})$ is* quasi-free *if there exists a cyclic quasi-free vector in $\mathcal{H}$.*

(4) *The form η is called the* covariance of the quasi-free state ψ, *and of the quasi-free vector Ψ.*

For a quasi-free state ψ on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$, let $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ be the corresponding GNS representation. Then, clearly, $\Omega_\psi \in \mathcal{H}_\psi$ is a quasi-free vector for the CCR representation $\mathcal{Y} \ni y \mapsto \pi_\psi\big(W(y)\big) \in U(\mathcal{H}_\psi)$.

The covariance defines the representation uniquely:

Proposition 17.2 *Let $\mathcal{Y} \ni y \mapsto W^i(y) \in U(\mathcal{H}_i)$, $i = 1, 2$, be quasi-free CCR representations with cyclic quasi-free vectors $\Psi_i \in \mathcal{H}_i$, both of covariance η. Then there exists a unique $U \in U(\mathcal{H}_1, \mathcal{H}_2)$ intertwining W^1 with W^2 and satisfying $U\Psi_1 = \Psi_2$.*

Let us note the following important special subclasses of quasi-free representations:

(1) If the pair $(2\eta, \omega)$ is Kähler, the corresponding quasi-free representation is Fock; see Thm. 17.13.
(2) Let $\omega = 0$. Then η can be an arbitrary positive definite form (see Prop. 17.5 below). Without loss of generality we can assume that $\mathcal{Y}$ is complete w.r.t. the scalar product given by η. Let $\mathcal{V} := \mathcal{Y}^{\#}$, so that $\mathcal{V}$ is a real Hilbert space with the scalar product η^{-1} and the generic variable v. Then the Hilbert space $\mathcal{H}$ can be identified with the Gaussian $\mathbf{L}^2$ space $\mathbf{L}^2(\mathcal{V}, \mathrm{e}^{-\frac{1}{2}v\cdot\eta^{-1}v}\mathrm{d}v)$, $W(y)$ are the operators of multiplication by $\mathrm{e}^{\mathrm{i}y\cdot v}$, and the function 1 is the corresponding quasi-free vector.

The following proposition follows from Prop. 8.11:

Proposition 17.3 *Every quasi-free representation is regular.*

We recall that the space $\mathcal{H}^{\infty,\pi}$ associated with a CCR representation W^π is defined in Subsect. 8.2.2. (It is the intersection of domains of products of field operators.)

Proposition 17.4 *A quasi-free vector Ψ for a CCR representation W^π belongs to the subspace $\mathcal{H}^{\infty,\pi}$. Moreover,*

$$\big(\Psi|\phi^\pi(y_1)\phi^\pi(y_2)\Psi\big) = y_1\cdot\eta y_2 + \frac{\mathrm{i}}{2}y_1\cdot\omega y_2, \quad y_1, y_2 \in \mathcal{Y}. \tag{17.5}$$

Proof We remove the superscript π to simplify notation. For any $y \in \mathcal{Y}$,

$$\big(\Psi|\mathrm{e}^{\mathrm{i}t\phi(y)}\Psi\big) = \mathrm{e}^{-\frac{t^2}{2}y\cdot\eta y}. \tag{17.6}$$

Hence, Ψ is an analytic vector for $\phi(y)$. It follows that, for any n, $\Psi \in \mathrm{Dom}\,\phi(y)^n$, hence $\Psi \in \mathcal{H}^\infty$.

To prove the second statement, we differentiate (17.6) w.r.t. t to get

$$\big(\Psi|\phi(y)^2\Psi\big) = y\cdot\eta y,$$

which, using linearity and the CCR, implies (17.5). □

Proposition 17.5 *Let $\eta \in L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{\#})$. Then the following are equivalent:*

(1) *$\mathcal{Y} \ni y \mapsto \mathrm{e}^{-\frac{1}{2}y\cdot\eta y}$ is a characteristic function in the sense of Def. 8.10, and hence there exists a quasi-free state satisfying (17.4).*
(2) *$\eta_{\mathbb{C}} + \frac{\mathrm{i}}{2}\omega_{\mathbb{C}} \geq 0$ on $\mathbb{C}\mathcal{Y}$, where $\eta_{\mathbb{C}}, \omega_{\mathbb{C}} \in L\big(\mathbb{C}\mathcal{Y}, (\mathbb{C}\mathcal{Y})^*\big)$ are the canonical sesquilinear extensions of η, ω.*

(3) $|y_1 \cdot \omega y_2| \leq 2(y_1 \cdot \eta y_1)^{\frac{1}{2}} (y_2 \cdot \eta y_2)^{\frac{1}{2}}, \ \ y_1, y_2 \in \mathcal{Y}$.

For the proof we will need the following fact:

Proposition 17.6 *Let* $A = [a_{jk}], \ \ B = [b_{jk}] \in B(\mathbb{C}^n)$*, with* $A, \ \ B \geq 0$*. Then* $[a_{jk} b_{jk}] =: C \geq 0$.

Proof Writing A and B as sums of positive rank one matrices, it suffices to prove the lemma if A and B are positive of rank one. In this case C is also positive of rank one. □

Corollary 17.7 *Let* $B = [b_{jk}] \in B(\mathbb{C}^n)$ *with* $B \geq 0$*. Then* $[\mathrm{e}^{b_{jk}}] \geq 0$.

Proof of Prop. 17.5. We work in the GNS representation and denote by Ψ the corresponding quasi-free vector.

(1) ⇒ (2). Using linearity, we deduce from (17.5) that

$$\big(\Psi | \phi(w)^* \phi(w) \Psi\big) = \overline{w} \cdot \eta_{\mathbb{C}} w + \frac{\mathrm{i}}{2} \overline{w} \cdot \omega_{\mathbb{C}} w, \quad w \in \mathbb{C}\mathcal{Y}. \tag{17.7}$$

It follows that the Hermitian form $\eta_{\mathbb{C}} + \frac{\mathrm{i}}{2} \omega_{\mathbb{C}}$ is positive semi-definite on $\mathbb{C}\mathcal{Y}$, which proves (2).

Conversely, let $y_1, \dots, y_n \in \mathcal{Y}$. Set

$$b_{jk} = y_j \cdot \eta y_k + \frac{\mathrm{i}}{2} y_j \cdot \omega y_k, \quad j, k = 1, \dots, n.$$

Then, for $\lambda_1, \dots, \lambda_n \in \mathbb{C}$,

$$\sum_{1 \leq j,k \leq n} \overline{\lambda_j} b_{jk} \lambda_k = \overline{w} \cdot \eta_{\mathbb{C}} w + \tfrac{\mathrm{i}}{2} \overline{w} \cdot \omega_{\mathbb{C}} w, \quad w = \sum_{j=1}^{n} \lambda_j y_j \in \mathbb{C}\mathcal{Y}.$$

By (2), the matrix $[b_{jk}]$ is positive. By Corollary 17.7, the matrix $[\mathrm{e}^{b_{jk}}]$ is positive, and hence the matrix $[\mathrm{e}^{-\frac{1}{2} y_j \cdot \eta y_j} b_{jk} \mathrm{e}^{-\frac{1}{2} y_k \cdot \eta y_k}]$ is positive. Thus

$$\begin{aligned} &\sum_{j,k=1}^{n} \mathrm{e}^{-\frac{1}{2}(y_k - y_j) \cdot \eta (y_k - y_j)} \mathrm{e}^{\frac{\mathrm{i}}{2} y_j \cdot \omega y_k} \overline{\lambda}_j \lambda_k \\ &= \sum_{j,k=1}^{n} \mathrm{e}^{-\frac{1}{2} y_j \cdot \eta y_j} \mathrm{e}^{b_{jk}} \mathrm{e}^{-\frac{1}{2} y_j \cdot \eta y_j} \overline{\lambda}_j \lambda_k \geq 0. \end{aligned}$$

Hence, by Def. 8.10, $\mathcal{Y} \ni y \mapsto \mathrm{e}^{-\frac{1}{2} y \cdot \eta y}$ is a characteristic function.

(2) ⇔ (3). We note that taking complex conjugates (2) implies that

$$\pm \frac{\mathrm{i}}{2} \omega_{\mathbb{C}} \leq \eta_{\mathbb{C}}, \text{ on } \mathbb{C}\mathcal{Y},$$

or equivalently

$$|\overline{w}_1 \cdot \omega_{\mathbb{C}} w_2| \leq 2(\overline{w}_1 \cdot \eta_{\mathbb{C}} w_1)^{\frac{1}{2}} (\overline{w}_2 \cdot \eta_{\mathbb{C}} w_2)^{\frac{1}{2}}, \quad w_1, w_2 \in \mathbb{C}\mathcal{Y}.$$

For $w_i = y_i \in \mathcal{Y}$, this implies (3).

Conversely, if (3) holds, then

$$2y_1{\cdot}\omega y_2 \leq y_1{\cdot}\eta y_1 + y_2{\cdot}\eta y_2,$$

which, setting $w = y_1 + \mathrm{i}y_2$, implies that $\overline{w}{\cdot}\eta_{\mathbb{C}}w + \frac{\mathrm{i}}{2}\overline{w}{\cdot}\omega_{\mathbb{C}}w \geq 0$. □

Let ψ be a quasi-free state on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$, η its covariance and $\mathcal{Y}^{\mathrm{cpl}}$ be the completion of $\mathcal{Y}$ w.r.t. η. Clearly, we can uniquely extend the pre-symplectic form ω to $\mathcal{Y}^{\mathrm{cpl}}$ so that it still satisfies the condition of Prop. 17.5 (3). We can also extend the state ψ uniquely to a quasi-free state on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}^{\mathrm{cpl}})$. Similarly, if W^{π} is a quasi-free CCR representation over $\mathcal{Y}$ satisfying (17.4), we can extend it uniquely to a quasi-free CCR representation over $\mathcal{Y}^{\mathrm{cpl}}$. Therefore, it will not restrict the generality to consider only quasi-free states and representations over $\mathcal{Y}$ complete w.r.t. η. Note, however, that ω may be degenerate on $\mathcal{Y}^{\mathrm{cpl}}$, even if it is non-degenerate on $\mathcal{Y}$.

Proposition 17.8 *Let $\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H})$ be a CCR representation. Let $\Psi \in \mathcal{H}$ be a unit vector. Then the following are equivalent:*

(1) *Ψ is a cyclic quasi-free vector.*
(2) *W^{π} is regular, $\Psi \in \mathcal{H}^{\infty,\pi}$ and, for $y_1, y_2, \ldots \in \mathcal{Y}$,*

$$\big(\Psi|\phi^{\pi}(y_1)\cdots\phi^{\pi}(y_{2m-1})\Psi\big) = 0,$$
$$\big(\Psi|\phi^{\pi}(y_1)\cdots\phi^{\pi}(y_{2m})\Psi\big) = \sum_{\sigma\in\mathrm{Pair}_{2m}} \prod_{j=1}^{m} \big(\Psi|\phi^{\pi}(y_{\sigma(2j-1)})\phi^{\pi}(y_{\sigma(2j)})\Psi\big).$$

Proof (2) ⇒ (1). Let $y \in \mathcal{Y}$. Since the number of elements of Pair_{2m} equals $\frac{1}{2^m}\frac{2m!}{m!}$, we have

$$\big(\Psi|\phi(y)^{2m+1}\Psi\big) = 0, \quad \big(\Psi|\phi(y)^{2m}\Psi\big) = \frac{1}{2^m}\frac{2m!}{m!}(y{\cdot}\eta y)^m,$$

for

$$y{\cdot}\eta y = \big(\Psi|\phi^2(y)\Psi\big). \tag{17.8}$$

Using the CCR, we see that the symmetric form η satisfies condition (2) of Prop. 17.5. Moreover, Ψ is an entire vector for $\phi(y)$, and

$$\big(\Psi|\mathrm{e}^{\mathrm{i}\phi(y)}\Psi\big) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^m}\frac{1}{m!}(y{\cdot}\eta y)^m = \mathrm{e}^{-\frac{1}{2}y\cdot\eta y}.$$

Hence, Ψ is a quasi-free vector.

(1) ⇒ (2). Let Ψ be a quasi-free vector. For $y_1, \ldots, y_n \in \mathcal{Y}$, $t_1, \ldots, t_n \in \mathbb{R}$, we have, using the CCR,

$$\prod_{j=1}^{n}\mathrm{e}^{\mathrm{i}t_j\phi(y_j)} = \exp\Big(-\frac{\mathrm{i}}{2}\sum_{1\leq j<k\leq n} t_j t_k y_j{\cdot}\omega y_k\Big)\exp\Big(\mathrm{i}\sum_{j=1}^{n} t_j\phi(y_j)\Big).$$

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Hence,

$$\Big(\Psi\Big|\prod_{j=1}^{n}\mathrm{e}^{\mathrm{i}t_j\phi(y_j)}\Psi\Big) = \exp\Big(-\frac{\mathrm{i}}{2}\sum_{1\leq j<k\leq n} t_jt_ky_j\cdot\omega y_k\Big)\exp\Big(-\frac{1}{2}\sum_{1\leq j,k\leq n} t_jt_ky_j\cdot\eta y_k\Big)$$

$$= \exp\Big(-\sum_{1\leq j<k\leq n} t_jt_k\big(y_j\cdot\eta y_k+\frac{\mathrm{i}}{2}y_j\cdot\omega y_k\big)\Big)\exp\Big(-\frac{1}{2}\sum_{j=1}^{n} t_j^2y_j\cdot\eta y_j\Big). \qquad (17.9)$$

From (17.7), we have

$$\big(\Psi|\phi(y_j)\phi(y_k)\Psi\big) = y_j\cdot\eta y_k + \frac{\mathrm{i}}{2}y_j\cdot\omega y_k =: r_{jk}.$$

Expanding the r.h.s. of (17.9), it follows that $\mathrm{i}^n\Big(\Psi|\prod_{j=1}^{n}\phi(y_j)\Psi\Big)$ is the coefficient of $t_1\cdots t_n$ in the product of the two formal power series

$$\sum_{p\in\mathbb{N}}\frac{1}{p!}\frac{(-1)^p}{2^p}\Big(\sum_{j<k}t_jt_kr_{jk}\Big)^p \times \sum_{p\in\mathbb{N}}\frac{1}{p!}\frac{(-1)^p}{2^p}\Big(\sum_{j=1}^{n}t_j^2y_j\cdot\eta y_j\Big)^p,$$

or equivalently in the formal power series

$$\sum_{p\in\mathbb{N}}\frac{1}{p!}\frac{(-1)^p}{2^p}\Big(\sum_{j<k}t_jt_kr_{jk}\Big)^p.$$

If n is odd, this coefficient vanishes. If $n=2m$, the only contributing term is

$$\frac{1}{m!}\frac{(-1)^m}{2^m}\Big(\sum_{j<k}t_jt_kr_{jk}\Big)^m,$$

which yields the coefficient

$$(-1)^m\sum_{\sigma\in\mathrm{Pair}_{2m}}\prod_{j=1}^{m}r_{\sigma(2j-1)\sigma(2j)},$$

as claimed. □

One could alternatively use the polynomial CCR algebra to describe bosonic quasi-free states. If we want to do this, there is a minor conceptual problem: these algebras are not C^*-algebras, hence strictly speaking the standard definition of a state is no longer valid. Fortunately, it is easy to extend the notion of a state to an arbitrary $*$-algebra by introducing the definition given below.

Definition 17.9 *Let $\mathfrak{A}$ be a unital $*$-algebra. A linear map $\psi:\mathfrak{A}\to\mathbb{C}$ is called a* state *if for any $A\in\mathfrak{A}$ we have $\psi(A^*A)\geq 0$ and $\psi(\mathbb{1})=1$.*

Note that, given a state on an arbitrary $*$-algebra, the GNS construction can be repeated verbatim from the C^*-algebraic theory.

The following definition is parallel to Def. 17.1 (1):

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Definition 17.10 *A state* ψ *on* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ *is* quasi-free *if*

$$\psi\big(\phi(y_1)\cdots\phi(y_{2m-1})\big) = 0,$$

$$\psi\big(\phi(y_1)\cdots\phi(y_{2m})\big) = \sum_{\sigma\in\mathrm{Pair}_{2m}}\prod_{j=1}^{m}\psi\big(\phi(y_{\sigma(2j-1)})\phi(y_{\sigma(2j)})\big).$$

Clearly, there is an obvious one-to-one correspondence between quasi-free states on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ and quasi-free states on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$.

17.1.2 Gauge-invariant bosonic quasi-free states

Let $(\mathcal{Y},\omega)$ be a symplectic space equipped with a pseudo-Kähler anti-involution j. The algebra $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ is then equipped with the one-parameter group of charge automorphisms, denoted $U(1)\ni\theta\mapsto\widehat{u_\theta}$, defined by

$$\widehat{u_\theta}\big(W(y)\big) = W(\mathrm{e}^{\mathrm{j}\theta}y).$$

Definition 17.11 *A state* ψ *on* $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ *is called* gauge-invariant *if it is invariant w.r.t.* $\widehat{u_\theta}$*, that is,*

$$\psi\big(W(y)\big) = \psi\big(W(\mathrm{e}^{\mathrm{j}\theta}y)\big),\qquad y\in\mathcal{Y},\quad \theta\in U(1). \tag{17.10}$$

In what follows we consider a gauge-invariant quasi-free state ψ with covariance η. Clearly, its gauge-invariance is equivalent to (η,j) being Kähler. (See Prop. 1.95 for a similar statement).

Let us stress that the fact that the two pairs (ω,j) and (η,j) are pseudo-Kähler does not imply that the triple (ω,η,j) is pseudo-Kähler.

Let us introduce the holomorphic space $\mathcal{Z}$ associated with the anti-involution j. Recall that $\mathbb{C}\mathcal{Y} = \mathcal{Z}\oplus\overline{\mathcal{Z}}$. The sesquilinear forms $\omega_{\mathbb{C}}$ and $\eta_{\mathbb{C}}$ can be reduced w.r.t. the direct sum $\mathcal{Z}\oplus\overline{\mathcal{Z}}$. Thus we can write

$$\omega_{\mathbb{C}} = \begin{bmatrix}\omega_{\mathcal{Z}} & 0\\ 0 & \overline{\omega}_{\mathcal{Z}}\end{bmatrix},\quad \eta_{\mathbb{C}} = \begin{bmatrix}\eta_{\mathcal{Z}} & 0\\ 0 & \overline{\eta}_{\mathcal{Z}}\end{bmatrix}, \tag{17.11}$$

where $\eta_{\mathcal{Z}}$ is Hermitian and $\omega_{\mathcal{Z}}$ anti-Hermitian. Note that the condition $\eta_{\mathbb{C}} + \frac{\mathrm{i}}{2}\omega_{\mathbb{C}}\geq 0$, which by Prop. 17.5 is necessary and sufficient for η to be the covariance of a quasi-free state, is equivalent to

$$\eta_{\mathcal{Z}} \pm \frac{\mathrm{i}}{2}\omega_{\mathcal{Z}} \geq 0. \tag{17.12}$$

If the pair (ω,j) is Kähler or, equivalently, $\mathrm{i}\omega_{\mathcal{Z}}\geq 0$, then (17.12) is equivalent to $\eta_{\mathcal{Z}}\geq\frac{\mathrm{i}}{2}\omega_{\mathcal{Z}}$.

Until the end of the subsection we assume that $(\mathcal{Y},\omega)$ is a pre-symplectic space and ψ is a quasi-free state on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ with covariance η. As explained in Subsect. 17.1.1, without loss of generality we can suppose that $\mathcal{Y}$ is complete for the metric given by η. We will see that under very general conditions there exists a Kähler anti-involution on $\mathcal{Y}$ that makes ψ gauge-invariant.

Theorem 17.12 (1) *Assume that* $\dim \operatorname{Ker} \omega$ *is even or infinite. Then there exists an anti-involution* j *such that* ψ *is gauge-invariant for the charge symmetry given by* j.

(2) *If* ω *is symplectic, then the anti-involution* j *described in (1) is unique if we demand in addition that it is Kähler on the symplectic space* $(\mathcal{Y}, \omega)$.

Proof By Prop. 17.5, we see that ω is a bilinear form on the real Hilbert space $(\mathcal{Y}, \eta)$ with norm less than 2. Hence, there exists $b \in B_{\mathrm{a}}(\mathcal{Y})$ (a bounded anti-symmetric operator on $\mathcal{Y}$) with $\|b\| \leq 1$ such that

$$y_1 \cdot \omega y_2 = 2 y_1 \cdot \eta b y_2. \tag{17.13}$$

Set $\mathcal{Y}_{\mathrm{sg}} := \operatorname{Ker} b$ and $\mathcal{Y}_{\mathrm{reg}} := \mathcal{Y}_{\mathrm{sg}}^{\perp}$. Since $b = -b^{\#}$, b preserves $\mathcal{Y}_{\mathrm{reg}}$ and we can set $b_{\mathrm{reg}} := b\big|_{\mathcal{Y}_{\mathrm{reg}}}$. From (17.13) we see that $\mathcal{Y}_{\mathrm{reg}}$ and $\mathcal{Y}_{\mathrm{sg}}$ are orthogonal for ω, and that $(\mathcal{Y}_{\mathrm{reg}}, \omega)$ is symplectic. Consider the polar decomposition $b_{\mathrm{reg}} =: -\mathrm{j}_{\mathrm{reg}} |b_{\mathrm{reg}}|$ of b_{reg}. Then both $(\eta\big|_{\mathcal{Y}_{\mathrm{reg}}}, \mathrm{j}_{\mathrm{reg}})$ and $(\omega\big|_{\mathcal{Y}_{\mathrm{reg}}}, \mathrm{j}_{\mathrm{reg}})$ are Kähler. Since $\dim \mathcal{Y}_{\mathrm{sg}}$ is even or infinite, there exists an orthogonal anti-involution j_{sg} on $\mathcal{Y}_{\mathrm{sg}}$. We now set $\mathrm{j} := \mathrm{j}_{\mathrm{reg}} \oplus \mathrm{j}_{\mathrm{sg}}$, which has the required properties. □

In the proof of the following theorem we will use the material developed in a later part of this section.

Theorem 17.13 *The GNS representation associated with* ψ *is*

(1) *factorial iff* ω *is non-degenerate on* $\mathcal{Y}$,
(2) *irreducible iff* $(2\eta, \omega)$ *is Kähler.*

Proof Set $\mathfrak{M} = \pi_\psi(\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}))''$. We easily see that $\pi_\psi\big(W(y)\big)$ is not proportional to the identity for $y \in \mathcal{Y} \backslash \{0\}$. If $y \in \operatorname{Ker} \omega$, then $\pi_\psi\big(W(y)\big) \in \mathfrak{M} \cap \mathfrak{M}'$. Therefore, if $\mathfrak{M}$ is a factor, then ω is non-degenerate. This proves (1) $\Rightarrow$.

Let us now discuss the GNS representation π_ψ when ω is non-degenerate. Let b and j be the operators constructed in the proof of Thm. 17.12. Recall that $b := (2\eta)^{-1}\omega \in B_{\mathrm{a}}(\mathcal{Y})$ and $b = -\mathrm{j}|b|$. Let $\mathcal{Z}$ be the corresponding holomorphic subspace. We have

$$\eta_{\mathcal{Z}} - \frac{\mathrm{i}}{2}\omega_{\mathcal{Z}} = \eta_{\mathcal{Z}} - \frac{1}{2}\omega_{\mathcal{Z}} \mathrm{j}_{\mathcal{Z}} = \eta_{\mathcal{Z}}(\mathbb{1} - |b_{\mathcal{Z}}|). \tag{17.14}$$

If we treat our CCR representation as a charged representation in the terminology of the next subsection, then (17.14) can be interpreted as the density ρ; see Def. 17.15.

We split $\mathcal{Y}$ as $\mathcal{Y}_1 \oplus \mathcal{Y}_2$, where

$$\mathcal{Y}_1 := \mathbb{1}_{\{1\}}(|b|)\mathcal{Y}, \quad \mathcal{Y}_2 := \mathbb{1}_{\mathbb{R}\backslash\{1\}}(|b|)\mathcal{Y},$$

and note that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are orthogonal for η and ω. For $i = 1, 2$ we set $\omega_i = \omega\big|_{\mathcal{Y}_i}$, $\eta_i = \eta\big|_{\mathcal{Y}_i}$, $\mathrm{j}_i = \mathrm{j}\big|_{\mathcal{Y}_i}$. We denote by ψ_i the quasi-free state on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_i)$ with covariance η_i, and by $\mathcal{Z}_i \subset \mathcal{Z}$ the holomorphic subspace associated with j_i. We set $\rho_i := \rho\big|_{\mathcal{Z}_i}$.

Note that ω_i are non-degenerate, and that the state ψ on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ can be identified to $\psi_1 \otimes \psi_2$ on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_1) \otimes \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_2)$. Therefore, the GNS representation associated with ψ is unitarily equivalent to the tensor product of the GNS representations associated with ψ_1 and ψ_2.

We have $\rho_1 = 0$. Hence, $(2\eta_1, \omega_1)$ is Kähler and the GNS representation associated with ψ_1 is the Fock representation associated with j_1.

Consider the Araki–Woods representation associated with ρ_2 (see Subsect. 17.1.5). By Thm. 17.24 (4), the vacuum Ω is a vector representative for ψ_2. By (17.14), $\mathrm{Ker}\,\rho_2 = \{0\}$, hence Ω is cyclic by Thm. 17.24 (6). Thus the Araki–Woods representation is the GNS representation for ψ_2.

We have $\mathfrak{M} = B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}_1)\big) \otimes \mathrm{CCR}_{\gamma_2,\mathrm{l}}$ (see Def. 17.23). Since by Thm. 17.24 (7), $1\!\!1 \otimes \mathrm{CCR}_{\gamma_2,\mathrm{r}} \subset \mathfrak{M}'$, we obtain that

$$\begin{aligned} B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}_1)\big) \otimes B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}_2 \oplus \overline{\mathcal{Z}}_2)\big) &\subset B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}_1)\big) \otimes (\mathrm{CCR}_{\gamma_2,\mathrm{l}} \cup \mathrm{CCR}_{\gamma_2,\mathrm{r}})'' \\ &\subset (\mathfrak{M} \cup \mathfrak{M}')'', \end{aligned}$$

hence $\mathfrak{M}$ is a factor. This proves (1) $\Leftarrow$.

Now note that the Kähler property implies that ω is non-degenerate. On the other hand, the irreducibility implies the factoriality, which by (1) implies that ω is non-degenerate. Therefore, to prove (2) we can assume the non-degeneracy of ω.

By the discussion above, the GNS representation associated with ψ is equal to the tensor product of the Fock representation associated with $(\mathcal{Y}_1, \omega_1, \mathrm{j}_1)$ and of the Araki–Woods representation associated with $(\mathcal{Z}_2, \rho_2)$, where $\rho_2 > 0$. Every Fock representation is irreducible, while an Araki–Woods representation for a non-zero particle density is not (see Thm. 17.24 (7)). Therefore, the GNS representation associated with ψ is irreducible iff $\mathcal{Y}_2 = \{0\}$ ie. $(2\eta, \omega)$ is Kähler. This proves (2). □

17.1.3 Quasi-free charged representations

The following subsection is essentially a translation of the previous subsection from the terminology of neutral CCR representation to that of charged CCR representations, which seems more convenient in the context of gauge invariance.

Let $(\mathcal{Y}, \omega)$ be a charged symplectic space. That means the symbols $\mathcal{Y}$ and ω slightly change their meanings compared with the previous subsection: $\mathcal{Y}$ is now a *complex* space and ω is a *charged* symplectic form. To go back to the framework of the previous subsection we need to take the space $\mathcal{Y}_{\mathbb{R}}$, the realification of $\mathcal{Y}$, and equip it with the symplectic form $y_1 \cdot \omega_{\mathrm{R}} y_2 := \mathrm{Re}\, y_1 \cdot \omega y_2$, the real part of the charged symplectic form.

Clearly, $\mathcal{Y}$ is equipped with a pseudo-Kähler anti-involution – the imaginary unit. Therefore, all the definitions of the previous subsections make sense. We will write $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$, resp. $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ to denote the algebra $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}_{\mathbb{R}})$, resp. $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}_{\mathbb{R}})$ equipped with the charge symmetry induced by $U(1) \ni \theta \mapsto \mathrm{e}^{\mathrm{i}\theta}$.

Note that we have a minor notational problem. Throughout our work, we consistently used the letter ψ to denote charged fields. In this chapter this letter is taken (and denotes a state). Therefore, we will use a different letter to denote charged fields – they will be denoted by the letter a, as annihilation operators. In particular, the algebra $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ is generated by the operators $a(y)$, $a^*(y)$, $y \in \mathcal{Y}$. Clearly, we can define the concepts of a gauge-invariant state and of a quasi-free state on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$.

We also have the corresponding notions on the algebra $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$, generated as usual by $W(y)$, $y \in \mathcal{Y}$. There is a one-to-one correspondence between gauge-invariant quasi-free states on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$ and $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ that can be derived from the formal relation

$$W(y) = \exp\Big((\mathrm{i}/\sqrt{2})\big(a^*(y) + a(y)\big)\Big).$$

However, when discussing charged CCR relations we prefer to use the polynomial algebra.

Proposition 17.14 (1) *A state ψ on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ is gauge-invariant if*

$$\psi\big(a^*(y_1)\cdots a^*(y_n)a(w_m)\cdots a(w_1)\big) = 0, \quad n \neq m, \quad y_1 \dots, y_n, w_m, \dots, w_1 \in \mathcal{Y}.$$

(2) *It is quasi-free if in addition, for any $y_1 \dots, y_n, w_n, \dots, w_1 \in \mathcal{Z}$,*

$$\psi\big(a^*(y_1)\cdots a^*(y_n)a(w_n)\cdots a(w_1)\big) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} \psi\big(a^*(y_j)a(w_{\sigma(j)})\big).$$

Definition 17.15 *If ψ is a gauge-invariant quasi-free state on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$, the positive semi-definite Hermitian form ρ on $\mathcal{Y}$ defined by*

$$(y_2|\rho y_1) := \psi\big(a^*(y_1)a(y_2)\big), \quad y_1, y_2 \in \mathcal{Y},$$

is called the density associated with ψ. *If $\mathrm{i}\omega$ is positive definite, we will also use the alternative name* one-particle density.

Recall that in the framework of neutral CCR relations one introduces the holomorphic space $\mathcal{Z}$. Charged CCR relations amount to identifying the space $\mathcal{Y}$ with $\mathcal{Z}$, as explained e.g. in Subsect. 8.2.5. Under this identification, the Hermitian form $\mathrm{i}\omega$ is transformed into $\mathrm{i}\omega_{\mathcal{Z}}$, and the density ρ into $\eta_{\mathcal{Z}} - \frac{\mathrm{i}}{2}\omega_{\mathcal{Z}}$ (see (17.11)). Therefore, (17.12) implies the following proposition.

Proposition 17.16 *A Hermitian form $\rho \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$ is the density of a gauge-invariant quasi-free state iff*

$$\rho \geq 0, \quad \rho + \mathrm{i}\omega \geq 0.$$

Assume that

$$\mathcal{Y} \ni y \mapsto a^{\pi *}(y) \in Cl(\mathcal{H}) \tag{17.15}$$

is a charged CCR representation. We have the obvious analogs of Def. 17.1 (2) and (3):

Definition 17.17 (1) $\Psi \in \mathcal{H}$ *is called a* gauge-invariant quasi-free vector *if* $\Psi \in \mathcal{H}^{\infty,\pi}$ *and*

$$\begin{aligned}&\psi\big(a^*(y_1)\cdots a^*(y_n)a(w_1)\cdots a(w_m)\big)\\ := &\big(\Psi|a^{\pi *}(y_1)\cdots a^{\pi *}(y_n)a^{\pi}(w_m)\cdots a^{\pi}(w_1)\Psi\big),\quad y_1,\dots,y_n,w_m,\dots,w_1\in\mathcal{Y},\end{aligned}$$

defines a gauge-invariant quasi-free state on $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$.

(2) *A charged CCR representation (17.15) is called* gauge-invariant quasi-free *if there exists a cyclic gauge-invariant quasi-free vector in* $\mathcal{H}$.

Recall that with every charged CCR representation (17.15) we can associate a unique regular neutral CCR representation

$$\mathcal{Y}_{\mathbb{R}} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H}) \tag{17.16}$$

such that

$$W^{\pi}(y) = \exp\Big((\mathrm{i}/\sqrt{2})\big(a^{\pi *}(y) + a^{\pi}(y)\big)\Big).$$

It is clear that a vector Ψ is gauge-invariant quasi-free w.r.t. W^{π} iff it is such w.r.t. $a^{\pi *}$. Likewise, the representation W^{π} is gauge-invariant quasi-free iff $a^{\pi *}$ is.

17.1.4 Gibbs states of bosonic quadratic Hamiltonians

Density matrix

Let $0 \leq \gamma \leq \mathbb{1}$ be a self-adjoint operator on a Hilbert space $\mathcal{Z}$ with $\mathrm{Ker}(\mathbb{1} - \gamma) = \{0\}$. We associate with γ the self-adjoint operator ρ, called the *one-particle density*, defined by

$$\rho := \gamma(\mathbb{1} - \gamma)^{-1}, \quad \gamma = \rho(\rho + \mathbb{1})^{-1}. \tag{17.17}$$

We assume in addition that γ is trace-class. This is equivalent to assuming that ρ is trace-class. Note the following identity:

$$\mathrm{Tr}\,\Gamma(\gamma) = \det(\mathbb{1} - \gamma)^{-1} = \det(\mathbb{1} + \rho).$$

Thus $\Gamma(\gamma)\det(\mathbb{1} - \gamma)$ is a density matrix (see Def. 2.41).

Definition 17.18 *The state* ψ_γ *on* $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ *is defined by*

$$\psi_\gamma(A) := \mathrm{Tr}\,A\Gamma(\gamma)\det(\mathbb{1} - \gamma), \quad A \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big).$$

We identify $\mathcal{Z}$ with $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ using the usual map $z \mapsto \frac{1}{\sqrt{2}}(z + \overline{z})$. We can faithfully represent the Weyl CCR algebra $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Z})$ in $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$. Note that we have a natural charge symmetry on $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ leaving invariant $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Z})$, implemented by $U(1) \ni \theta \mapsto \mathrm{e}^{\mathrm{i}\theta N}$.

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Proposition 17.19 *The state ψ_γ restricted to* $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Z})$ *is gauge-invariant quasi-free. We have*

$$\psi_\gamma\big(W(z)\big) = \exp\Big(-\frac{1}{4}(z|z) - \frac{1}{2}(z|\rho z)\Big) = \exp\Big(-\frac{1}{4}\Big(z|\frac{\mathbb{1}+\gamma}{\mathbb{1}-\gamma}z\Big)\Big), \quad z\in\mathcal{Z}.$$

The "2-point functions" are

$$\begin{aligned}\psi_\gamma\big(a^*(z_1)a(z_2)\big) &= (z_2|\rho z_1),\\ \psi_\gamma\big(a(z_1)a^*(z_2)\big) &= (z_1|z_2) + (z_1|\rho z_2),\\ \psi_\gamma\big(a(z_1)a(z_2)\big) = \psi_\gamma\big(a^*(z_1)a^*(z_2)\big) &= 0, \quad z_1, z_2\in\mathcal{Z}.\end{aligned}$$

Proof We can find an o.n. basis $(e_1, e_2, \dots)$ diagonalizing the trace-class operator γ. Using the identification

$$\Gamma_{\mathrm{s}}(\mathcal{Z}) \simeq \mathop{\otimes}_{i=1}^{\infty}\big(\Gamma_{\mathrm{s}}(\mathbb{C}e_i), \Omega\big), \tag{17.18}$$

we can confine ourselves to the case of one degree of freedom, which is a well-known computation involving summing up a geometric series. □

Suppose now that γ is non-degenerate. In this case, the state ψ_γ is faithful. If in addition we fix $\beta > 0$ and $\gamma = \mathrm{e}^{-\beta h}$ for some operator h bounded from below, then

$$\Gamma(\gamma)\det(\mathbb{1}-\gamma) = \mathrm{e}^{-\beta \mathrm{d}\Gamma(h)}/\mathrm{Tr}\,\mathrm{e}^{-\beta\mathrm{d}\Gamma(h)}.$$

Thus, in this case, ψ_γ is the Gibbs state at the inverse temperature β for the dynamics generated by the Hamiltonian $\mathrm{d}\Gamma(h)$.

Standard representations on Hilbert–Schmidt operators

Consider the Hilbert space $B^2\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$. It will be convenient to introduce an alternative notation for the Hermitian conjugation: $JB := B^*$.

Recall the representations of $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ and $\overline{B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)}$ on $B^2\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ introduced in Subsect. 6.4.5:

$$\pi_{\mathrm{l}}(A)B = AB, \quad \pi_{\mathrm{r}}(\overline{A})B := BA^*, \quad B\in B^2\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big), \quad A\in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big).$$

Clearly, $J\pi_{\mathrm{l}}(A)J^* = \pi_{\mathrm{r}}(A)$.

Thus we can introduce two commuting charged CCR representations,

$$\mathcal{Z}\ni z\mapsto \pi_{\mathrm{l}}\big(a^*(z)\big)\in Cl\left(B^2\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)\right), \tag{17.19}$$
$$\overline{\mathcal{Z}}\ni \overline{z}\mapsto \pi_{\mathrm{r}}\big(\overline{a^*(z)}\big)\in Cl\left(B^2\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)\right). \tag{17.20}$$

They are interchanged by the operator J:

$$J\pi_{\mathrm{l}}\big(a^*(z)\big)J^* = \pi_{\mathrm{r}}\big(\overline{a^*(z)}\big).$$

The vector

$$\Psi_\gamma := \det(\mathbb{1}-\gamma)^{\frac{1}{2}}\Gamma(\gamma^{\frac{1}{2}})$$

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is gauge-invariant quasi-free for the representations (17.19) and (17.20) and the one-particle density ρ. Both (17.19) and (17.20) are gauge-invariant quasi-free charged CCR representations.

Standard representations on the double Fock spaces

Note the following chain of identifications:

$$\begin{aligned} B^2\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big) &\simeq \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \overline{\Gamma_{\mathrm{s}}(\mathcal{Z})} \\ &\simeq \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}(\overline{\mathcal{Z}}) \simeq \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}). \end{aligned} \tag{17.21}$$

We denote by $T_{\mathrm{s}} : B^2\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big) \to \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ the unitary map given by (17.21). Introduce the anti-unitary map

$$\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni (z_1, \overline{z}_2) \mapsto \epsilon(z_1, \overline{z}_2) := (z_2, \overline{z_1}) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}. \tag{17.22}$$

Proposition 17.20 $T_{\mathrm{s}} J T_{\mathrm{s}}^* = \Gamma(\epsilon)$.

By applying T_{s} to (17.19) and (17.20), we obtain two new commuting charged CCR representations

$$\mathcal{Z} \ni z \mapsto T_{\mathrm{s}} \pi_{\mathrm{l}}\big(a^*(z)\big) T_{\mathrm{s}}^* = a^*(z, 0) \in Cl\big(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big), \tag{17.23}$$

$$\overline{\mathcal{Z}} \ni \overline{z} \mapsto T_{\mathrm{s}} \pi_{\mathrm{r}}\big(\overline{a^*(z)}\big) T_{\mathrm{s}}^* = a^*(0, \overline{z}) \in Cl\big(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big). \tag{17.24}$$

They are interchanged by the operator $\Gamma(\epsilon)$:

$$\Gamma(\epsilon) a^*(z, 0) \Gamma(\epsilon)^* = a^*(0, \overline{z}).$$

Again using a basis diagonalizing γ, as in (17.18), the double Fock space on the right of (17.21) can be written as an infinite tensor product,

$$\mathop{\otimes}_{i=1}^{\infty} \big(\Gamma_{\mathrm{s}}(\mathbb{C}e_i \oplus \mathbb{C}\overline{e}_i), \Omega\big). \tag{17.25}$$

We have $T_{\mathrm{s}} \Psi_\gamma = \Omega_\gamma$, where

$$\Omega_\gamma := \mathop{\otimes}_{i=1}^{\infty} (1 - \gamma_i)^{\frac{1}{2}} \mathrm{e}^{\gamma_i^{\frac{1}{2}} a^*(e_i) a^*(\overline{e}_i)} \Omega$$

is a gauge-invariant quasi-free vector for the representations (17.23) and (17.24), and the one-particle density ρ. Clearly, both (17.23) and (17.24) are gauge-invariant quasi-free CCR representations.

Note that if we set

$$c = \begin{bmatrix} 0 & \gamma^{\frac{1}{2}} \\ \overline{\gamma}^{\frac{1}{2}} & 0 \end{bmatrix} \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}} \oplus \mathcal{Z}, \mathcal{Z} \oplus \overline{\mathcal{Z}}), \tag{17.26}$$

then

$$\Omega_\gamma = \det(\mathbb{1} - cc^*)^{\frac{1}{4}} \mathrm{e}^{\frac{1}{2} a^*(c)} \Omega,$$

so this is an example of a bosonic Gaussian vector introduced in (11.33), where it was denoted Ω_c.

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Araki–Woods form of standard representation

Using the infinite tensor product decomposition (17.25), we define the following transformation on $\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$:

$$R_\gamma := \mathop{\otimes}_{i=1}^{\infty}(1-\gamma_i)^{\frac12}\mathrm{e}^{-\gamma_i^{\frac12}a^*(e_i)a^*(\overline{e}_i)}\Gamma\big((1-\gamma_i)^{\frac12}1\!\!1\big)\mathrm{e}^{\gamma_i^{\frac12}a(e_i)a(\overline{e}_i)}. \qquad (17.27)$$

Theorem 17.21 *R_γ is a unitary operator satisfying*

$$\begin{aligned}
R_\gamma\phi(z_1,\overline{z_2})R_\gamma^* &= \phi\big((\rho+1\!\!1)^{\frac12}z_1+\rho^{\frac12}z_2,\overline{\rho}^{\frac12}\overline{z_1}+(\overline{\rho}+1\!\!1)^{\frac12}\overline{z_2}\big),\\
R_\gamma a(z_1,\overline{z_2})R_\gamma^* &= a\big((\rho+1\!\!1)^{\frac12}z_1+\rho^{\frac12}z_2,0\big)\\
&\qquad\qquad + a^*\big(0,\overline{\rho}^{\frac12}\overline{z_1}+(\overline{\rho}+1\!\!1)^{\frac12}\overline{z_2}\big),\\
R_\gamma a^*(z_1,\overline{z_2})R_\gamma^* &= a^*\big((\rho+1\!\!1)^{\frac12}z_1+\rho^{\frac12}z_2,0\big)\\
&\qquad\qquad + a\big(0,\overline{\rho}^{\frac12}\overline{z_1}+(\overline{\rho}+1\!\!1)^{\frac12}\overline{z_2}\big), \quad (z_1,\overline{z}_2)\in\mathcal{Z}\oplus\overline{\mathcal{Z}},\\
R_\gamma\Gamma(\epsilon)R_\gamma^* &= \Gamma(\epsilon),\\
R_\gamma\Omega_\gamma &= \Omega,\\
R_\gamma\mathrm{d}\Gamma(h,-\overline{h})R_\gamma^* &= \mathrm{d}\Gamma(h,-\overline{h}).
\end{aligned}$$

Proof Let c be defined as in (17.26). Using

$$\Gamma(1\!\!1-cc^*) = \Gamma\big((1\!\!1-\gamma)\oplus(1\!\!1-\overline{\gamma})\big),$$

we see that

$$R_\gamma := \det(1\!\!1-cc^*)^{\frac14}\mathrm{e}^{-\frac12 a^*(c)}\Gamma(1\!\!1-cc^*)^{\frac12}\mathrm{e}^{\frac12 a(c)}.$$

Thus R_γ is an example of an operator whose properties we studied in detail in Sect. 11.3. Thus all the identities that we need to show follow from the fact that R_γ is a unitary operator implementing a positive symplectic map given in (11.50). □

By applying R_γ to (17.23) and (17.24), we obtain two new commuting charged CCR representations

$$\begin{aligned}
\mathcal{Z}\ni z\mapsto a_{\gamma,\mathrm{l}}^*(z) &:= R_\gamma a^*(z,0)R_\gamma^*\\
&= a^*\big((\rho+1\!\!1)^{\frac12}z,0\big)+a\big(0,\overline{\rho}^{\frac12}\overline{z}\big)\in Cl\big(\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big),\\
\overline{\mathcal{Z}}\ni\overline{z}\mapsto a_{\gamma,\mathrm{r}}^*(\overline{z}) &:= R_\gamma a^*(0,\overline{z})R_\gamma^*\\
&= a\big(\rho^{\frac12}z,0\big)+a^*\big(0,(\overline{\rho}+1\!\!1)^{\frac12}\overline{z}\big)\in Cl\big(\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big).
\end{aligned}$$

They are interchanged by the operator $\Gamma(\epsilon)$:

$$\Gamma(\epsilon)a_{\gamma,\mathrm{l}}^*(z)\Gamma(\epsilon)^* = a_{\gamma,\mathrm{r}}^*(\overline{z}).$$

We have $R_\gamma\Omega_\gamma=\Omega$, hence the Fock vacuum Ω is a gauge-invariant quasi-free vector for the representations $a_{\gamma,\mathrm{l}}^*$ and $a_{\gamma,\mathrm{r}}^*$, and the one-particle density ρ. Both $a_{\gamma,\mathrm{l}}^*$ and $a_{\gamma,\mathrm{r}}^*$ are gauge-invariant quasi-free CCR representations. They are special cases of *Araki–Woods CCR representations*, which we will consider in the next subsection.

17.1.5 Araki–Woods representations

In this subsection we will see that $a^*_{\gamma,\mathrm{l}}$ and $a^*_{\gamma,\mathrm{r}}$ can be defined more generally, as compared with the framework of the previous subsection.

Let $\mathcal{Z}$ be a Hilbert space. We introduce the operators γ and ρ as at the beginning of Subsect. 17.1.4, except that we do not assume that they are trace-class.

Definition 17.22 *For $z \in \mathrm{Dom}\,\rho^{\frac12}$ we define the following closed operators on $\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$, called the* Araki–Woods creation operators*:*

$$a^*_{\gamma,\mathrm{l}}(z) := a^*\left((\rho+\mathbb{1})^{\frac12}z, 0\right) + a\left(0, \overline{\rho}^{\frac12}\,\overline{z}\right),$$
$$a^*_{\gamma,\mathrm{r}}(\overline{z}) := a^*\left(\rho^{\frac12}z, 0\right) + a\left(0, (\overline{\rho}+\mathbb{1})^{\frac12}\,\overline{z}\right).$$

For completeness, let us write down the adjoints of $a^*_{\gamma,\mathrm{l/r}}(z)$, called the *Araki–Woods annihilation operators*:

$$a_{\gamma,\mathrm{l}}(z) := a\left((\rho+\mathbb{1})^{\frac12}z, 0\right) + a^*\left(0, \overline{\rho}^{\frac12}\,\overline{z}\right),$$
$$a_{\gamma,\mathrm{r}}(\overline{z}) := a\left(\rho^{\frac12}z, 0\right) + a^*\left(0, (\overline{\rho}+\mathbb{1})^{\frac12}\,\overline{z}\right).$$

We also have the *Araki–Woods Weyl operators*:

$$W_{\gamma,\mathrm{l}}(z) := \exp\left((\mathrm{i}/\sqrt{2})\left(a^*_{\gamma,\mathrm{l}}(z) + a_{\gamma,\mathrm{l}}(z)\right)\right) = W\left((\mathbb{1}+\rho)^{\frac12}z, \overline{\rho}^{\frac12}\,\overline{z}\right),$$
$$W_{\gamma,\mathrm{r}}(\overline{z}) := \exp\left((\mathrm{i}/\sqrt{2})\left(a^*_{\gamma,\mathrm{r}}(z) + a_{\gamma,\mathrm{r}}(z)\right)\right) = W\left(\rho^{\frac12}z, (\mathbb{1}+\overline{\rho})^{\frac12}\,\overline{z}\right).$$

Definition 17.23 *The von Neumann algebra generated by*

$$\left\{W_{\gamma,\mathrm{l}}(z) \ :\ z\in\mathrm{Dom}\,\rho^{\frac12}\right\} \quad \textit{resp.} \quad \left\{W_{\gamma,\mathrm{r}}(\overline{z}) \ :\ z\in\mathrm{Dom}\,\rho^{\frac12}\right\}$$

will be denoted by $\mathrm{CCR}_{\gamma,\mathrm{l}}$*, resp.* $\mathrm{CCR}_{\gamma,\mathrm{r}}$*, and called the* left*, resp.* right Araki–Woods CCR algebra.

Theorem 17.24 (1) *The map*

$$\mathcal{Z}\ni z\mapsto a^*_{\gamma,\mathrm{l}}(z)\in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big)$$

is a charged CCR representation. In particular,

$$[a_{\gamma,\mathrm{l}}(z_1), a^*_{\gamma,\mathrm{l}}(z_2)] = (z_1|z_2)\mathbb{1},$$
$$[a^*_{\gamma,\mathrm{l}}(z_1), a^*_{\gamma,\mathrm{l}}(z_2)] = [a_{\gamma,\mathrm{l}}(z_1), a_{\gamma,\mathrm{l}}(z_2)] = 0.$$

It will be called the left Araki–Woods (charged CCR) representation*.*

(2) *The map*

$$\overline{\mathcal{Z}}\ni \overline{z}\mapsto a^*_{\gamma,\mathrm{r}}(\overline{z})\in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big)$$

is a charged CCR representation. In particular,

$$[a_{\gamma,\mathrm{r}}(\overline{z}_1), a^*_{\gamma,\mathrm{r}}(\overline{z}_2)] = \overline{(z_1|z_2)}\mathbb{1},$$
$$[a^*_{\gamma,\mathrm{r}}(\overline{z}_1), a^*_{\gamma,\mathrm{r}}(\overline{z}_2)] = [a_{\gamma,\mathrm{r}}(\overline{z}_1), a_{\gamma,\mathrm{r}}(\overline{z}_2)] = 0.$$

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It will be called the right Araki–Woods (charged CCR) representation.

(3) *Set*

$$J_{\mathrm{s}} := \Gamma(\epsilon). \tag{17.28}$$

Then

$$J_{\mathrm{s}} a^*_{\gamma,\mathrm{l}}(z) J_{\mathrm{s}} = a^*_{\gamma,\mathrm{r}}(\overline{z}),$$
$$J_{\mathrm{s}} a_{\gamma,\mathrm{l}}(z) J_{\mathrm{s}} = a_{\gamma,\mathrm{r}}(\overline{z}).$$

(4) *The vacuum Ω is a bosonic quasi-free vector for $a^*_{\gamma,\mathrm{l}}$ with the 2-point functions equal to*

$$\begin{aligned}
\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= \big(z_1|(\rho+\mathbb{1})z_2\big) = \big(z_1|(\mathbb{1}-\gamma)^{-1}z_2\big),\\
\big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= (z_2|\rho z_1) = \big(z_2|\gamma(\mathbb{1}-\gamma)^{-1}z_1\big),\\
\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= \big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) \ = \ 0.
\end{aligned}$$

(5) $\mathrm{CCR}_{\gamma,\mathrm{l}}$ *is a factor.*

(6) $\mathrm{Ker}\,\gamma = \{0\}$ *iff Ω is separating for* $\mathrm{CCR}_{\gamma,\mathrm{l}}$ *iff Ω is cyclic for* $\mathrm{CCR}_{\gamma,\mathrm{l}}$. *If this is the case, the modular conjugation for Ω is equal to J_{s} and the modular operator for Ω is $\Delta = \Gamma(\gamma \oplus \overline{\gamma}^{-1})$.*

(7) *We have*

$$\mathrm{CCR}'_{\gamma,\mathrm{l}} = \mathrm{CCR}_{\gamma,\mathrm{r}}.$$

Proof (1)–(4) follow by straightforward computations. Let us prove (5).

We check that $[W_{\gamma,\mathrm{l}}(z_1), W_{\gamma,\mathrm{r}}(\overline{z}_2)] = 0$ for $z_1, z_2 \in \mathrm{Dom}\,\rho^{\frac{1}{2}}$, which implies that $\mathrm{CCR}_{\gamma,\mathrm{r}} \subset \mathrm{CCR}'_{\gamma,\mathrm{l}}$.

Clearly, $(\mathrm{CCR}_{\gamma,\mathrm{l}} \cup \mathrm{CCR}_{\gamma,\mathrm{r}})''$ is equal to $\{W(w),\ w \in \mathcal{E}\}''$, for

$$\mathcal{E} = \mathrm{Span}\left\{\Big((\rho+\mathbb{1})^{\frac{1}{2}}z_1 + \rho^{\frac{1}{2}}z_2, \overline{\rho}^{\frac{1}{2}}\overline{z_1} + (\overline{\rho}+\mathbb{1})^{\frac{1}{2}}\overline{z_2}\Big),\ z_1, z_2 \in \mathrm{Dom}\,\rho^{\frac{1}{2}}\right\}.$$

Clearly, $\mathcal{E}$ is dense in $\mathcal{Z} \oplus \overline{\mathcal{Z}}$. Recall that, by Thm. 9.5, Weyl operators on a Fock space depend strongly continuously on their arguments. Therefore, $\big\{W(w),\ w \in \mathcal{E}\big\}'' = \big\{W(w),\ w \in \mathcal{Z} \oplus \overline{\mathcal{Z}}\big\}'' = B\big(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big)$. Thus

$$(\mathrm{CCR}_{\gamma,\mathrm{l}} \cup \mathrm{CCR}'_{\gamma,\mathrm{l}})'' \supset (\mathrm{CCR}_{\gamma,\mathrm{l}} \cup \mathrm{CCR}_{\gamma,\mathrm{r}})'' = B\big(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big),$$

which implies that $\mathrm{CCR}_{\gamma,\mathrm{l}}$ is a factor.

Let us prove the $\Rightarrow$ part of (6). Assume that $\mathrm{Ker}\,\gamma = \{0\}$ and set $\tau^t(A) = \Gamma(\gamma, \overline{\gamma}^{-1})^{\mathrm{i}t} A \Gamma(\gamma, \overline{\gamma}^{-1})^{-\mathrm{i}t}$. We have

$$\tau^t(W_{\gamma,\mathrm{l}}(z)) = W_{\gamma,\mathrm{l}}(\gamma^{\mathrm{i}t}z),$$

hence τ^t is a W^*-dynamics on $\mathrm{CCR}_{\gamma,\mathrm{l}}$. We claim that Ω is a $(\tau, 1)$-KMS vector on $\mathrm{CCR}_{\gamma,\mathrm{l}}$. In fact, we have

$$\big(\Omega|W_{\gamma,\mathrm{l}}(z_1)W_{\gamma,\mathrm{l}}(\gamma^{\mathrm{i}t}z_2)\Omega\big) = \mathrm{e}^{-\frac{1}{4}F(t,z_1,z_2)},$$

for

$$F(t,z_1,z_2)=\Big(z_1|\frac{\gamma+1\!\!1}{1\!\!1-\gamma}z_1\Big)+\Big(z_2|\frac{\gamma+1\!\!1}{1\!\!1-\gamma}z_2\Big)+\Big(z_1|\gamma^{\mathrm{i}t}\frac{2\gamma}{1\!\!1-\gamma}z_2\Big)$$
$$+\Big(z_2|\gamma^{-\mathrm{i}t}\frac{2}{1\!\!1-\gamma}z_1\Big),$$

which proves the $(\tau,1)$-KMS condition for the Weyl operators. By linearity, it holds also for the $*$-algebra of finite linear combinations of $W_{\gamma,\mathrm{l}}(z)$ and, by Prop. 6.64, for $\mathrm{CCR}_{\gamma,\mathrm{l}}$.

Applying then Prop. 6.65 to the factor $\mathrm{CCR}_{\gamma,\mathrm{l}}$, we obtain that Ω is separating for $\mathrm{CCR}_{\gamma,\mathrm{l}}$.

We denote by $\mathcal{H}$ the closure of $\mathrm{CCR}_{\gamma,\mathrm{l}}\Omega$. We would like to show that $\mathcal{H}=\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$, which means that Ω is cyclic for $\mathrm{CCR}_{\gamma,\mathrm{l}}$. As a byproduct of this proof, we will develop the modular theory of $\mathrm{CCR}_{\gamma,\mathrm{l}}$.

Clearly, $\mathcal{H}$ is invariant under $\mathrm{CCR}_{\gamma,\mathrm{l}}$, Ω is cyclic and separating for $\mathrm{CCR}_{\gamma,\mathrm{l}}$ restricted to $\mathcal{H}$. Let us compute the operators S,Δ and J of the modular theory for Ω and $\mathrm{CCR}_{\gamma,\mathrm{l}}$ restricted to $\mathcal{H}$.

Let us set

$$\mathcal{H}_1:=\mathrm{Span}\big\{W_{\gamma,\mathrm{l}}(z)\Omega\ :\ z\in\mathrm{Dom}\,\rho^{\frac12}\big\}=\mathrm{Span}\big\{\Psi_z\ :\ z\in\mathrm{Dom}\,\rho^{\frac12}\big\},$$

for

$$\Psi_z=\mathrm{e}^{\mathrm{i}a^*((\rho+1\!\!1)^{\frac12}z,\overline{\rho}^{\frac12}\overline{z})}\Omega.$$

We have

$$\Gamma(\gamma,\overline{\gamma}^{-1})^{\mathrm{i}t}\Psi_z=\Psi_{\gamma^{\mathrm{i}t}z},\tag{17.29}$$

which implies that the self-adjoint operator $\Gamma(\gamma,\overline{\gamma}^{-1})$ preserves $\mathcal{H}$. Moreover, the r.h.s. of (17.29) extends analytically in t to $t=-\mathrm{i}/2$. This shows that $\Psi_z\in\mathrm{Dom}\,\Gamma(\gamma,\overline{\gamma}^{-1})^{\frac12}$ and

$$\Gamma(\gamma,\overline{\gamma}^{-1})^{\frac12}\Psi_z=\mathrm{e}^{\mathrm{i}a^*\left(\rho^{\frac12}z,(\overline{\rho}+1\!\!1)^{\frac12}\overline{z}\right)}\Omega.$$

Moreover,

$$\begin{aligned}S\Psi_z&=\mathrm{e}^{-\mathrm{i}a^*\left((\rho+1\!\!1)^{\frac12}z,\overline{\rho}^{\frac12}\overline{z}\right)}\Omega\\&=J_{\mathrm{s}}\Gamma(\gamma,\overline{\gamma}^{-1})^{\frac12}\Psi_z.\end{aligned}\tag{17.30}$$

Clearly, $\mathcal{H}_1$ is dense in $\mathcal{H}$.

$$\mathfrak{A}:=\mathrm{Span}\big\{W_{\gamma,\mathrm{l}}(z)\ :\ z\in\mathrm{Dom}\,\rho^{\frac12}\big\}$$

is a $*$-algebra $*$-strongly dense in $\mathrm{CCR}_{\gamma,\mathrm{l}}$ and $\mathcal{H}_1=\mathfrak{A}\Omega$; therefore, by Subsect. 6.4.2, $\mathcal{H}_1$ is an essential domain of S. Therefore, we can extend (17.30) by density to the whole $\mathcal{H}$, using that J_{s} is isometric. We obtain

$$S=J_{\mathrm{s}}\Gamma(\gamma,\overline{\gamma}^{-1})^{\frac12}\big|_{\mathcal{H}}.\tag{17.31}$$

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Since $\operatorname{Ker}\gamma = \{0\}$, the range of $\Gamma(\gamma, \overline{\gamma}^{-1})^{\frac{1}{2}}\big|_{\mathcal{H}}$ is dense in $\mathcal{H}$. Using (17.31), this implies that J_{s} preserves $\mathcal{H}$. Thus

$$S = J_{\mathrm{s}}\big|_{\mathcal{H}}\Gamma(\gamma, \overline{\gamma}^{-1})^{\frac{1}{2}}\big|_{\mathcal{H}}$$

is the polar decomposition of S, defining the modular operator and modular conjugation. Next we see that

$$W_{\gamma,\mathrm{l}}(z_1)J_{\mathrm{s}}W_{\gamma,\mathrm{l}}(z_2)\Omega = W\left((\rho+\mathbb{1})^{\frac{1}{2}}z_1 + \rho^{\frac{1}{2}}z_2, \overline{\rho}^{\frac{1}{2}}\overline{z_1} + (\overline{\rho}+\mathbb{1})^{\frac{1}{2}}\overline{z_2}\right)\Omega.$$

Therefore, $\mathrm{CCR}_{\gamma,\mathrm{l}}J_{\mathrm{s}}\mathrm{CCR}_{\gamma,\mathrm{l}}\Omega$ is dense in $\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. Since $\mathrm{CCR}_{\gamma,\mathrm{l}}J_{\mathrm{s}}\mathrm{CCR}_{\gamma,\mathrm{l}}\Omega \subset \mathcal{H}$, this proves that $\mathcal{H} = \Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$, and hence Ω is cyclic for $\mathrm{CCR}_{\gamma,\mathrm{l}}$. This proves the $\Rightarrow$ part of (6). The proof of the $\Leftarrow$ part of (6) will be given after the proof of (7).

Let us now prove (7). Assume first that $\operatorname{Ker}\gamma = \{0\}$. By the $\Rightarrow$ part of (6), we can apply the Tomita–Takesaki theory to $(\mathrm{CCR}_{\gamma,\mathrm{l}}, \Omega)$ and obtain that $\mathrm{CCR}'_{\gamma,\mathrm{l}} = J_{\mathrm{s}}\mathrm{CCR}_{\gamma,\mathrm{l}}J_{\mathrm{s}}$. By (3), we have $J_{\mathrm{s}}\mathrm{CCR}_{\gamma,\mathrm{l}}J_{\mathrm{s}} = \mathrm{CCR}_{\gamma,\mathrm{r}}$.

For a general γ, we write $\mathcal{Z} = \mathcal{Z}_0 \oplus \mathcal{Z}_1$, for $\mathcal{Z}_0 = \operatorname{Ker}\gamma$. Then $\gamma_1 = \gamma\big|_{\mathcal{Z}_1}$ is injective. Using the exponential law of Subsect. 3.3.7, we have

$$\begin{aligned}\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}) &\simeq \Gamma_{\mathrm{s}}(\mathcal{Z}_0)\otimes\Gamma_{\mathrm{s}}(\mathcal{Z}_1\oplus\overline{\mathcal{Z}_1})\otimes\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}_0}),\\ W_{\gamma,\mathrm{l}}(z) &\simeq W(z_0)\otimes W_{\gamma_1,\mathrm{l}}(z_1)\otimes\mathbb{1},\\ W_{\gamma,\mathrm{r}}(\overline{z}) &\simeq \mathbb{1}\otimes W_{\gamma_1,\mathrm{r}}(\overline{z}_1)\otimes W(\overline{z_0}),\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{CCR}_{\gamma,\mathrm{l}} &\simeq B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}_0)\big)\otimes\mathrm{CCR}_{\gamma_1,\mathrm{l}}\otimes\mathbb{C}\mathbb{1},\\ \mathrm{CCR}_{\gamma,\mathrm{r}} &\simeq \mathbb{C}\mathbb{1}\otimes\mathrm{CCR}_{\gamma_1,\mathrm{r}}\otimes B\big(\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}_0})\big),\end{aligned}\tag{17.32}$$

which shows that $\mathrm{CCR}'_{\gamma,\mathrm{l}} = \mathrm{CCR}_{\gamma,\mathrm{r}}$ and completes the proof of (7).

From (17.32), we see that if $\operatorname{Ker}\gamma \neq \{0\}$, Ω is neither cyclic nor separating. This proves the $\Leftarrow$ part of (6). □

17.1.6 Quasi-free CCR representations as Araki–Woods representations

Recall that in Thm. 17.12 we showed that every neutral quasi-free CCR representation over a symplectic space can be reinterpreted as a charged quasi-free CCR representation over a charged symplectic space with $\mathrm{i}\omega$ positive definite. Under minor technical assumptions, such representations are unitarily equivalent to Araki–Woods representations. This is described in the theorem below.

Theorem 17.25 *Let $(\mathcal{Y}, \omega)$ be a charged symplectic space such that $\mathrm{i}\omega$ is positive definite on $\mathcal{Y}$. Let*

$$\mathcal{Y} \ni y \mapsto a^{\pi *}(y) \in Cl(\mathcal{H})$$

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be a charged CCR representation with a cyclic gauge-invariant quasi-free vector Ψ. *Let* ρ *be given by*

$$\overline{y}_1 \cdot \rho y_2 = \big(\Psi | a^{\pi *}(y_2) a^{\pi}(y_1)\Psi\big), \quad y_1, y_2 \in \mathcal{Y}.$$

Assume that $\mathcal{Y}$ *is complete for the scalar product* $2\rho + \mathrm{i}\omega$. *Let* $\mathcal{Z}$ *be the completion of* $\mathcal{Y}$ *w.r.t.* $\mathrm{i}\omega$. *Note that* ρ *can be interpreted as a positive self-adjoint operator on* $\mathcal{Z}$ *such that* $\mathcal{Y} = \mathrm{Dom}\, \rho^{\frac{1}{2}}$. *Set* $\gamma = \rho(\mathbb{1} + \rho)^{-1}$. *Then there exists a unique isometry* $U : \mathcal{H} \to \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ *such that*

$$\begin{aligned} U\Psi &= \Omega, \\ Ua^{\pi *}(y) &= a^*_{\gamma,\mathrm{l}}(y)U, \quad y \in \mathcal{Y}. \end{aligned} \tag{17.33}$$

17.1.7 Free Bose gas at positive temperatures

In this subsection we would like to describe in general terms how quasi-free bosonic states usually arise in quantum physics. We will also discuss various mathematical formalisms used in this context.

Let h be a positive self-adjoint operator on a Hilbert space $\mathcal{Z}$. Consider a quantum system described by the Hamiltonian $H := \mathrm{d}\Gamma(h)$ acting on the Hilbert space $\Gamma_{\mathrm{s}}(\mathcal{Z})$. Note that $(\Omega| \cdot \Omega)$ describes the ground state of the system. On the algebra $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ we have the dynamics

$$\tau^t(A) := \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}, \quad A \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big), \quad t \in \mathbb{R}.$$

We also have a natural charged CCR representation $\mathcal{Z} \ni z \mapsto a^*(z) \in Cl\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ and the corresponding neutral CCR representation $\mathcal{Z} \ni z \mapsto W(z) = \exp\big(\mathrm{i}\frac{a^*(z)+a(z)}{\sqrt{2}}\big) \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$. They satisfy

$$\tau^t\big(a^*(z)\big) = a^*(\mathrm{e}^{\mathrm{i}th}z), \quad \tau^t\big(W(z)\big) = W(\mathrm{e}^{\mathrm{i}th}z), \quad z \in \mathcal{Z}.$$

Suppose that we consider the above quantum system at a positive temperature. Let $\beta \geq 0$ denote the inverse temperature. If

$$\mathrm{Tr}\, \mathrm{e}^{-\beta h} < \infty, \tag{17.34}$$

we can consider the Gibbs state given by the density matrix

$$\mathrm{e}^{-\beta \mathrm{d}\Gamma(h)} / \mathrm{Tr}\, \mathrm{e}^{-\beta \mathrm{d}\Gamma(h)}. \tag{17.35}$$

Positive-temperature systems are especially interesting for infinitely extended physical systems. For such systems $\mathrm{e}^{-\beta h}$ is rarely trace-class – in fact, typically, h has a continuous spectrum, which rules out (17.34). Therefore, the formalism based on the Gibbs state with the density matrix (17.35) breaks down.

As a typical example of such a system we can consider the (non-relativistic) free Bose gas. Its one-particle Hilbert space and the one-particle Hamiltonian

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are

$$\mathcal{Z} := L^2(\mathbb{R}^d), \quad h := -\Delta. \tag{17.36}$$

Clearly, in this case (17.34) is not satisfied. Therefore, we need a different formalism to describe positive-temperature systems in this situation.

In the literature one can distinguish three approaches to positive temperatures for infinitely extended systems:

(1) the thermodynamic limit,
(2) the W^* approach,
(3) the C^* approach.

Thermodynamic limit

The thermodynamic limit consists in approximating our system by a sequence of systems in finite volume. Thus we have a sequence of one-particle Hilbert spaces $\mathcal{Z}_L$ with one-particle Hamiltonians h_L. We also need to identify $\mathcal{Z}_{L_1}$ as a subspace of $\mathcal{Z}_{L_2}$ for $L_1 < L_2$, which allows us to embed the corresponding observable algebras $B\big(\Gamma(\mathcal{Z}_{L_1})\big) \subset B\big(\Gamma(\mathcal{Z}_{L_2})\big)$. Typically, for finite L, the condition

$$\operatorname{Tr} \mathrm{e}^{-\beta h_L} < \infty \tag{17.37}$$

is satisfied, and so we can use the corresponding Gibbs state. Then we expect that for a fixed L_0 and a large class of observables $A \in B\big(\Gamma(\mathcal{Z}_{L_0})\big)$, the expectation value

$$\operatorname{Tr} A\mathrm{e}^{-\beta \mathrm{d}\Gamma(h_L)} / \operatorname{Tr} \mathrm{e}^{-\beta \mathrm{d}\Gamma(h_L)}$$

converges to a limit as $L \to \infty$.

In the case of (17.36), we typically take $\mathcal{Z}_L := L^2\big([-L, L]^d\big)$, and h_L is the Laplacian with some conditions on the boundary of the box $[-L, L]^d$. For many purposes the choice of boundary conditions should not matter. The Dirichlet or Neumann boundary conditions seem more relevant physically, whereas the periodic boundary conditions might be more convenient mathematically.

Note that this approach involves a significant amount of arbitrariness. One needs to introduce a lot of additional structure, which in the end is irrelevant.

W^* approach

We can describe temperature states by using the Araki–Woods representations. In fact, consider the space $\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. For $z \in \mathrm{Dom}(\mathrm{e}^{\beta h/2})$, define

$$a_\beta^*(z) := a^*\big((\mathbb{1} - \mathrm{e}^{-\beta h})^{-\frac{1}{2}} z, 0\big) + a\big(0, (\mathrm{e}^{\beta \overline{h}} - \mathbb{1})^{-\frac{1}{2}} \overline{z}\big), \tag{17.38}$$

that is, the Araki–Woods representation for the *Planck density* $(\mathrm{e}^{\beta h} - \mathbb{1})^{-1}$. Then

$$\mathrm{Dom}(\mathrm{e}^{\beta h/2}) \ni z \mapsto a_\beta^*(z) \in Cl\big(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big)$$

is a charged CCR representation. The von Neumann algebra generated by (17.38) will be denoted by CCR_β. Set

$$L := \mathrm{d}\Gamma\left(h \oplus (-\overline{h})\right).$$

Then

$$\tau_\beta^t(A) := \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}, \quad A \in \mathrm{CCR}_\beta, \quad t \in \mathbb{R},$$

is a W^*-dynamics on CCR_β and L is its Liouvillean. The state

$$\omega_\beta := (\Omega|A\Omega), \quad A \in \mathrm{CCR}_\beta,$$

is a β-KMS state for the W^*-dynamics τ_β.

Thus we obtain a family of W^*-dynamical systems $(\mathfrak{M}_\beta, \tau_\beta)$ equipped with the state ω_β. One can argue that all of them describe the same physical system and differ only by the temperature. In concrete situations, one can derive the family $(\mathfrak{M}_\beta, \tau_\beta, \omega_\beta)$ using the thermodynamic limit.

Note that the W^* approach does not involve any additional structure (unlike the thermodynamic limit). It is often used in the mathematical physics literature. Implicitly, it is also widely used in the theoretical physics literature.

C^* approach

Consider the C^*-algebra $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Z})$, where $\mathcal{Z}$ is equipped with the symplectic structure $\mathrm{Im}(\cdot|\cdot)$, as well as the charge symmetry $z \mapsto \mathrm{e}^{\mathrm{i}\theta} z$, $\theta \in [0, 2\pi[$. Define the dynamics on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Z})$ by setting

$$\tau^t\left(W(z)\right) := W(\mathrm{e}^{\mathrm{i}th} z), \quad z \in \mathcal{Z}, \quad t \in \mathbb{R}.$$

It is easy to see that, for any $\beta \in]0, \infty]$, there exists on $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Z})$ a unique state β-KMS for the dynamics τ. It is given by

$$\omega_\beta\left(W(z)\right) = \exp\left(-\frac{1}{4}\left(z\Big|\frac{1\!\!1 + \exp(-\beta h)}{1\!\!1 - \exp(-\beta h)} z\right)\right), \quad z \in \mathcal{Z}.$$

We can then pass to the GNS representation $(\mathcal{H}_\beta, \pi_\beta, \Omega_\beta)$ and construct the Liouvillean L_β.

In the case of $\beta = \infty$ (the zero temperature), we obtain, up to unitary equivalence, $\mathcal{H}_\infty = \Gamma_{\mathrm{s}}(\mathcal{Z})$, $\pi_\infty\left(W(z)\right) = W(z)$, $\Omega_\infty = \Omega$ and $L_\infty = H$. This is the quantum system that we started with at the beginning of the subsection.

In the case $\beta < \infty$ (positive temperatures), we obtain the Araki–Woods representation for $\gamma = \mathrm{e}^{-\beta h}$ described in (17.38).

The main advantage of this approach is its conceptual and mathematical elegance. Its starting point is a *single* system, and various temperature states arise naturally by the application of a general principle.

This approach has also a serious disadvantage. The choice of the algebra of observables $\mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Z})$ is rather arbitrary. In principle, one could replace it by another $*$-algebra related to the CCR over $\mathcal{Z}$, e.g. one of those described

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in Sect. 8.3. The choice of the CCR algebra does not have much relevance as long as the dynamics is free (that is, as long as it is described by Bogoliubov transformations). The problem becomes more serious when we try to consider a system with a non-trivial interaction. Then, in concrete situations, it is usually not easy to find a C^*-algebra preserved by a given dynamics, and the C^* approach is difficult to apply.

17.2 Fermionic quasi-free states

In this section we describe the theory of fermionic quasi-free states. It is in many ways parallel to that of bosonic quasi-free states. Therefore, each subsection about fermionic quasi-free states has its counterpart in the previous bosonic section.

17.2.1 Definition of fermionic quasi-free states

Let $(\mathcal{Y}, \nu)$ be a real Hilbert space. Recall that $\mathrm{CAR}^{C^*}(\mathcal{Y})$ denotes the CAR C^*-algebra over $\mathcal{Y}$, that is, the C^*-algebra generated by $\phi(y)$, $y \in \mathcal{Y}$, satisfying the CAR relations; see Subsect. 12.5.2.

Definition 17.26 (1) *A state ψ on $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is called* quasi-free *if*

$$\psi\big(\phi(y_1)\cdots\phi(y_{2m-1})\big) = 0,$$

$$\psi\big(\phi(y_1)\cdots\phi(y_{2m})\big) = \sum_{\sigma\in\mathrm{Pair}_{2m}} \mathrm{sgn}(\sigma)\prod_{j=1}^{m}\psi\big(\phi(y_{\sigma(2j-1)})\phi(y_{\sigma(2j)})\big),$$

for all $y_1, y_2, \dots \in \mathcal{Y}$, $m \in \mathbb{N}$.

(2) *If $\mathcal{Y} \ni y \mapsto \phi^\pi(y) \in B_\mathrm{h}(\mathcal{H})$ is a CAR representation, $\Psi \in \mathcal{H}$ is called a* quasi-free vector *if*

$$\psi\big(\phi(y_1)\cdots\phi(y_n)\big) := \big(\Psi|\phi^\pi(y_1)\cdots\phi^\pi(y_n)\Psi\big), \quad y_1,\dots,y_n \in \mathcal{Y}, \quad n \in \mathbb{N},$$

defines a quasi-free state on $\mathrm{CAR}^{C^*}(\mathcal{Y})$.

(3) *A CAR representation ϕ^π on a Hilbert space $\mathcal{H}$ is* quasi-free *if there exists a cyclic quasi-free vector in $\mathcal{H}$.*

(4) *The anti-symmetric form $\beta \in L_\mathrm{a}(\mathcal{Y}, \mathcal{Y}^\#)$ given by*

$$y_1\cdot\beta y_2 := \frac{1}{\mathrm{i}}\psi\big([\phi(y_1),\phi(y_2)]\big), \quad y_1, y_2 \in \mathcal{Y}. \tag{17.39}$$

is called the covariance of the quasi-free state ψ, *and* of the quasi-free vector Ψ.

For a quasi-free state ψ on $\mathrm{CAR}^{C^*}(\mathcal{Y})$, let $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ be the corresponding GNS representation. Then clearly $\Omega_\psi \in \mathcal{H}_\psi$ is a quasi-free vector for the CAR representation $\mathcal{Y} \ni y \mapsto \pi_\psi\big(\phi(y)\big) \in B_\mathrm{h}\big(\mathcal{H}_\psi\big)$.

The covariance defines the representation uniquely:

Proposition 17.27 *Let $\mathcal{Y} \ni y \mapsto \phi^i(y) \in B_\mathrm{h}(\mathcal{H}_i)$, $i = 1, 2$, be quasi-free CAR representations with cyclic quasi-free vectors $\Psi_i \in \mathcal{H}_1$, both of covariance β. Then there exists a unique $U \in U(\mathcal{H}_1, \mathcal{H}_2)$ intertwining ϕ^1 with ϕ^2 satisfying $U\Psi_1 = \Psi_2$.*

Let us note the following important special subclasses of quasi-free representations:

(1) If the pair ν and $\frac{1}{2}\beta$ is Kähler, the corresponding representation is Fock; see Thm. 17.31.
(2) If $\beta = 0$, the corresponding representation is unitarily equivalent to the real-wave (or tracial) representation, discussed already in Subsects. 12.4.2 and 13.2.1.

From the CAR it follows that

$$\psi\big(\phi(y_1)\phi(y_2)\big) = y_1\cdot\nu y_2 + \frac{\mathrm{i}}{2} y_1\cdot\beta y_2, \quad y_1, y_2 \in \mathcal{Y}. \tag{17.40}$$

(17.40) implies the following proposition:

Proposition 17.28 *Let $\beta \in L_\mathrm{a}(\mathcal{Y}, \mathcal{Y}^\#)$. Then the following are equivalent:*

(1) *There exists a quasi-free state ψ such that (17.40) holds.*
(2) $\nu_\mathbb{C} + \frac{\mathrm{i}}{2}\beta_\mathbb{C} \geq 0$ *on* $\mathbb{C}\mathcal{Y}$
(3) $|y_1\cdot\beta y_2| \leq 2(y_1\cdot\nu y_1)^{\frac{1}{2}}(y_2\cdot\nu y_2)^{\frac{1}{2}}$, $y_1, y_2 \in \mathcal{Y}$.

Proof The equivalence of (2) and (3) is shown as in Prop. 17.8. To prove (1) $\Rightarrow$ (2) we compute

$$\psi\big(\phi^*(w)\phi(w)\big) = \overline{w}\cdot\nu_\mathbb{C} w + \frac{\mathrm{i}}{2}\overline{w}\cdot\beta_\mathbb{C} w \geq 0, \quad w \in \mathbb{C}\mathcal{Y}.$$

Let us prove (3) $\Rightarrow$ (1). We fix $\beta \in L_\mathrm{a}(\mathcal{Y}, \mathcal{Y}^\#)$ satisfying (3). From Def. 17.26, we obtain a linear functional ψ on the $*$-algebra generated by the $\phi(y)$, $y \in \mathcal{Y}$. It clearly suffices to show that ψ is positive. To check this we may assume that $\mathcal{Y}$ is finite-dimensional.

Using Corollary 2.85 we can find an o.n. basis $(e_1, \dots, e_{2m}, f_1, \dots, f_d)$ of $\mathcal{Y}$ such that

$$\beta e_{2j-1} = \lambda_j e_{2j}, \quad \beta e_{2j} = -\lambda_j e_{2j-1}, \quad \beta f_j = 0,$$

for $\lambda_1, \dots, \lambda_m > 0$. Condition (3) for β is equivalent to $|\lambda_1|, \dots, |\lambda_m| \leq 2$.

Assume first that $\dim \mathcal{Y} = 2n$. Then, allowing some λ_j to be equal to 0, we can assume that $m = n$. We set $\phi_j = \phi(e_j)$ and use the Jordan–Wigner representation of $\mathrm{CAR}(\mathbb{R}^{2n})$ on $\otimes^n\mathbb{C}^2$ defined in Subsect. 12.2.3. We note that if $|\lambda| \leq 2$, then

$$\rho(\lambda) = \frac{1}{2}\begin{bmatrix} 1 - \lambda/2 & 0 \\ 0 & 1 + \lambda/2 \end{bmatrix}$$

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satisfies

$$\begin{aligned} &\rho(\lambda) \geq 0, \quad \operatorname{Tr} \rho(\lambda) = 1, \\ &\operatorname{Tr}(\rho(\lambda)\sigma_1) = \operatorname{Tr}(\rho(\lambda)\sigma_2) = 0, \quad \operatorname{Tr}(\rho(\lambda)\sigma_3) = -\lambda/2. \end{aligned} \tag{17.41}$$

We set

$$\rho = \rho(\lambda_1) \otimes \cdots \otimes \rho(\lambda_n).$$

We will prove that

$$\psi(A) = \operatorname{Tr}(\rho A), \quad A \in \operatorname{CAR}(\mathbb{R}^{2n}), \tag{17.42}$$

which implies that ψ is positive.

We first see that

$$\operatorname{Tr}(\rho \phi_{2j-1}\phi_{2j}) = -\mathrm{i}\lambda_j/2, \quad \operatorname{Tr}(\rho\phi_j\phi_k) = 0 \ \text{ if } \ |j-k| \geq 2, \tag{17.43}$$

hence

$$\psi\big(\phi(y_1)\phi(y_2)\big) = \operatorname{Tr}\big(\rho\phi(y_1)\phi(y_2)\big), \quad y_1, y_2 \in \mathcal{Y}.$$

We claim now that

$$\operatorname{Tr}(\rho\phi_{i_1}\cdots\phi_{i_k}) = 0, \ \text{ if } \ k \ \text{ is odd}. \tag{17.44}$$

We can assume, using the CAR, that $i_1 < \cdots < i_k$. Let i_l be one of the indices. If $l = 2j-1$, then the j factor of $\phi_{i_1}\cdots\phi_{i_k}$ is equal to $-\mathrm{i}\sigma_2$, except if $i_{l+1} = i_l + 1$, and if $l = 2j$, the j factor of $\phi_{i_1}\cdots\phi_{i_k}$ is equal to $\mathrm{i}\sigma_1$, except if $i_{l-1} = i_l - 1$. It follows from (17.41) that $\operatorname{Tr}(\rho\phi_{i_1}\cdots\phi_{i_k}) = 0$, except when for each $1 \leq l \leq k$ one has $i_{l+1} = i_l + 1$ or $i_{l-1} = i_l - 1$. This condition is not satisfied if k is odd, which proves (17.44). We claim that

$$\operatorname{Tr}(\rho\phi_{i_1}\cdots\phi_{i_{2m}}) = \sum_{\sigma\in\mathrm{Pair}_{2m}} \operatorname{sgn}(\sigma)\prod_{j=1}^{m}\operatorname{Tr}(\rho\phi_{i_{\sigma(2j-1)}}\phi_{i_{\sigma(2j)}}), \tag{17.45}$$

which combined with (17.44) implies (17.42).

The same argument as above shows that the l.h.s. of (17.45) is zero if $(i_1,\dots,i_{2m})$ is not a collection of pairs $(2j-1, 2j)$. The same holds for the r.h.s., since in this case, for all $\sigma \in \mathrm{Pair}_{2m}$, at least one of the factors vanishes. It remains to consider the case when

$$(i_1,\dots,i_{2m}) = (2j_1-1, 2j_1, \dots, 2j_m-1, 2j_m),$$

for $j_1 < \cdots < j_m$. In this case

$$\phi_{i_1}\cdots\phi_{i_{2m}} = (\mathrm{i}\sigma_3)^{(j_1)}\cdots(\mathrm{i}\sigma_3)^{j_m},$$

and hence the l.h.s. of (17.45) equals

$$\operatorname{Tr}(\rho\phi_{i_1}\cdots\phi_{i_{2m}}) = \prod_{k=1}^{m}(-\mathrm{i}\lambda_{j_k})/2.$$

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Since the only pairing contributing to the r.h.s. is $(2j_1-1, 2j_1), \ldots, (2j_m-1, 2j_m)$, we see using (17.43) that (17.45) holds.

Assume now that $\dim \mathcal{Y}$ is odd. Then we set $\mathcal{Y}_1 = \mathcal{Y} \oplus \mathbb{R}$, and consider the (reducible) representation of $\mathrm{CAR}(\mathbb{R}^{2n+1})$ in $\otimes^{n+1}\mathbb{C}^2$ obtained from the Jordan–Wigner representation of $\mathrm{CAR}(\mathbb{R}^{2(n+1)})$. We are then reduced to the previous case. This completes the proof of the proposition. □

17.2.2 Gauge-invariant fermionic quasi-free states

Suppose that the real Hilbert space $(\mathcal{Y}, \nu)$ is equipped with a Kähler anti-involution j. As in Subsect. 1.3.11, $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is equipped with the one-parameter group of charge automorphisms, denoted $U(1) \ni \theta \mapsto \widehat{u_\theta}$, and defined by

$$\widehat{u_\theta}\big(\phi(y)\big) = \phi(\mathrm{e}^{\mathrm{j}\theta} y).$$

Definition 17.29 *A state ψ on $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is called* gauge-invariant *if it is invariant w.r.t. $\widehat{u_\theta}$.*

Consider a fermionic gauge-invariant quasi-free state with covariance β.

Let us introduce the holomorphic space $\mathcal{Z}$ associated with the anti-involution j, so that $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. The sesquilinear forms $\nu_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$ can be reduced w.r.t. the direct sum $\mathcal{Z} \oplus \overline{\mathcal{Z}}$. Thus we can write

$$\nu_{\mathbb{C}} = \begin{bmatrix} \nu_{\mathcal{Z}} & 0 \\ 0 & \overline{\nu}_{\mathcal{Z}} \end{bmatrix}, \quad \beta_{\mathbb{C}} = \begin{bmatrix} \beta_{\mathcal{Z}} & 0 \\ 0 & \overline{\beta}_{\mathcal{Z}} \end{bmatrix}, \tag{17.46}$$

where $\nu_{\mathcal{Z}}$ is Hermitian and $\beta_{\mathcal{Z}}$ anti-Hermitian. Note that the condition $\nu_{\mathbb{C}} + \frac{\mathrm{i}}{2}\beta_{\mathbb{C}} \geq 0$, which by Prop. 17.28 is necessary and sufficient for β to be a covariance of a quasi-free state, is equivalent to

$$\nu_{\mathcal{Z}} \pm \frac{\mathrm{i}}{2}\beta_{\mathcal{Z}} \geq 0. \tag{17.47}$$

Until the end of this subsection we assume that $(\mathcal{Y}, \nu)$ is a real Hilbert space and ψ a quasi-free state on $\mathrm{CAR}^{C^*}(\mathcal{Y})$ with covariance $\beta \in L_{\mathrm{a}}(\mathcal{Y}, \mathcal{Y}^{\#})$.

Theorem 17.30 (1) *Assume that* $\mathrm{Ker}\,\beta$ *is even- or infinite-dimensional. Then there exists an anti-involution* j *such that ψ is gauge-invariant for the complex structure given by* j.

(2) *If* $\mathrm{Ker}\,\beta = \{0\}$*, then the anti-involution* j *given by (1) is unique if we demand in addition that* (β, j) *is Kähler.*

Proof By Prop. 17.28, there exists an anti-symmetric operator b such that $\|b\| \leq 1$ and

$$y_1 \cdot \beta y_2 = 2 y_1 \cdot \nu b y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

Let $\mathcal{Y}_{\mathrm{sg}} := \operatorname{Ker} b$ and $\mathcal{Y}_{\mathrm{reg}} = \mathcal{Y}_{\mathrm{sg}}^{\perp}$. On $\mathcal{Y}_{\mathrm{reg}}$ we use the polar decomposition

$$b_{\mathrm{reg}} = -|b_{\mathrm{reg}}| \mathrm{j}_{\mathrm{reg}} = -\mathrm{j}_{\mathrm{reg}} |b_{\mathrm{reg}}|,$$

so that $\mathrm{j}_{\mathrm{reg}}$ is a Kähler anti-involution on $\mathcal{Y}_{\mathrm{reg}}$ both for $\nu\big|_{\mathcal{Y}_{\mathrm{reg}}}$ and $\beta\big|_{\mathcal{Y}_{\mathrm{reg}}}$. If $\dim \mathcal{Y}_{\mathrm{reg}}$ is even or infinite, we can extend $\mathrm{j}_{\mathrm{reg}}$ to a Kähler anti-involution on $(\mathcal{Y}, \nu)$. □

The following theorem is the fermionic analog of Thm. 17.13 (2).

Theorem 17.31 *The GNS representation associated with ψ is irreducible iff $(\nu, \frac{1}{2}\beta)$ is Kähler.*

17.2.3 Charged quasi-free CAR representations

The following subsection is essentially a translation of the previous subsection from the terminology of neutral CAR representation to that of charged CAR representations, which seems more convenient in the context of gauge invariance.

Let $\big(\mathcal{Y}, (\cdot|\cdot)\big)$ be a complex Hilbert space. On $\mathcal{Y}_{\mathbb{R}}$, that is, on the realification of $\mathcal{Y}$, we introduce the real scalar product $\nu := \frac{1}{2}\mathrm{Re}(\cdot|\cdot)$.

Clearly, $\mathcal{Y}$ is equipped with a Kähler anti-involution – the imaginary unit. Therefore, all the definitions off the previous subsections make sense. In particular, the CAR algebra $\mathrm{CAR}^{C^*}(\mathcal{Y}_{\mathbb{R}})$ is equipped with a charge symmetry and we can define the notion of a gauge-invariant state. We will write $\mathrm{CAR}^{C^*}(\mathcal{Y})$ to denote the algebra $\mathrm{CAR}^{C^*}(\mathcal{Y}_{\mathbb{R}})$ equipped with this charge symmetry.

As in the bosonic case, we will denote charged fields using the letter a, and not the usual ψ. Clearly, $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is generated as a $*$-algebra by $a(y) = \frac{1}{2}\big(\phi(y) - \mathrm{i}\phi(\mathrm{i}y)\big)$, $y \in \mathcal{Y}$.

Proposition 17.32 (1) *A state ψ on $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is gauge-invariant if*

$$\psi\big(a^*(y_1)\cdots a^*(y_n)a(w_m)\cdots a(w_1)\big) = 0,\ n \neq m,\ y_1 \ldots, y_n, w_m, \ldots, w_1 \in \mathcal{Y}.$$

(2) *It is quasi-free if in addition, for any $y_1 \ldots, y_n, w_n, \ldots, w_1 \in \mathcal{Y}$,*

$$\psi\big(a^*(y_1)\cdots a^*(y_n)a(w_n)\cdots a(w_1)\big) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} \psi\big(a^*(y_j)a(w_{\sigma(j)})\big).$$

Definition 17.33 *If ψ is a gauge-invariant quasi-free state on $\mathrm{CAR}^{C^*}(\mathcal{Y})$, the positive Hermitian operator χ on $\mathcal{Y}$ defined by*

$$(y_2|\chi y_1) := \psi\big(a^*(y_1)a(y_2)\big),\ \ y_1, y_2 \in \mathcal{Y},$$

is called the one-particle density *of ψ.*

Recall that in the framework of neutral CAR relations one introduces the holomorphic space $\mathcal{Z}$. Charged CAR relations amount to identifying the space $\mathcal{Y}$ with $\mathcal{Z}$, as explained e.g. in Subsect. 12.1.7. Under this identification, the scalar product on $\mathcal{Y}$ is transformed into $2\nu_{\mathcal{Z}}$ and the Hermitian form defined by the one-particle density χ is transformed into $\nu_{\mathcal{Z}} - \frac{\mathrm{i}}{2}\beta_{\mathcal{Z}}$. Therefore, (17.47) implies the following proposition.

Proposition 17.34 *A Hermitian operator $\chi \in B_{\mathrm{h}}(\mathcal{Y})$ is the one-particle density of a gauge-invariant quasi-free state iff*

$$0 \leq \chi \leq \mathbb{1}.$$

Suppose now that

$$\mathcal{Y} \ni y \mapsto a^{\pi *}(y) \in B(\mathcal{H}) \tag{17.48}$$

is a charged CAR representation.

Definition 17.35 (1) *$\Psi \in \mathcal{H}$ is called a* gauge-invariant quasi-free vector *if*

$$\begin{aligned}&\psi\big(a^*(y_1)\cdots a^*(y_n)a(w_m)\cdots a(w_1)\big)\\ :=\ &\big(\Psi|a^{\pi *}(y_1)\cdots a^{\pi *}(y_n)a^{\pi}(w_m)\cdots a^{\pi}(w_1)\Psi\big), \quad y_1,\dots,y_n,w_1,\dots,w_m \in \mathcal{Y},\end{aligned}$$

defines a gauge-invariant quasi-free state on $\mathrm{CAR}^{C^*}(\mathcal{Y})$.

(2) *A charged CAR representation (17.48) is* gauge-invariant quasi-free *if there exists a cyclic gauge-invariant quasi-free vector in $\mathcal{H}$.*

Recall that with every neutral CAR representation over a unitary space we associate a charged CAR representation $\mathcal{Y} \ni y \mapsto a^{\pi *}(y) \in B(\mathcal{H})$, such that

$$\phi^{\pi}(y) = a^{\pi *}(y) + a^{\pi}(y).$$

It is clear that a vector Ψ is gauge-invariant quasi-free w.r.t. ϕ^{π} iff it is such w.r.t. $a^{\pi *}$. Likewise, the representation ϕ^{π} is gauge-invariant quasi-free iff $a^{\pi *}$ is.

17.2.4 Gibbs states of fermionic quadratic Hamiltonians

Density matrix

Let $0 \leq \gamma$ be a self-adjoint operator on a Hilbert space $\mathcal{Z}$. We associate with γ the self-adjoint operator $0 \leq \chi < \mathbb{1}$, called the *one-particle density*, defined by

$$\chi := \gamma(\mathbb{1}+\gamma)^{-1}, \ \gamma = \chi(\mathbb{1}-\chi)^{-1}. \tag{17.49}$$

Note in passing that replacing γ with γ^{-1} is equivalent to replacing χ with $\mathbb{1}-\chi$.

We assume in addition that γ is trace-class. This is equivalent to assuming that χ is trace-class. Note the following identity:

$$\mathrm{Tr}\Gamma(\gamma) = \det(\mathbb{1}+\gamma) = \det(\mathbb{1}-\chi)^{-1}.$$

Thus $\Gamma(\gamma)\det(\mathbb{1}+\gamma)^{-1}$ is a density matrix.

Definition 17.36 *We define the state ψ_γ on $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ by*

$$\psi_\gamma(A) := \mathrm{Tr}\, A\Gamma(\gamma)\det(\mathbb{1}+\gamma)^{-1}, \quad A \in \mathrm{CAR}^{C^*}(\mathcal{Z}).$$

We identify $\mathcal{Z}$ with $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ using the usual map $z \mapsto (z, \overline{z})$. We can faithfully represent the algebra $\mathrm{CAR}^{C^*}(\mathcal{Z})$ in $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$.

Proposition 17.37 *The state ψ_γ restricted to $\mathrm{CAR}^{C^*}(\mathcal{Z})$ is gauge-invariant quasi-free. We have*

$$\begin{aligned}
\psi_\gamma\big(a^{\pi*}(z_1)a^{\pi}(z_2)\big) &= (z_2|\chi z_1),\\
\psi_\gamma\big(a^{\pi}(z_1)a^{\pi*}(z_2)\big) &= (z_1|z_2) - (z_1|\chi z_2),\\
\psi_\gamma\big(a^{\pi}(z_1)a^{\pi}(z_2)\big) &= \psi_\gamma\big(a^{\pi*}(z_1)a^{\pi*}(z_2)\big) \;=\; 0, \quad z_1, z_2 \in \mathcal{Z}.
\end{aligned}$$

Proof We can find an o.n. basis $(e_1, e_2, \dots)$ diagonalizing the trace-class operator γ. Using the identification

$$\Gamma_{\mathrm{a}}(\mathcal{Z}) \simeq \mathop{\otimes}_{i=1}^{\infty}\big(\Gamma_{\mathrm{a}}(\mathbb{C}e_i), \Omega\big), \tag{17.50}$$

we can confine ourselves to the case of one degree of freedom. □

Suppose now that γ is non-degenerate. In this case, the state ψ_γ is faithful. If in addition we fix $\beta \in \mathbb{R}$, we can write $\gamma = \mathrm{e}^{-\beta h}$ for some self-adjoint operator h. Then

$$\Gamma(\gamma)\det(\mathbb{1}+\gamma)^{-1} = \mathrm{e}^{-\beta \mathrm{d}\Gamma(h)}/\mathrm{Tr}\,\mathrm{e}^{-\beta\mathrm{d}\Gamma(h)}.$$

Thus in this case ψ_γ is the Gibbs state for the dynamics $\mathrm{d}\Gamma(h)$ at the inverse temperature β.

Standard representations on Hilbert–Schmidt operators

Consider the Hilbert space $B^2\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. As in the bosonic case, we will use an alternative notation for the Hermitian conjugation: $JB := B^*$.

We will use the representations of $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ and $\overline{B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)}$ on $B^2\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ introduced in Subsect. 6.4.5:

$$\pi_{\mathrm{l}}(A)B = AB, \quad \pi_{\mathrm{r}}(\overline{A})B := BA^*, \quad B \in B^2\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big), \quad A \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big).$$

Again, $J\pi_{\mathrm{l}}(A)J^* = \pi_{\mathrm{r}}(A)$.

Thus we can introduce two commuting charged CAR representations

$$\mathcal{Z} \ni z \mapsto \pi_{\mathrm{l}}\big(a^*(z)\big) \in B\left(B^2\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)\right), \tag{17.51}$$

$$\overline{\mathcal{Z}} \ni \overline{z} \mapsto \pi_{\mathrm{r}}\big(\overline{a^*(z)}\big) \in B\left(B^2\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)\right). \tag{17.52}$$

They are interchanged by the operator J:

$$J\pi_{\mathrm{l}}\big(a^*(z)\big)J^* = \pi_{\mathrm{r}}\big(\overline{a^*(z)}\big).$$

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The vector

$$\Psi_\gamma := \det(\mathbb{1}+\gamma)^{-\frac{1}{2}}\Gamma(\gamma^{\frac{1}{2}})$$

is gauge-invariant quasi-free for the representations (17.51) and (17.52), and the one-particle density χ. If γ is non-degenerate, then both (17.51) and (17.52) are gauge-invariant quasi-free CAR representations.

Standard representations on double Fock spaces

We need to identify the complex conjugate of the Fock space $\overline{\Gamma_{\rm a}(\mathcal{Z})}$ with the Fock space over the complex conjugate $\Gamma_{\rm a}(\overline{\mathcal{Z}})$. Recall that in the bosonic case this is straightforward. In the fermionic case, however, we will not use the naive identification, but the identification that "reverses the order of particles", consistent with the convention adopted in (3.4). More precisely, if $z_1, \dots, z_n \in \mathcal{Z}$, then the identification looks as follows:

$$\overline{\Gamma_{\rm a}(\mathcal{Z})} \ni \overline{z_1 \otimes_{\rm a} \cdots \otimes_{\rm a} z_n} \ \mapsto \ \overline{z}_n \otimes_{\rm a} \cdots \otimes_{\rm a} \overline{z}_1 \in \Gamma_{\rm a}(\overline{\mathcal{Z}}). \tag{17.53}$$

(Thus this identification equals Λ times the naive, "non-reversing" identification.)

Note the following chain of identifications:

$$\begin{aligned} B^2\big(\Gamma_{\rm a}(\mathcal{Z})\big) &\simeq \Gamma_{\rm a}(\mathcal{Z}) \otimes \overline{\Gamma_{\rm a}(\mathcal{Z})} \\ &\simeq \Gamma_{\rm a}(\mathcal{Z}) \otimes \Gamma_{\rm a}(\overline{\mathcal{Z}}) \simeq \Gamma_{\rm a}(\mathcal{Z} \oplus \overline{\mathcal{Z}}). \end{aligned} \tag{17.54}$$

We denote by $T_{\rm a} : B^2\big(\Gamma_{\rm a}(\mathcal{Z})\big) \to \Gamma_{\rm a}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ the unitary map given by (17.54).

Proposition 17.38 $T_{\rm a} J T_{\rm a}^* = \Lambda\Gamma(\epsilon)$.

Proof Consider $z_1, \dots, z_n, w_1, \dots, w_m \in \mathcal{Z}$ and

$$B = |z_1 \otimes_{\rm a} \cdots \otimes_{\rm a} z_n)(w_1 \otimes_{\rm a} \cdots \otimes_{\rm a} w_m| \in B^2\big(\Gamma_{\rm a}(\mathcal{Z})\big).$$

This corresponds to

$$\begin{aligned} &\sqrt{(n+m)!}z_1 \otimes_{\rm a} \cdots \otimes_{\rm a} z_n \otimes_{\rm a} \overline{w_1 \otimes_{\rm a} \cdots \otimes_{\rm a} w_m} \\ =\ &\sqrt{(n+m)!}z_1 \otimes_{\rm a} \cdots \otimes_{\rm a} z_n \otimes_{\rm a} \overline{w}_m \otimes_{\rm a} \cdots \otimes_{\rm a} \overline{w}_1 \in \Gamma_{\rm a}(\mathcal{Z} \oplus \overline{\mathcal{Z}}). \end{aligned}$$

On the other hand,

$$B^* = |w_1 \otimes_{\rm a} \cdots \otimes_{\rm a} w_m)(z_1 \otimes_{\rm a} \cdots \otimes_{\rm a} z_n|$$

corresponds to

$$\begin{aligned} &\sqrt{(n+m)!}w_1 \otimes_{\rm a} \cdots \otimes_{\rm a} w_m \otimes_{\rm a} \overline{z_1 \otimes_{\rm a} \cdots \otimes_{\rm a} z_n} \\ =\ &\sqrt{(n+m)!}w_1 \otimes_{\rm a} \cdots \otimes_{\rm a} w_m \otimes_{\rm a} \overline{z}_n \otimes_{\rm a} \cdots \otimes_{\rm a} \overline{z}_1 \\ =\ &(-1)^{\frac{n(n-1)}{2}+\frac{m(m-1)}{2}+nm}\sqrt{(n+m)!}\overline{z}_1 \otimes_{\rm a} \cdots \otimes_{\rm a} \overline{z}_n \otimes_{\rm a} w_m \otimes_{\rm a} \cdots \otimes_{\rm a} w_1 \\ =\ &\Lambda\Gamma(\epsilon)\sqrt{(n+m)!}z_1 \otimes_{\rm a} \cdots \otimes_{\rm a} z_n \otimes_{\rm a} \overline{w}_m \otimes_{\rm a} \cdots \otimes_{\rm a} \overline{w}_1, \end{aligned}$$

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where at the last step we used $\Gamma(\epsilon)z_i = \overline{z}_i$, $\Gamma(\epsilon)\overline{w}_i = w_i$ and

$$\tfrac{n(n-1)}{2} + \tfrac{m(m-1)}{2} + nm = \tfrac{(n+m)(n+m-1)}{2}.$$

□

By applying T_a to (17.51) and (17.52), we obtain two new commuting charged CAR representations

$$\mathcal{Z} \ni z \mapsto T_\mathrm{a}\pi_\mathrm{l}\big(a^*(z)\big)T_\mathrm{a}^* = a^*(z,0) \in B\left(\Gamma_\mathrm{a}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\right), \tag{17.55}$$

$$\overline{\mathcal{Z}} \ni \overline{z} \mapsto T_\mathrm{a}\pi_\mathrm{r}\big(\overline{a^*(z)}\big)T_\mathrm{a}^* = \Lambda a^*(0,\overline{z})\Lambda \in B\big(\Gamma_\mathrm{a}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big). \tag{17.56}$$

They are interchanged by the operator $\Lambda\Gamma(\epsilon)$:

$$\Lambda\Gamma(\epsilon)a^*(z,0)\Gamma(\epsilon)^*\Lambda^* = \Lambda a^*(0,\overline{z})\Lambda, \quad z \in \mathcal{Z}.$$

Again using a basis diagonalizing γ, as in (17.50), the double Fock space on the right of (17.54) can be written as an infinite tensor product

$$\mathop{\otimes}_{i=1}^{\infty}\big(\Gamma_\mathrm{a}(\mathbb{C}e_i \oplus \mathbb{C}\overline{e}_i), \Omega\big). \tag{17.57}$$

We have $T_\mathrm{a}\Psi_\gamma = \Omega_\gamma$, where

$$\Omega_\gamma := \mathop{\otimes}_{i=1}^{\infty}(1+\gamma_i)^{-\frac{1}{2}}\mathrm{e}^{\gamma_i^{\frac{1}{2}}a^*(e_i)a^*(\overline{e}_i)}\Omega$$

is gauge-invariant quasi-free for the representations (17.55) and (17.56), and the one-particle density χ. Clearly, both (17.55) and (17.56) are gauge-invariant quasi-free CAR representation.

Note that if we set

$$c = \begin{bmatrix} 0 & \gamma^{\frac{1}{2}} \\ -\overline{\gamma}^{\frac{1}{2}} & 0 \end{bmatrix} \in B_\mathrm{a}^2(\overline{\mathcal{Z}}\oplus\mathcal{Z}, \mathcal{Z}\oplus\overline{\mathcal{Z}}), \tag{17.58}$$

then

$$\Omega_\gamma = \det(\mathbb{1} + c^*c)^{-\frac{1}{4}}\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega,$$

so this is an example of a fermionic Gaussian vector introduced in Def. 16.35, where it was denoted Ω_c.

Araki–Wyss form of standard representation

Using the infinite tensor product decomposition (17.57), we define the following transformation on $\Gamma_\mathrm{a}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$:

$$R_\gamma := \mathop{\otimes}_{i=1}^{\infty}(1+\gamma_i)^{-\frac{1}{2}}\mathrm{e}^{\gamma_i^{\frac{1}{2}}a^*(e_i)a^*(\overline{e}_i)}\Gamma\big((1+\gamma_i)^{\frac{1}{2}}\mathbb{1}\big)\mathrm{e}^{-\gamma_i^{\frac{1}{2}}a(e_i)a(\overline{e}_i)}. \tag{17.59}$$

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Theorem 17.39 *R_γ is a unitary operator satisfying*

$$\begin{aligned}
R_\gamma \phi(z_1, \overline{z_2}) R_\gamma^* &= \phi\big((\mathbb{1}-\chi)^{\frac{1}{2}} z_1 + \chi^{\frac{1}{2}} z_2, \overline{\chi}^{\frac{1}{2}} \overline{z_1} + (\mathbb{1}-\overline{\chi})^{\frac{1}{2}} \overline{z_2}\big), \\
R_\gamma a(z_1, \overline{z_2}) R_\gamma^* &= a\big((\mathbb{1}-\chi)^{\frac{1}{2}} z_1 + \chi^{\frac{1}{2}} z_2, 0\big) \\
&\quad + a^*\big(0, \overline{\chi}^{\frac{1}{2}} \overline{z_1} + (\mathbb{1}-\overline{\chi})^{\frac{1}{2}} \overline{z_2}\big), \\
R_\gamma a^*(z_1, \overline{z_2}) R_\gamma^* &= a^*\big((\mathbb{1}-\chi)^{\frac{1}{2}} z_1 + \chi^{\frac{1}{2}} z_2, 0\big) \\
&\quad + a\big(0, \overline{\chi}^{\frac{1}{2}} \overline{z_1} + (\mathbb{1}-\overline{\chi})^{\frac{1}{2}} \overline{z_2}\big), \quad (z_1, \overline{z}_2) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}, \\
R_\gamma \Gamma(\epsilon) R_\gamma^* &= \Gamma(\epsilon), \\
R_\gamma \Omega_\gamma &= \Omega, \\
R_\gamma \mathrm{d}\Gamma(h, -\overline{h}) R_\gamma^* &= \mathrm{d}\Gamma(h, -\overline{h}).
\end{aligned}$$

Proof Let c be defined as in (17.58). Using

$$\Gamma(\mathbb{1} + cc^*) = \Gamma\big((\mathbb{1}+\gamma) \oplus (\mathbb{1}+\overline{\gamma})\big),$$

we see that

$$R_\gamma := \det(\mathbb{1} + cc^*)^{-\frac{1}{4}} \mathrm{e}^{\frac{1}{2} a^*(c)} \Gamma(\mathbb{1}+cc^*)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2} a(c)}.$$

Thus R_γ belongs to a class of operators that we know very well and we can easily show the properties mentioned in the theorem: it is the unitary operator implementing a j-positive orthogonal transformation given in (16.63). □

By applying R_γ to (17.55) and (17.56), we obtain two new commuting charged CAR representations

$$\begin{aligned}
\mathcal{Z} \ni z \mapsto a_{\gamma,\mathrm{l}}^*(z) &:= R_\gamma a^*(z, 0) R_\gamma^* \\
&= a^*\big((\mathbb{1}-\chi)^{\frac{1}{2}} z, 0\big) + a\big(0, \overline{\chi}^{\frac{1}{2}} \overline{z}\big) \in B\left(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\right), \\
\overline{\mathcal{Z}} \ni \overline{z} \mapsto a_{\gamma,\mathrm{r}}^*(\overline{z}) &:= R_\gamma \Lambda a^*(0, \overline{z}) \Lambda R_\gamma^* \\
&= \Lambda\big(a(\chi^{\frac{1}{2}} z, 0) + a^*(0, (\mathbb{1}-\chi)^{\frac{1}{2}} \overline{z})\big)\Lambda \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big).
\end{aligned}$$

They are interchanged by the operator $\Lambda\Gamma(\epsilon)$:

$$\Lambda\Gamma(\epsilon) a_{\gamma,\mathrm{l}}^*(z) \Gamma(\epsilon)^* \Lambda^* = a_{\gamma,\mathrm{r}}^*(\overline{z}), \quad z \in \mathcal{Z}.$$

We have $R_\gamma \Omega_\gamma = \Omega$, hence the Fock vacuum Ω is a quasi-free vector for the representations $a_{\gamma,\mathrm{l}}^*$ and $a_{\gamma,\mathrm{r}}^*$, and the one-particle density χ. Thus, if γ is non-degenerate, then both are gauge-invariant quasi-free CAR representations. They are special cases of Araki–Wyss charged CAR representations, which we consider more generally in the next subsection.

17.2.5 Araki–Wyss representations

In this subsection we will see that Araki–Wyss representations can be defined more generally, as compared with the framework of the previous subsection.

Let $\mathcal{Z}$ be a Hilbert space. We assume that we are given the operators γ and χ linked by the relation (17.49). This time we drop the condition that γ is trace-class. We assume only that γ is positive, possibly with a non-dense domain, and $0 \leq \chi \leq \mathbb{1}$.

Note that $\operatorname{Dom}\gamma = \operatorname{Ran}(\mathbb{1}-\chi) = \operatorname{Ker}(\mathbb{1}-\chi)^{\perp}$. We have $\operatorname{Ker}\gamma = \operatorname{Ker}\chi$, and set $\operatorname{Ker}\gamma^{-1} := \operatorname{Ker}(\mathbb{1}-\chi)$, which amounts to setting $(z|\gamma z) = +\infty$ for $z \notin \operatorname{Dom}\gamma$.

Definition 17.40 *For $z \in \mathcal{Z}$, we define the* Araki–Wyss creation operators *on* $\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$:

$$\begin{aligned} a^*_{\gamma,\mathrm{l}}(z) &:= a^*\big((\mathbb{1}-\chi)^{\frac12} z, 0\big) + a\big(0, \overline{\chi}^{\frac12}\overline{z}\big),\\ a^*_{\gamma,\mathrm{r}}(\overline{z}) &:= \Big(-a\big(\chi^{\frac12} z, 0\big) + a^*\big(0, (\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\big)\Big) I\\ &= \Lambda\Big(a\big(\chi^{\frac12} z, 0\big) + a^*\big(0, (\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\big)\Big)\Lambda \;=\; \Lambda a_{\gamma^{-1},\mathrm{l}}(z)\Lambda. \end{aligned}$$

For completeness let us write down the adjoints of Araki–Wyss creation operators, called *Araki–Wyss annihilation operators*:

$$\begin{aligned} a_{\gamma,\mathrm{l}}(z) &:= a\big((\mathbb{1}-\chi)^{\frac12} z, 0\big) + a^*\big(0, \overline{\chi}^{\frac12}\overline{z}\big),\\ a_{\gamma,\mathrm{r}}(\overline{z}) &:= \Big(a^*\big(\chi^{\frac12} z, 0\big) - a\big(0, (\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\big)\Big) I\\ &= \Lambda\Big(a^*\big(\chi^{\frac12} z, 0\big) + a\big(0, (\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\big)\Big)\Lambda \;=\; \Lambda a^*_{\gamma^{-1},\mathrm{l}}(z)\Lambda. \end{aligned}$$

We also have *Araki–Wyss field operators*:

$$\begin{aligned} \phi_{\gamma,\mathrm{l}}(z) &:= a^*_{\gamma,\mathrm{l}}(z) + a_{\gamma,\mathrm{l}}(z) = \phi\big((\mathbb{1}-\chi)^{\frac12} z, \overline{\chi}^{\frac12}\overline{z}\big),\\ \phi_{\gamma,\mathrm{r}}(\overline{z}) &:= a^*_{\gamma,\mathrm{r}}(\overline{z}) + a_{\gamma,\mathrm{r}}(\overline{z}) = -\mathrm{i}\phi\big(\mathrm{i}\chi^{\frac12} z, \mathrm{i}(\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\big) I\\ &= \Lambda\phi\big(\chi^{\frac12} z, (\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\big)\Lambda = \Lambda\phi_{\gamma^{-1},\mathrm{l}}(z)\Lambda. \end{aligned}$$

(See (3.30) for identities concerning Λ.)

Definition 17.41 *The von Neumann algebras generated by $\{a^*_{\gamma,\mathrm{l}}(z): \ z \in \mathcal{Z}\}$, resp. $\{a^*_{\gamma,\mathrm{r}}(\overline{z}): \ z \in \mathcal{Z}\}$ will be denoted by $\mathrm{CAR}_{\gamma,\mathrm{l}}$, resp. $\mathrm{CAR}_{\gamma,\mathrm{r}}$ and called the* left*, resp.* right Araki–Wyss algebras.

Clearly,

$$\mathrm{CAR}_{\gamma,\mathrm{r}} = \Lambda \mathrm{CAR}_{\gamma,\mathrm{l}} \Lambda.$$

Theorem 17.42 (1) *The map*

$$\mathcal{Z} \ni z \mapsto a^*_{\gamma,\mathrm{l}}(z) \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big)$$

is a charged CAR representation. In particular

$$\begin{aligned} [a_{\gamma,\mathrm{l}}(z_1), a^*_{\gamma,\mathrm{l}}(z_2)]_+ &= (z_1|z_2),\\ [a^*_{\gamma,\mathrm{l}}(z_1), a^*_{\gamma,\mathrm{l}}(z_2)]_+ &= [a_{\gamma,\mathrm{l}}(z_1), a_{\gamma,\mathrm{l}}(z_2)]_+ = 0. \end{aligned}$$

It will be called the left Araki–Wyss (charged CAR) representation.

(2) *The map*

$$\overline{\mathcal{Z}} \ni \overline{z} \mapsto a^*_{\gamma,\mathrm{r}}(\overline{z}) \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big)$$

is a charged CAR representation. In particular

$$\begin{aligned}
&[a_{\gamma,\mathrm{r}}(z_1), a^*_{\gamma,\mathrm{r}}(z_2)]_+ = \overline{(z_1|z_2)},\\
&[a^*_{\gamma,\mathrm{r}}(z_1), a^*_{\gamma,\mathrm{r}}(z_2)]_+ = [a_{\gamma,\mathrm{r}}(z_1), a_{\gamma,\mathrm{r}}(z_2)]_+ = 0.
\end{aligned}$$

It will be called the right Araki–Wyss (charged CAR) representation.

(3) *Set*

$$J_{\mathrm{a}} := \Lambda\Gamma(\epsilon). \tag{17.60}$$

Then

$$\begin{aligned}
J_{\mathrm{a}} a^*_{\gamma,\mathrm{l}}(z) J_{\mathrm{a}} &= a^*_{\gamma,\mathrm{r}}(\overline{z}),\\
J_{\mathrm{a}} a_{\gamma,\mathrm{l}}(z) J_{\mathrm{a}} &= a_{\gamma,\mathrm{r}}(\overline{z}).
\end{aligned}$$

(4) *The vacuum* Ω *is a fermionic quasi-free vector for* $a^*_{\gamma,\mathrm{l}}$ *with the 2-point functions*

$$\begin{aligned}
\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= \big(z_1|(\mathbb{1}-\chi)z_2\big) = \big(z_1|(\mathbb{1}+\gamma)^{-1}z_2\big),\\
\big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= (z_2|\chi z_1) \quad = \quad \big(z_2|\gamma(\mathbb{1}+\gamma)^{-1}z_1\big),\\
\big(\Omega|a_{\gamma,\mathrm{l}}(z_1)a_{\gamma,\mathrm{l}}(z_2)\Omega\big) &= \big(\Omega|a^*_{\gamma,\mathrm{l}}(z_1)a^*_{\gamma,\mathrm{l}}(z_2)\Omega\big) \ = \ 0.
\end{aligned}$$

(5) $\mathrm{CAR}_{\gamma,\mathrm{l}}$ *is a factor.*

(6) $\operatorname{Ker}\gamma = \operatorname{Ker}\gamma^{-1} = \{0\}$ *(equivalently,* $\operatorname{Ker}\chi = \operatorname{Ker}(\mathbb{1}-\chi) = \{0\}$*) iff* Ω *is separating for* $\mathrm{CAR}_{\gamma,\mathrm{l}}$ *iff* Ω *is cyclic for* $\mathrm{CAR}_{\gamma,\mathrm{l}}$*. If this is the case, then* J_{a} *and* $\Delta = \Gamma(\gamma \oplus \overline{\gamma}^{-1})$ *are the modular conjugation and modular operator for* $(\mathrm{CAR}_{\gamma,\mathrm{l}}, \Omega)$*.*

(7) *We have*

$$\mathrm{CAR}'_{\gamma,\mathrm{l}} = \mathrm{CAR}_{\gamma,\mathrm{r}}. \tag{17.61}$$

(8) *If* $\chi = \frac{1}{2}\mathbb{1}$ *(or, equivalently,* $\gamma = \mathbb{1}$*), then the Araki–Wyss representation coincides with the real-wave representation and* $\mathrm{CAR}_{\gamma,\mathrm{l}}$ *coincides with* $\mathrm{CAR}^{W^*}(\mathcal{Z})$*.*

Proof Items (1) to (4) follow by straightforward computations.

The proof of (5) uses Prop. 6.44. First note that

$$[\phi_{\gamma,\mathrm{l}}(z_1), \phi_{\gamma,\mathrm{r}}(\overline{z}_2)] = 0.$$

Consequently $\mathrm{CAR}_{\gamma,\mathrm{l}}$ and $\mathrm{CAR}_{\gamma,\mathrm{r}}$ commute with one another. Therefore,

$$(\mathrm{CAR}_{\gamma,\mathrm{l}} \cup \mathrm{CAR}'_{\gamma,\mathrm{l}})'' \supset \mathrm{CAR}_{\gamma,\mathrm{l}} \cup \mathrm{CAR}_{\gamma,\mathrm{r}}.$$

It is easy to see that Ω is cyclic for $\mathrm{CAR}_{\gamma,\mathrm{l}} \cup \mathrm{CAR}_{\gamma,\mathrm{r}}$, which means that Condition (1) of Prop. 6.44 is satisfied for the vector Ω.

Set

$$\begin{aligned} b(z) &= a_{\gamma,\mathrm{l}}\big((\mathbb{1}-\chi)^{\frac12} z\big) + a^*_{\gamma,\mathrm{r}}\big(\overline{\chi}^{\frac12}\overline{z}\big) \\ &= a(z,0) + \Big(-a\big(\chi z\big) + a^*\big(0,(\mathbb{1}-\overline{\chi})^{\frac12}\overline{\chi}^{\frac12}\overline{z}\big)\Big)(\mathbb{1}-I), \\ b(\overline{z}) &= a_{\gamma,\mathrm{l}}\big(\chi^{\frac12} z\big) - a^*_{\gamma,\mathrm{r}}\big((\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\big) \\ &= a(0,\overline{z}) + \Big(a^*\big((\mathbb{1}-\chi)^{\frac12}\chi^{\frac12}z,0\big) - a\big(0,(\mathbb{1}-\overline{\chi})\overline{z}\big)\Big)(\mathbb{1}-I). \end{aligned}$$

For Condition (2) of Prop. 6.44, the set $\mathfrak{L}$ is defined as

$$\mathfrak{L} := \big\{b(z)\ :\ z\in\mathcal{Z}\big\}\cup\big\{b(\overline{z})\ :\ z\in\mathcal{Z}\big\}.$$

Suppose that Ψ is annihilated by all elements of $\mathfrak{L}$. All of them anti-commute with I; therefore they separately annihilate the even and odd parts of Ψ, i.e. $\Psi_\pm := \frac12(\mathbb{1}+I)\Psi$. We have

$$\begin{aligned} b(z)\Psi_+ &= a(z,0)\Psi_+ = 0, \\ b(\overline{z})\Psi_+ &= a(0,\overline{z})\Psi_+ = 0. \end{aligned}$$

Therefore, Ψ_+ is proportional to Ω, the Fock vacuum. Moreover,

$$\begin{aligned} b(z)\Psi_- &= \Big(a\big((\mathbb{1}-2\chi)z,0\big) + a^*\big(0,(\mathbb{1}-\overline{\chi})^{\frac12}\overline{\chi}^{\frac12}\overline{z}\big)\Big)\Psi_- = 0, \\ b(\overline{z})\Psi_- &= \Big(a^*\big((\mathbb{1}-\chi)^{\frac12}\chi^{\frac12}z,0\big) - a\big(0,(\mathbb{1}-2\overline{\chi})\overline{z}\big)\Big)\Psi_- = 0. \end{aligned} \tag{17.62}$$

Define $\mathcal{Z}_0 := \mathrm{Ker}(\chi-\frac12\mathbb{1})$, and $\mathcal{Z}_1 := \mathcal{Z}_0^\perp$, so that $\mathcal{Z} = \mathcal{Z}_0\oplus\mathcal{Z}_1$. We can rewrite (17.62) as

$$\begin{aligned} \Big(a\big(w_1,0\big) + a^*\big(0,(\mathbb{1}-\overline{\chi})^{\frac12}\overline{\chi}^{\frac12}(\mathbb{1}-2\chi)^{-1}\overline{w}_1\big)\Big)\Psi_- = 0, &\quad w_1\in\mathcal{Z}_1, \\ \Big(a^*\big((\mathbb{1}-\chi)^{\frac12}\chi^{\frac12}(-\mathbb{1}+2\chi)^{-1}w_1,0\big) + a\big(0,\overline{w}_1\big)\Big)\Psi_- = 0, &\quad w_1\in\mathcal{Z}_1, \\ a^*(0,\overline{w}_0)\Psi_- = a^*(w_0,0)\Psi_- = 0, &\quad w_0\in\mathcal{Z}_0. \end{aligned} \tag{17.63}$$

By Lemma 16.46, Ψ_- can be non-zero only if $\dim\mathcal{Z}_0$ is finite. If this is the case, by Thm. 16.36 and arguments of Subsect. 16.3.5, Ψ_- is proportional to a fermionic Gaussian vector tensored with an even ceiling vector. In any case, this means that Ψ_- is even. But we know that Ψ_- is odd. Hence, $\Psi_- = 0$.

Therefore, Ψ is proportional to Ω. Hence, Condition (2) of Prop. 6.44 is satisfied. This proves that $\mathrm{CAR}_{\gamma,\mathrm{l}}$ is a factor and ends the proof of (5).

Let us now prove (6). Assume that $\mathrm{Ker}\,\gamma = \mathrm{Ker}\,\gamma^{-1} = \{0\}$. Set

$$\tau^t(A) := \Gamma(\gamma\oplus\overline{\gamma}^{-1})^{\mathrm{i}t}A\Gamma(\gamma\oplus\overline{\gamma}^{-1})^{-\mathrm{i}t}.$$

We first see that $\tau^t(\phi_{\gamma,\mathrm{l}}(z)) = \phi_{\gamma,\mathrm{l}}(\gamma^{\mathrm{i}t}z)$, hence τ^t preserves $\mathrm{CAR}_{\gamma,\mathrm{l}}$ and is a W^*-dynamics on $\mathrm{CAR}_{\gamma,\mathrm{l}}$. Next we check that Ω is a $(\tau,-1)$-KMS vector. This is straightforward for the field operators $\phi_{\gamma,\mathrm{l}}(z)$. For products of field operators we use the identities of Prop. 17.32. By Prop. 6.64 we extend the KMS condition to

$\mathrm{CAR}_{\gamma,1}$. Applying Prop. 6.65 to the factor $\mathrm{CAR}_{\gamma,1}$, we obtain that Ω is separating for $\mathrm{CAR}_{\gamma,1}$.

We denote by $\mathcal{H}$ the closure of $\mathrm{CAR}_{\gamma,1}\Omega$. The vector Ω is cyclic and separating for $\mathrm{CAR}_{\gamma,1}$ restricted to $\mathcal{H}$. Therefore, we can compute the operators that belong to the modular theory for Ω, as operators on $\mathcal{H}$.

We fix an o.n. basis $\{f_j\}_{j\in J}$ of $\mathcal{Z}$. Since $\operatorname{Ker}\gamma = \operatorname{Ker}\gamma^{-1} = \{0\}$, we can moreover assume that $f_j \in \operatorname{Dom}\gamma^{\frac{1}{2}} \cap \operatorname{Dom}\gamma^{-\frac{1}{2}}$. Clearly, the family $\{e_i\}_{i\in I} = \{f_j, \mathrm{i}f_j\}_{j\in J}$ is an o.n. basis of $\mathcal{Z}$ for the Euclidean scalar product $\mathrm{Re}(\cdot|\cdot)$. Set

$$\mathcal{H}_1 := \mathrm{Span}\Big\{ \prod_{i\in I_1} \phi_{\gamma,1}(e_i)\Omega,\ I_1 \subset I \text{ finite}\Big\}.$$

Clearly, $\mathcal{H}_1$ is a dense subspace of $\mathcal{H}$. We will prove that

$$S = J_{\mathrm{a}}\Gamma(\gamma \oplus \overline{\gamma}^{-1})^{\frac{1}{2}} \text{ on } \mathcal{H}_1. \tag{17.64}$$

Let $(e_1,\dots,e_n)$ be a finite family in $\{e_i\}_{i\in I}$ and

$$\Phi := \prod_{i=1}^{n} \phi_{\gamma,1}(e_i)\Omega.$$

We have

$$S\Phi = \prod_{i=n}^{1} \phi_{\gamma,1}(e_i)\Omega = (-1)^{n(n-1)/2} \prod_{i=1}^{n} \phi_{\gamma,1}(e_i)\Omega.$$

Note that

$$\Phi = \prod_{i=1}^{n} \big(a^*(u_i) + a(u_i)\big)\Omega,$$

for $u_i = \big((\mathbb{1}-\chi)^{\frac{1}{2}} e_i, \overline{\chi}^{\frac{1}{2}}\overline{e_i}\big)$. To compute $\Gamma(\gamma \oplus \overline{\gamma}^{-1})^{\frac{1}{2}}\Phi$, we apply Prop. 3.53 (1). We obtain

$$\begin{aligned}&\Gamma(\gamma \oplus \overline{\gamma}^{-1})^{\frac{1}{2}}\Phi\\ &= \prod_{i=1}^{n}\Big(a^*\big(\chi^{\frac{1}{2}} e_i, (\mathbb{1}-\overline{\chi})^{\frac{1}{2}}\overline{e_i}\big) + a\big(\chi^{-\frac{1}{2}}(\mathbb{1}-\chi)e_i, \overline{\chi}(\mathbb{1}-\overline{\chi})^{-\frac{1}{2}}\overline{e_i}\big)\Big)\Omega,\end{aligned}$$

and hence

$$\begin{aligned}&\Gamma(\epsilon)\Gamma(\gamma \oplus \overline{\gamma}^{-1})^{\frac{1}{2}}\Phi\\ &= \prod_{i=1}^{n}\Big(a^*\big((\mathbb{1}-\chi)^{\frac{1}{2}} e_i, \overline{\chi}^{\frac{1}{2}}\overline{e_i}\big) + a\big(\chi(\mathbb{1}-\chi)^{-\frac{1}{2}} e_i, \overline{\chi}^{-\frac{1}{2}}(\mathbb{1}-\overline{\chi})\overline{e_i}\big)\Big)\Omega.\end{aligned}$$

Using (3.30), we finally get that

$$\begin{aligned}&\Lambda\Gamma(\epsilon)\Gamma(\gamma \oplus \overline{\gamma}^{-1})^{\frac{1}{2}}\Phi\\ &= (-1)^{n(n-1)/2}\prod_{i=1}^{n}\Big(a^*\big((\mathbb{1}-\chi)^{\frac{1}{2}} e_i, \overline{\chi}^{\frac{1}{2}}\overline{e_i}\big) - a\big(\chi(\mathbb{1}-\chi)^{-\frac{1}{2}} e_i, \overline{\chi}^{-\frac{1}{2}}(\mathbb{1}-\overline{\chi})\overline{e_i}\big)\Big)\Omega.\end{aligned}$$

Hence, to prove that $S\Phi = J_{\mathrm{a}}\Gamma(\gamma\oplus\overline{\gamma}^{-1})^{\frac{1}{2}}\Phi$, it remains to check that

$$\begin{aligned}&\prod_{i=1}^{n}\Big(a^*\big((1\!\!1-\chi)^{\frac{1}{2}}e_i,\overline{\chi}^{\frac{1}{2}}\overline{e_i}\big)+a\big((1\!\!1-\chi)^{\frac{1}{2}}e_i,\overline{\chi}^{\frac{1}{2}}\overline{e_i}\big)\Big)\Omega\\ =&\prod_{i=1}^{n}\Big(a^*\big((1\!\!1-\chi)^{\frac{1}{2}}e_i,\overline{\chi}^{\frac{1}{2}}\overline{e_i}\big)-a\big(\chi(1\!\!1-\chi)^{-\frac{1}{2}}e_i,\overline{\chi}^{-\frac{1}{2}}(1\!\!1-\overline{\chi})\overline{e_i}\big)\Big)\Omega.\end{aligned}\tag{17.65}$$

We can Wick-order both sides of (17.65) by moving annihilation operators to the right until they act on Ω. For the l.h.s., we pick terms coming from the anti-commutation relations that are products of

$$\begin{aligned}L(i,k):=&\Big((1\!\!1-\chi)^{\frac{1}{2}}e_i|(1\!\!1-\chi)^{\frac{1}{2}}e_k\Big)_{\mathcal{Z}}+\Big(\overline{\chi}^{\frac{1}{2}}\overline{e_i}|\overline{\chi}^{\frac{1}{2}}\overline{e_k}\Big)_{\overline{\mathcal{Z}}}\\ =&\,(e_i|e_k)_{\mathcal{Z}}-(e_i|\chi e_k)_{\mathcal{Z}}+(e_k|\chi e_i)_{\mathcal{Z}}.\end{aligned}$$

For the r.h.s., we obtain identical terms with $L(i,k)$ replaced with

$$\begin{aligned}R(i,k):=&-\Big(\chi(1\!\!1-\chi)^{-\frac{1}{2}}e_i|(1\!\!1-\chi)^{\frac{1}{2}}e_k\Big)_{\mathcal{Z}}-\Big(\overline{\chi}^{-\frac{1}{2}}(1\!\!1-\overline{\chi})\overline{e_i}|\overline{\chi}^{\frac{1}{2}}\overline{e_k}\Big)_{\overline{\mathcal{Z}}}\\ =&-(e_k|e_i)_{\mathcal{Z}}-(e_i|\chi e_k)_{\mathcal{Z}}+(e_k|\chi e_i)_{\mathcal{Z}}.\end{aligned}$$

Therefore,

$$L(i,k)-R(i,k)=2\mathrm{Re}(e_i|e_k)=2\delta_{i,k}.$$

This ends the proof of (17.64).

By Prop. 6.59, we know that the closure of $S\big|_{\mathcal{H}_1}$ equals S. Moreover, we easily see that $\Gamma(\gamma\oplus\overline{\gamma}^{-1})^{\frac{1}{2}}$ preserves $\mathcal{H}$ and is essentially self-adjoint on $\mathcal{H}_1$. Since J_{a} is isometric, (17.64) implies that $S=J_{\mathrm{a}}\Gamma(\gamma\oplus\overline{\gamma}^{-1})^{\frac{1}{2}}$, as an identity between closed operators on $\mathcal{H}$. It also proves that the modular conjugation is given by $J_{\mathrm{a}}\big|_{\mathcal{H}}$ and the modular operator is given by $\Gamma(\gamma,\overline{\gamma}^{-1})\big|_{\mathcal{H}}$.

Now,

$$\begin{aligned}&\phi_{\gamma,\mathrm{l}}(z_1)\cdots\phi_{\gamma,\mathrm{l}}(z_n)J_{\mathrm{a}}\phi_{\gamma,\mathrm{l}}(w_m)\cdots\phi_{\gamma,\mathrm{l}}(w_1)\Omega\\ =&\,\phi_{\gamma,\mathrm{l}}(z_1)\cdots\phi_{\gamma,\mathrm{l}}(z_n)\phi_{\gamma,\mathrm{r}}(\overline{w}_m)\cdots\phi_{\gamma,\mathrm{r}}(\overline{w}_1)\Omega.\end{aligned}$$

This easily implies that $\mathrm{CAR}_{\gamma,\mathrm{l}}J_{\mathrm{a}}\mathrm{CAR}_{\gamma,\mathrm{l}}\Omega$ is dense in $\Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. But

$$\mathrm{CAR}_{\gamma,\mathrm{l}}J_{\mathrm{a}}\mathrm{CAR}_{\gamma,\mathrm{l}}\Omega\subset\mathcal{H},$$

hence $\mathcal{H}$ is dense in $\Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$, which proves that Ω is cyclic. This ends the proof of the $\Rightarrow$ part of (6), as well as giving the formulas for the modular conjugation and the modular operator.

We first prove (7) under the assumption that $\mathrm{Ker}\,\gamma=\mathrm{Ker}\,\gamma^{-1}=\{0\}$. By the $\Rightarrow$ part of (6), we know that Ω is cyclic and separating for $\mathrm{CAR}_{\gamma,\mathrm{l}}$, and J_{a} is the modular conjugation for Ω. Applying the modular theory, we have $\mathrm{CAR}_{\gamma,\mathrm{l}}'=J_{\mathrm{a}}\mathrm{CAR}_{\gamma,\mathrm{l}}J_a=\mathrm{CAR}_{\gamma,\mathrm{r}}$ by (3).

To prove the general case, we will invoke some of the results to be proven only in the next section. Set $\mathcal{Z}_0=\mathrm{Ker}\,\chi$, $\mathcal{Z}_2=\mathrm{Ker}(1\!\!1-\chi)$, and write

$$\mathcal{Z}=\mathcal{Z}_0\oplus\mathcal{Z}_1\oplus\mathcal{Z}_2.\tag{17.66}$$

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We set

$$\mathcal{V} := \left\{ \left((1\!\!1 - \chi)^{\frac{1}{2}} z, \overline{\chi}^{\frac{1}{2}} \overline{z} \right) \ : \ z \in \mathcal{Z} \right\},$$

which is a closed real subspace of $\mathcal{W} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. From (17.66) we obtain

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2,$$

for

$$\mathcal{V}_0 = \{(z_0, 0) \ : \ z_0 \in \mathcal{Z}_0\}, \qquad \mathcal{V}_2 = \{(0, \overline{z}_2) \ : z_2 \in \mathcal{Z}_2\},$$
$$\mathcal{V}_1 = \left\{ \left((1\!\!1 - \chi_1)^{\frac{1}{2}} z_1, \overline{\chi_1}^{\frac{1}{2}} \overline{z}_1 \right) \ : \ z_1 \in \mathcal{Z}_1 \right\}.$$

We have, with the notation in Subsect. 17.3.1,

$$\mathrm{i}\mathcal{V}_0^{\mathrm{perp}} = \{(0, \overline{z}_0) \ : \ z_0 \in \mathcal{Z}_0\}, \qquad \mathrm{i}\mathcal{V}_2^{\mathrm{perp}} = \{(z_2, 0) \ : z_2 \in \mathcal{Z}_2\},$$
$$\mathrm{i}\mathcal{V}_1^{\mathrm{perp}} = \left\{ \left(\chi_1^{\frac{1}{2}} z_1, (1\!\!1 - \overline{\chi_1})^{\frac{1}{2}} \overline{z}_1 \right) \ : \ z_1 \in \mathcal{Z}_1 \right\}.$$

With the notation of Subsect. 17.3.5, $\mathrm{CAR}_{\gamma,1}$ is identified with $\mathrm{CAR}(\mathcal{V})$, hence (7) follows from Thm. 17.61.

It remains to prove the $\Leftarrow$ part of (6). If $\mathrm{Ker}\,\chi \neq \{0\}$, then

$$\Gamma_{\mathrm{a}}(\{0\} \oplus \overline{\mathcal{Z}_0}) \perp \mathrm{CAR}_{\gamma,1}\Omega$$

and $a_{\gamma,1}(z_0)\Omega = 0$ for $z_0 \in \mathcal{Z}_0$, hence Ω is neither cyclic not separating for $\mathrm{CAR}_{\gamma,1}$.

Similarly, if $\mathrm{Ker}(1\!\!1 - \chi) \neq \{0\}$, then $\Gamma_{\mathrm{a}}(\mathcal{Z}_2 \oplus \{0\}) \perp \mathrm{CAR}_{\gamma,1}\Omega$ and $a^*_{\gamma,1}(z_2)\Omega = 0$ for $z_2 \in \mathcal{Z}_2$. This completes the proof of (6). □

17.2.6 Quasi-free CAR representations as Araki–Wyss representations

Every quasi-free charged CAR representation can be realized as an Araki–Wyss representation.

Theorem 17.43 *Let $\mathcal{Z}$ be a Hilbert space. Let*

$$\mathcal{Z} \ni z \mapsto a^{\pi *}(z) \in B(\mathcal{H})$$

be a charged CAR representation with a gauge-invariant cyclic quasi-free vector Ψ. Let χ be defined by

$$\overline{z}_1 \cdot \chi z_2 = \left(\Psi | a^{\pi *}(z_2) a^{\pi}(z_1) \Psi \right), \quad z_1, z_2 \in \mathcal{Z}.$$

Then, for $\gamma := \chi(1\!\!1 - \chi)^{-1}$, there exists an isometry $U : \mathcal{H} \to \Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ such that

$$U\Psi = \Omega,$$
$$U a^{\pi *}(z) = a^*_{\gamma,1}(z) U, \quad z \in \mathcal{Z}.$$

17.2.7 Free Fermi gas at positive temperatures

This subsection is parallel to Subsect. 17.1.7 about the free Bose gas. We start with h, a positive self-adjoint operator on a Hilbert space $\mathcal{Z}$. Consider a quantum system described by the Hamiltonian $H := \mathrm{d}\Gamma(h)$ on the Hilbert space $\Gamma_{\mathrm{a}}(\mathcal{Z})$. Clearly, $(\Omega| \cdot \Omega)$ describes the ground state of the system. On the algebra $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ we have the dynamics

$$\tau^t(A) := \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}, \quad A \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big), \quad t \in \mathbb{R}.$$

We also have a natural charged CAR representation $\mathcal{Z} \ni z \mapsto a^*(z) \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ and the corresponding neutral CAR representation $\mathcal{Z} \ni z \mapsto \phi(z) = a^*(z) + a(z) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. They satisfy

$$\tau^t\big(a^*(z)\big) = a^*(\mathrm{e}^{\mathrm{i}th} z), \quad \tau^t\big(\phi(z)\big) = \phi(\mathrm{e}^{\mathrm{i}th} z), \quad z \in \mathcal{Z}.$$

Suppose that we consider the above quantum system at a positive temperature. Let $\beta \geq 0$ denote the inverse temperature. If

$$\mathrm{Tr}\, \mathrm{e}^{-\beta h} < \infty, \tag{17.67}$$

we can consider the Gibbs state given by the density matrix

$$\mathrm{e}^{-\beta \mathrm{d}\Gamma(h)} / \mathrm{Tr}\, \mathrm{e}^{-\beta \mathrm{d}\Gamma(h)}. \tag{17.68}$$

Again, the formalism based on the Gibbs state with the density matrix (17.68) breaks down at infinite volume, for instance in the case of (17.36).

As in the case of the Bose gas, we distinguish three possible formalisms for infinitely extended systems:

(1) the thermodynamic limit,
(2) the W^* approach,
(3) the C^* approach.

The general framework of the thermodynamic limit in the Fermi case is analogous to that in the Bose case. Therefore, we do not describe it separately.

W^* approach

The W^*-approach to free Fermi systems is also analogous to that for Bose systems. We just replace Araki–Woods representations with Araki–Wyss representations. Let us, however, describe this in detail, apologizing to the reader for almost verbatim repetitions from the bosonic case.

Consider the space $\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. For $z \in \mathcal{Z}$, define

$$a^*_\beta(z) := a^*\big((\mathbb{1} + \mathrm{e}^{-\beta h})^{-\frac{1}{2}} z, 0\big) + a\big(0, (\mathbb{1} + \mathrm{e}^{\beta \overline{h}})^{-\frac{1}{2}} \overline{z}\big). \tag{17.69}$$

Then

$$\mathcal{Z} \ni z \mapsto a^*_\beta(z) \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\big)$$

is a charged CAR representation. In fact, it is the Araki–Wyss representation for the *Fermi–Dirac density* $(\mathbb{1}+\mathrm{e}^{\beta h})^{-1}$. The von Neumann algebra generated by (17.69) will be denoted by CAR_β. Set

$$L := \mathrm{d}\Gamma\left(h \oplus (-\overline{h})\right).$$

Then

$$\tau_\beta^t(A) := \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}, \quad A \in \mathrm{CAR}_\beta,$$

is a W^*-dynamics on CAR_β. The state

$$\omega_\beta(A) := (\Omega|A\Omega), \quad A \in \mathrm{CAR}_\beta,$$

is a β-KMS state for the W^*-dynamics τ_β.

C^* approach

Again, the C^* approach for fermions follows the same lines as the C^* approach for bosons. There is, however, a difference: there exists a natural choice of a C^*-algebra, which seemed not to be the case for bosons.

Consider the C^*-algebra $\mathrm{CAR}^{C^*}(\mathcal{Z})$, where $\mathcal{Z}$ is equipped with the Euclidean structure $\frac{1}{2}\mathrm{Re}(\cdot|\cdot)$, as well as the usual charge symmetry. Define the dynamics on $\mathrm{CAR}^{C^*}(\mathcal{Z})$ by setting

$$\tau^t\left(a^*(z)\right) := a^*(\mathrm{e}^{\mathrm{i}th} z), \quad z \in \mathcal{Z}.$$

It is easy to see that, for any $\beta \in [-\infty, \infty]$, there exists on $\mathrm{CAR}^{C^*}(\mathcal{Z})$ a unique state β-KMS for the dynamics τ. It is the gauge-invariant quasi-free state given by

$$\omega_\beta\left(a(z_1)a^*(z_2)\right) = \left(z_1|(\mathbb{1}+\mathrm{e}^{-\beta h})^{-1} z_2\right), \quad z_1, z_2 \in \mathcal{Z}.$$

We can then pass to the GNS representation, obtaining $(\mathcal{H}_\beta, \pi_\beta, \Omega_\beta)$, and the Liouvillean L_β.

In the case of $\beta = \infty$ (the zero temperature), we obtain, up to unitary equivalence, $\mathcal{H}_\infty = \Gamma_\mathrm{s}(\mathcal{Z})$, $\pi_\infty\left(W(z)\right) = W(z)$, $\Omega_\infty = \Omega$ and $L_\infty = H$. This is the quantum system that we started with at the beginning of the subsection.

In the case $-\infty < \beta < \infty$ (positive temperatures), we obtain the Araki–Wyss representation for $\gamma = \mathrm{e}^{-\beta h}$ described in (17.69).

Note that in the fermionic case the C^*-algebraic approach is better justified than in the bosonic case. The algebra $\mathrm{CAR}^{C^*}(\mathcal{Z})$ can be viewed as a natural algebra to describe observables of a fermionic system. Because of the boundedness of fermionic fields, it is more likely that we will be able to define a dynamics on this algebra, even in the presence of non-trivial interactions.

17.3 Lattices of von Neumann algebras on a Fock space

Let $\mathcal{W}$ be a complex Hilbert space. With every real closed subspace $\mathcal{V}$ of $\mathcal{W}$ we can naturally associate the von Neumann sub-algebra $\mathfrak{M}(\mathcal{V})$ of $B\big(\Gamma_{\mathrm{s/a}}(\mathcal{W})\big)$ generated by fields based on $\mathcal{V}$. These von Neumann sub-algebras form a complete lattice. Properties of this lattice are studied in this section. They have important applications in quantum field theory.

The material of this section is closely related to the Araki–Woods and Araki–Wyss representations. In fact, the algebras $\mathrm{CCR}_{\gamma,1}$ and $\mathrm{CAR}_{\gamma,1}$ coincide with the algebras $\mathfrak{M}(\mathcal{V})$ for appropriate real subspaces $\mathcal{V}$ inside $\mathcal{Z} \oplus \overline{\mathcal{Z}}$.

17.3.1 Pair of subspaces in a Hilbert space

In this subsection we consider one of the classic problems of the theory of Hilbert spaces: how to describe a relative position of two closed subspaces.

Suppose that $\mathcal{Y}$ is a real or complex Hilbert space and $\mathcal{P}$, $\mathcal{Q}$ are closed subspaces in $\mathcal{Y}$. Let p, resp. q be the orthogonal projections onto $\mathcal{P}$, resp. $\mathcal{Q}$.

Proposition 17.44 $(\mathcal{P} \cap \mathcal{Q}) + (\mathcal{P}^{\perp} \cap \mathcal{Q}^{\perp}) = \mathrm{Ker}(p - q)$.

Proof The $\subset$ part is obvious.

Let $y \in \mathcal{Y}$. If $(p-q)y = 0$, then $y = py + (\mathbb{1} - q)y$, where $py = qy \in \mathcal{P} \cap \mathcal{Q}$ and $(\mathbb{1} - p)y = (\mathbb{1} - q)y \in \mathcal{P}^{\perp} \cap \mathcal{Q}^{\perp}$. This shows the $\supset$ part. □

Proposition 17.45 *The following conditions are equivalent:*

(1) $\mathrm{Ker}(p - q) = \{0\}$.
(2) $\mathcal{P} \cap \mathcal{Q} = \mathcal{P}^{\perp} \cap \mathcal{Q}^{\perp} = \{0\}$.
(3) $\mathcal{P} \cap \mathcal{Q} = \{0\}$ *and* $\mathcal{P} + \mathcal{Q}$ *is dense in* $\mathcal{Y}$.

Proof The equivalence of (1) and (2) follows by Prop. 17.44.

The equivalence of (2) and (3) follows by

$$\{0\} = (\mathcal{P} + \mathcal{Q})^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{Q}^{\perp}.$$

□

Definition 17.46 *We say that a pair* $(\mathcal{P}, \mathcal{Q})$ *is in* generic position *if*

$$\mathcal{P} \cap \mathcal{Q} = \mathcal{P}^{\perp} \cap \mathcal{Q}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{Q} = \mathcal{P}^{\perp} \cap \mathcal{Q} = \{0\}.$$

Set $m := p + q - \mathbb{1}$, $n := p - q$, which are bounded self-adjoint operators. The following relations are immediate:

$$\begin{aligned} n^2 &= \mathbb{1} - m^2 &&= p + q - pq - qp, \\ nm &= -mn &&= qp - pq, \\ -\mathbb{1} \le m \le \mathbb{1},\ &-\mathbb{1} \le n \le \mathbb{1}\,. \end{aligned} \tag{17.70}$$

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Proposition 17.47 *$(\mathcal{P},\mathcal{Q})$ are in generic position iff*

$$\operatorname{Ker} m = \operatorname{Ker} n = \{0\}. \tag{17.71}$$

If this is the case, we also have

$$\operatorname{Ker}(m \pm \mathbb{1}) = \{0\}, \quad \operatorname{Ker}(n \pm \mathbb{1}) = \{0\}. \tag{17.72}$$

Proof The following identities follow from Prop. 17.44:

$$\begin{aligned} \operatorname{Ker} n &= \operatorname{Ker} p \cap \operatorname{Ker} q + \operatorname{Ker}(\mathbb{1}-p) \cap \operatorname{Ker}(\mathbb{1}-q), \\ \operatorname{Ker} m &= \operatorname{Ker} p \cap \operatorname{Ker}(\mathbb{1}-q) + \operatorname{Ker}(\mathbb{1}-p) \cap \operatorname{Ker} q. \end{aligned}$$

This yields (17.71). We also obviously have

$$\begin{aligned} \operatorname{Ker}(n-\mathbb{1}) &= \operatorname{Ker}(\mathbb{1}-p) \cap \operatorname{Ker} q, & \operatorname{Ker}(n+\mathbb{1}) &= \operatorname{Ker} p \cap \operatorname{Ker}(\mathbb{1}-q), \\ \operatorname{Ker}(m-\mathbb{1}) &= \operatorname{Ker}(\mathbb{1}-p) \cap \operatorname{Ker}(\mathbb{1}-q), & \operatorname{Ker}(m+\mathbb{1}) &= \operatorname{Ker} p \cap \operatorname{Ker} q, \end{aligned}$$

which proves (17.72). □

The following result is immediate:

Proposition 17.48 *Set*

$$\begin{aligned} \mathcal{Y}_0 &:= \left(\mathcal{P}\cap\mathcal{Q} + \mathcal{P}^\perp\cap\mathcal{Q}^\perp + \mathcal{P}^\perp\cap\mathcal{Q} + \mathcal{P}\cap\mathcal{Q}^\perp\right)^\perp, \\ \mathcal{P}_0 &:= \mathcal{P}\cap\mathcal{Y}_0, \quad \mathcal{Q}_0 := \mathcal{Q}\cap\mathcal{Y}_0. \end{aligned}$$

Then the following direct sum decomposition holds:

$$\begin{aligned} \mathcal{Y} &= \mathcal{P}\cap\mathcal{Q} \oplus \mathcal{P}^\perp\cap\mathcal{Q}^\perp \oplus \mathcal{P}^\perp\cap\mathcal{Q} \oplus \mathcal{P}\cap\mathcal{Q}^\perp \oplus \mathcal{Y}_0, \\ \mathcal{P} &= \mathcal{P}\cap\mathcal{Q} \oplus \quad \{0\} \quad \oplus \quad \{0\} \quad \oplus \mathcal{P}\cap\mathcal{Q}^\perp \oplus \mathcal{P}_0, \\ \mathcal{Q} &= \mathcal{P}\cap\mathcal{Q} \oplus \quad \{0\} \quad \oplus \mathcal{P}^\perp\cap\mathcal{Q} \oplus \quad \{0\} \quad \oplus \mathcal{Q}_0. \end{aligned}$$

Moreover, the pair $(\mathcal{P}_0,\mathcal{Q}_0)$ is in generic position in $\mathcal{Y}_0$.

Theorem 17.49 *Let $(\mathcal{P},\mathcal{Q})$ be a pair of subspaces in generic position. Then the following is true:*

(1) *There exists a unitary (orthogonal in the real case) involution ϵ, a subspace $\mathcal{Z}$ of $\mathcal{Y}$ such that $\mathcal{Z}^\perp = \epsilon\mathcal{Z}$, and a self-adjoint operator χ on $\mathcal{Z}$ satisfying*

$$\begin{aligned} &0 < \chi < \tfrac{1}{2}\mathbb{1}, \\ &\left\{\left((\mathbb{1}-\chi)^{\frac12} z, \epsilon\chi^{\frac12} z\right) \ : \ z \in \mathcal{Z}\right\} = \mathcal{P}, \\ &\left\{\left(\chi^{\frac12} z, \epsilon(\mathbb{1}-\chi)^{\frac12} z\right) \ : \ z \in \mathcal{Z}\right\} = \mathcal{Q}. \end{aligned}$$

(2) *Set $\rho := \chi(\mathbb{1}-2\chi)^{-1}$. Then*

$$\begin{aligned} &\rho > 0, \\ &\left\{\left((\mathbb{1}+\rho)^{\frac12} z, \epsilon\rho^{\frac12} z\right) \ : \ z \in \operatorname{Dom}\rho^{\frac12}\right\} = \mathcal{P}, \\ &\left\{\left(\rho^{\frac12} z, \epsilon(\mathbb{1}+\rho)^{\frac12} z\right) \ : \ z \in \operatorname{Dom}\rho^{\frac12}\right\} = \mathcal{Q}. \end{aligned}$$

Proof We introduce the polar decompositions of n and m:

$$n = |n|\epsilon = \epsilon|n|, \quad m = \kappa|m| = |m|\kappa.$$

Clearly, ϵ, κ are unitary/orthogonal operators satisfying $\epsilon^2 = \kappa^2 = \mathbb{1}$. Moreover, using (17.70) we obtain

$$\kappa\epsilon = -\epsilon\kappa, \quad \epsilon m = -m\epsilon, \quad \kappa n = -n\kappa. \tag{17.73}$$

Set

$$\mathcal{Z} := \mathrm{Ker}(\kappa - \mathbb{1}) = \mathrm{Ran}\, \mathbb{1}_{]0,1[}(m).$$

We have

$$\epsilon\mathcal{Z} = \mathrm{Ker}(\kappa + \mathbb{1}) = \mathrm{Ran}\, \mathbb{1}_{]-1,0[}(m),$$

hence $\epsilon\mathcal{Z} = \mathcal{Z}^\perp$.

Let $\mathbb{1}_\mathcal{Z}$ be the orthogonal projection from $\mathcal{Y}$ onto $\mathcal{Z}$. Clearly,

$$\mathbb{1}_\mathcal{Z} = \mathbb{1}_{]0,1[}(m), \quad \epsilon\mathbb{1}_\mathcal{Z}\epsilon = \mathbb{1} - \mathbb{1}_\mathcal{Z} = \mathbb{1}_{]-1,0[}(m).$$

We claim that $\mathcal{P}$ is the closure of $p\mathcal{Z}$. Indeed, $\mathcal{P}$ is closed and contains $p\mathcal{Z}$. Let $y \in \mathcal{P} \cap (p\mathcal{Z})^\perp$. Then

$$0 = (y|p\mathbb{1}_\mathcal{Z}y) = (y|\mathbb{1}_\mathcal{Z}y) = \|\mathbb{1}_\mathcal{Z}y\|^2,$$

hence $y \in \mathcal{Z}^\perp$. Therefore, using $q = m + \mathbb{1} - p$, we obtain

$$(y|qy) = (y|my) \le 0.$$

Hence, $qy = 0$, and so $y \in \mathcal{Q}^\perp$. Remember that $y \in \mathcal{P}$, hence, by the generic position, $y = 0$.

Set $\chi := \frac{1}{2}\mathbb{1}_\mathcal{Z}(\mathbb{1} - m)$. Clearly, $0 < \chi < \frac{1}{2}\mathbb{1}$. Using $p = \frac{m+n+\mathbb{1}}{2}$, we obtain

$$\begin{aligned} p\mathbb{1}_\mathcal{Z} &= \frac{m+\mathbb{1}}{2}\mathbb{1}_\mathcal{Z} + \frac{\epsilon|n|}{2}\mathbb{1}_\mathcal{Z} \\ &= \frac{m+\mathbb{1}}{2}\mathbb{1}_\mathcal{Z} + \frac{\epsilon(\mathbb{1} - m^2)^{\frac{1}{2}}}{2}\mathbb{1}_\mathcal{Z} \\ &= \left((\mathbb{1} - \chi) + \epsilon\chi^{\frac{1}{2}}(\mathbb{1} - \chi)^{\frac{1}{2}}\right)\mathbb{1}_\mathcal{Z}. \end{aligned}$$

Thus

$$p\mathcal{Z} = \left((\mathbb{1} - \chi)^{\frac{1}{2}} + \epsilon\chi^{\frac{1}{2}}\right)(\mathbb{1} - \chi)^{\frac{1}{2}}\mathcal{Z}. \tag{17.74}$$

The operator

$$(\mathbb{1} - \chi)^{\frac{1}{2}}\mathbb{1}_\mathcal{Z} + \epsilon\chi^{\frac{1}{2}}\mathbb{1}_\mathcal{Z}$$

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is an isometry from $\mathcal{Z}$ into $\mathcal{Y}$. Therefore,

$$\left((\mathbb{1}-\chi)^{\frac{1}{2}}+\epsilon\chi^{\frac{1}{2}}\right)\mathcal{Z} \tag{17.75}$$

is closed. $(\mathbb{1}-\chi)^{\frac{1}{2}}\mathcal{Z}$ is dense in $\mathcal{Z}$. Therefore, (17.75) is the closure of (17.74).

We proved that $\mathcal{P}$ is the closure of $p\mathcal{Z}$. Hence, (17.75) equals $\mathcal{P}$. This completes the proof of the first identity of (1).

To prove (2), we note that every $z \in \mathcal{Z}$ can be written as

$$z = (\mathbb{1}+2\rho)^{\frac{1}{2}}z_1, \quad z_1 \in \operatorname{Dom}\rho^{\frac{1}{2}}.$$

We then have

$$\begin{aligned}(\mathbb{1}-\chi)^{\frac{1}{2}}z+\epsilon\chi^{\frac{1}{2}}z &= (\mathbb{1}+\rho)^{\frac{1}{2}}z_1+\epsilon\rho^{\frac{1}{2}}z_1,\\ \chi^{\frac{1}{2}}z+\epsilon(\mathbb{1}-\chi)^{\frac{1}{2}}z &= \rho^{\frac{1}{2}}z_1+\epsilon(\mathbb{1}+\rho)^{\frac{1}{2}}z_1,\end{aligned}$$

which immediately implies (2). □

17.3.2 Real subspaces in a Hilbert space

This subsection is devoted to another classic problem, closely related to the previous subsection: how to describe the position of a closed real subspace in a complex Hilbert space. This analysis will then be used in both the bosonic and the fermionic case.

Let $(\mathcal{W}, (\cdot|\cdot))$ be a complex Hilbert space. Then $(\mathcal{W}_{\mathbb{R}}, \operatorname{Re}(\cdot|\cdot))$ is a real Hilbert space and $(\mathcal{W}_{\mathbb{R}}, \operatorname{Im}(\cdot|\cdot))$ is a symplectic space. Clearly, if $\mathcal{V} \subset \mathcal{W}$ is a real vector space, $\mathcal{V}\cap \mathrm{i}\mathcal{V}$ and $\mathcal{V}+\mathrm{i}\mathcal{V}$ are complex vector spaces.

Definition 17.50 *If $\mathcal{U} \subset \mathcal{W}$, then we have three kinds of complements of $\mathcal{U}$:*

$$\begin{aligned}\mathcal{U}^{\perp} &:= \left\{w \in \mathcal{W} \ : \ (v|w)=0,\ v\in\mathcal{U}\right\},\\ \mathcal{U}^{\mathrm{perp}} &:= \left\{w \in \mathcal{W} \ : \ \operatorname{Re}(v|w)=0,\ v\in\mathcal{U}\right\},\\ \mathrm{i}\mathcal{U}^{\mathrm{perp}} &= \left\{w \in \mathcal{W} \ : \ \operatorname{Im}(v|w)=0,\ v\in\mathcal{U}\right\} = (\mathrm{i}\mathcal{U})^{\mathrm{perp}}.\end{aligned}$$

$\mathcal{U}^{\perp}$, $\mathcal{U}^{\mathrm{perp}}$, resp. $\mathrm{i}\mathcal{U}^{\mathrm{perp}}$ will be called the complex orthogonal*, the* real orthogonal*, resp. the* symplectic complement *of $\mathcal{U}$.*

Clearly, $\mathcal{U}^{\mathrm{perp}}$ and $\mathrm{i}\mathcal{U}^{\mathrm{perp}}$ are closed real vector subspaces of $\mathcal{W}$. If $\mathcal{V}$ is a complex vector subspace, then $\mathcal{V}^{\mathrm{perp}} = \mathrm{i}\mathcal{V}^{\mathrm{perp}} = \mathcal{V}^{\perp}$.

Let $\mathcal{V}$ be a closed real subspace of $\mathcal{W}$. Let us remark that $(\mathrm{i}\mathcal{V})^{\mathrm{perp}} = \mathrm{i}(\mathcal{V})^{\mathrm{perp}}$. Moreover, $\mathrm{i}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})^{\mathrm{perp}} = \mathcal{V}$.

Definition 17.51 *We will say that $\mathcal{V} \subset \mathcal{W}$* is in generic position *if*

$$\mathcal{V}\cap\mathrm{i}\mathcal{V} = \mathcal{V}\cap\mathrm{i}\mathcal{V}^{\mathrm{perp}} = \{0\}.$$

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Proposition 17.52 *The following conditions are equivalent:*

(1) $\mathcal{V}$ *is in generic position.*
(2) $(\mathcal{V}, \mathrm{i}\mathcal{V})$ *is in generic position in* $\mathcal{W}_{\mathbb{R}}$.
(3) $(\mathcal{V}, \mathrm{i}\mathcal{V}^{\mathrm{perp}})$ *is in generic position in* $\mathcal{W}_{\mathbb{R}}$.

The following result is an analog of Prop. 17.48:

Proposition 17.53 *Let* $\mathcal{V}$ *be a closed real subspace of* $\mathcal{W}$. *Set*

$$\mathcal{W}_1 := \mathcal{V} \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}} + (\mathrm{i}\mathcal{V} \cap \mathcal{V}^{\mathrm{perp}}), \quad \mathcal{V}_1 := \mathcal{V} \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}},$$

$$\mathcal{W}_+ := \mathcal{V} \cap \mathrm{i}\mathcal{V}, \qquad \mathcal{W}_- := \mathcal{V}^{\mathrm{perp}} \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}},$$

$$\mathcal{W}_0 := (\mathcal{W}_+ + \mathcal{W}_- + \mathcal{W}_1)^{\perp}, \qquad \mathcal{V}_0 := \mathcal{V} \cap \mathcal{W}_0.$$

Then the following is true:

(1) $\mathcal{W}_-$, $\mathcal{W}_+$, $\mathcal{W}_0$, $\mathcal{W}_1$ *are closed complex subspaces of* $\mathcal{W}$.
(2) *The following direct sum decompositions hold:*

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_+ \oplus \mathcal{W}_- \oplus \mathcal{W}_0,$$
$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{W}_+ \oplus \{0\} \oplus \mathcal{V}_0,$$
$$\mathrm{i}\mathcal{V}^{\mathrm{perp}} = \mathcal{V}_1 \oplus \{0\} \oplus \mathcal{W}_- \oplus \mathrm{i}\mathcal{V}_0^{\mathrm{perp}},$$

where $\mathcal{V}_0^{\mathrm{perp}}$ *is the real orthogonal of* $\mathcal{V}_0$ *inside* $\mathcal{W}_0$.
(3) $\mathcal{W}_1 \cap \mathcal{V} = \mathcal{W}_1 \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}} = \mathcal{V}_1 = \mathrm{i}\mathcal{V}_1^{\mathrm{perp}}$, *where* $\mathcal{V}_1^{\mathrm{perp}}$ *is the real orthogonal complement of* $\mathcal{V}_1$ *inside* $\mathcal{W}_1$.
(4) $\mathcal{W}_+ \cap \mathcal{V} = \mathcal{W}_+$, $\quad \mathcal{W}_+ \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}} = \{0\}$.
(5) $\mathcal{W}_- \cap \mathcal{V} = \{0\}$, $\quad \mathcal{W}_- \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}} = \mathcal{W}_-$.
(6) $\mathcal{W}_0 \cap \mathcal{V} = \mathcal{V}_0$, $\quad \mathcal{W}_0 \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}} = \mathrm{i}\mathcal{V}_0^{\mathrm{perp}}$. *Moreover,* $\mathcal{V}_0$ *is in generic position in* $\mathcal{W}_0$.

In other words, given a closed real subspace $\mathcal{V} \subset \mathcal{W}$, one can decompose $\mathcal{W}$ into four complex subspaces such that $\mathcal{V}$ decomposes into subspaces which are respectively complex, in generic position, Lagrangian and zero.

We can define the operators m, n for the pair of subspaces $\mathcal{V}$, $\mathcal{V}^{\mathrm{perp}}$, as in the previous subsection. They are self-adjoint in the sense of the real Hilbert space $\mathcal{W}_{\mathbb{R}}$. m is linear, whereas n is anti-linear on $\mathcal{W}$. Therefore, κ is unitary and ϵ is anti-unitary. We can use $\epsilon\big|_{\mathcal{Z}}$ as the (external) conjugation and identify $\epsilon\mathcal{Z}$ with $\overline{\mathcal{Z}}$. This gives a unitary identification

$$\mathcal{W} \simeq \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

Note that

$$\epsilon(z_1, \overline{z}_2) := (z_2, \overline{z}_1)$$

and $\epsilon\mathcal{V} = \mathrm{i}\mathcal{V}^{\mathrm{perp}}$.

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Now Thm. 17.49 (1) can be reformulated in the following way, which is adapted to the Araki–Wyss representation:

$$\begin{aligned}&\left\{\left((\mathbb{1}-\chi)^{\frac12}z,\overline{\chi}^{\frac12}\overline{z}\right)\ :\ z\in\mathcal{Z}\right\}=\mathcal{V},\\&\left\{\left(\chi^{\frac12}z,(\mathbb{1}-\overline{\chi})^{\frac12}\overline{z}\right)\ :\ z\in\mathcal{Z}\right\}=\mathrm{i}\mathcal{V}^{\mathrm{perp}}.\end{aligned}\tag{17.76}$$

Thm. 17.49 (2) can be reformulated as follows, which is adapted to the Araki–Woods representation:

$$\begin{aligned}&\left\{\left((\mathbb{1}+\rho)^{\frac12}z,\overline{\rho}^{\frac12}\overline{z}\right)\ :\ z\in\mathrm{Dom}\,\rho^{\frac12}\right\}=\mathcal{V},\\&\left\{\left(\rho^{\frac12}z,(\mathbb{1}+\overline{\rho})^{\frac12}\overline{z}\right)\ :\ z\in\mathrm{Dom}\,\rho^{\frac12}\right\}=\mathrm{i}\mathcal{V}^{\mathrm{perp}}.\end{aligned}\tag{17.77}$$

In the following proposition, which follows immediately from (17.76) and (17.77), for typographical reasons we will write τz for $\overline{z}$, where $z\in\mathcal{Z}$. We consider $\mathcal{W}$ as a Kähler space with the Euclidean, resp. symplectic form given by $\mathrm{Re}(\cdot|\cdot)$, resp. $\mathrm{Im}(\cdot|\cdot)$. It is equipped with an anti-involution and conjugation

$$\mathrm{j}=\begin{bmatrix}\mathrm{i}\mathbb{1}&0\\0&-\mathrm{i}\mathbb{1}\end{bmatrix},\quad \epsilon=\begin{bmatrix}0&\tau^{-1}\\\tau&0\end{bmatrix}.$$

We also have the operators χ and ρ on $\mathcal{Z}$. Recall that the notion of a j-positive orthogonal transformation was defined in Def. 16.8.

Proposition 17.54 *Let $\mathcal{V}$ be a closed real vector subspace of a complex Hilbert space $\mathcal{W}$ in generic position.*

(1) *Define the operator r_{a} on $\mathcal{W}$ by*

$$r_{\mathrm{a}}=\begin{bmatrix}(\mathbb{1}-\chi)^{\frac12}&\chi^{\frac12}\tau^{-1}\\-\overline{\chi}^{\frac12}\tau&(\mathbb{1}-\overline{\chi})^{\frac12}\end{bmatrix}.$$

Then r_{a} is a j*-positive orthogonal transformation on $\mathcal{W}$ commuting with ϵ, and $r_{\mathrm{a}}\mathcal{Z}=\mathcal{V}$.*

(2) *Define the operator $r_{\mathrm{s}}:\mathrm{Dom}(\rho^{\frac12})\oplus\overline{\mathrm{Dom}(\rho^{\frac12})}\to\mathcal{W}$ by*

$$r_{\mathrm{s}}=\begin{bmatrix}(\mathbb{1}+\rho)^{\frac12}&\rho^{\frac12}\tau^{-1}\\\overline{\rho}^{\frac12}\tau&(\mathbb{1}+\overline{\rho})^{\frac12}\end{bmatrix}.$$

Then r_{s} is a positive symplectic transformation on $\mathcal{W}$ commuting with ϵ, and $r_{\mathrm{s}}\mathrm{Dom}(\rho^{\frac12})=\mathcal{V}$.

Note that the transformations r_{s}, resp. r_{a} yield the Bogoliubov rotations implemented by the operators (17.27) and (17.59), which were used in Subsect. 17.1.4, resp. 17.2.4 to introduce the Araki–Woods, resp. Araki–Wyss representations.

17.3.3 Complete lattices

In this subsection we recall some definitions about abstract lattices. They provide a convenient language that can be used to express some properties of a class of von Neumann algebras acting on a Fock space.

Definition 17.55 *Let $(X, \leq)$ be an ordered set. Let $\{x_i \ : \ i \in I\}$ be a non-empty subset of X. One says that $u \in X$ is a* largest minorant of $\{x_i \ : \ i \in I\}$ *if*

(1) *$i \in I$ implies $u \leq x_i$,*
(2) *$v \leq x_i$ for all $i \in I$ implies $v \leq u$.*

If $\{x_i \ : \ i \in I\}$ has a largest minorant, then it is unique. The largest minorant of a set $\{x_i \ : \ i \in I\}$ is usually denoted by $\underset{i\in I}{\wedge} x_i$.

We define similarly the smallest majorant of $\{x_i \ : \ i \in I\}$, which is usually denoted by $\underset{i\in I}{\vee} x_i$.

One says that $(X, \leq)$ is a complete lattice *if every non-empty subset of X has the largest minorant and the smallest majorant. It is then equipped with the operations $\wedge$ and $\vee$.*

Definition 17.56 *One says that the complete lattice $(X, \leq)$ is* complemented *if it is equipped with a map $X \ni x \mapsto \sim x \in X$ such that*

(1) *$\sim(\sim x) = x$,*
(2) *$x_1 \leq x_2$ implies $\sim x_2 \leq \sim x_1$,*
(3) *$\sim \underset{i\in I}{\wedge} x_i = \underset{i\in I}{\vee} \sim x_i$.*

The operation $\sim$ will be called a complementation.

Let us give some examples of complemented lattices that will be useful in the sequel.

Example 17.57 (1) *Let $\mathcal{W}$ be a topological vector space. Then the set* $\mathrm{Subsp}(\mathcal{W})$ *of closed vector subspaces of $\mathcal{W}$ equipped with the order $\subset$ is a complete lattice with*

$$\underset{i\in I}{\wedge} \mathcal{V}_i = \underset{i\in I}{\cap} \mathcal{V}_i, \quad \underset{i\in I}{\vee} \mathcal{V}_i = \Big(\sum_{i\in I} \mathcal{V}_i \Big)^{\mathrm{cl}}.$$

(2) *If $\mathcal{W}$ is a (real or complex) Hilbert space, then the map $\mathcal{V} \mapsto \mathcal{V}^{\perp}$ is a complementation on $(\mathrm{Subsp}(\mathcal{W}), \subset)$.*
(3) *If $\mathcal{W}$ is a complex Hilbert space, then $(\mathrm{Subsp}(\mathcal{W}_{\mathbb{R}}), \subset)$ denotes the lattice of closed real subspaces of $\mathcal{W}$. Then $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{perp}}$ and $\mathcal{V} \mapsto \mathrm{i}\mathcal{V}^{\mathrm{perp}}$ are complementations on this lattice.*
(4) *Now let $\mathcal{H}$ be a Hilbert space and $\mathrm{vN}(\mathcal{H})$ be the set of von Neumann algebras in $B(\mathcal{H})$ equipped with the order $\subset$. Then $(\mathrm{vN}(\mathcal{H}), \subset)$ is also a complete*

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lattice with

$$\mathop{\wedge}_{i\in I}\mathfrak{M}_i = \mathop{\cap}_{i\in I}\mathfrak{M}_i, \quad \mathop{\vee}_{i\in I}\mathfrak{M}_i = \Big(\mathop{\cup}_{i\in I}\mathfrak{M}_i\Big)''.$$

The map $\mathfrak{M} \mapsto \mathfrak{M}'$ *is a complementation on* $(\mathrm{vN}(\mathcal{H}), \subset)$.

17.3.4 Lattice of von Neumann algebras on a bosonic Fock space

Let $\mathcal{W}$ be a complex Hilbert space. We identify $\mathcal{W}$ with $\mathrm{Re}(\mathcal{W}\oplus\overline{\mathcal{W}})$ using $w \mapsto \frac{1}{\sqrt{2}}(w, \overline{w})$; see (1.29). Consider the Hilbert space $\Gamma_{\mathrm{s}}(\mathcal{W})$ and the corresponding Fock representation $\mathcal{W} \ni w \mapsto W(w) \in U\big(\Gamma_{\mathrm{s}}(\mathcal{W})\big)$.

Definition 17.58 *For a real subspace* $\mathcal{V} \subset \mathcal{W}$, *we define the von Neumann algebra*

$$\mathfrak{M}_{\mathrm{s}}(\mathcal{V}) := \big\{W(w) \ : \ w \in \mathcal{V}\big\}'' \subset B\big(\Gamma_{\mathrm{s}}(\mathcal{W})\big).$$

Using von Neumann's density theorem and the fact that $\mathcal{W} \in w \mapsto W(w)$ is strongly continuous (see Thm. 9.5), we see that $\mathfrak{M}_{\mathrm{s}}(\mathcal{V}) = \mathfrak{M}_{\mathrm{s}}(\mathcal{V}^{\mathrm{cl}})$. Therefore, in the sequel it suffices to consider closed real subspaces of $\mathcal{W}$.

Theorem 17.59 (1) $\mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1) = \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_2)$ *iff* $\mathcal{V}_1 = \mathcal{V}_2$;
(2) $\mathcal{V}_1 \subset \mathcal{V}_2$ *implies* $\mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1) \subset \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_2)$;
(3) $\mathfrak{M}_{\mathrm{s}}(\mathcal{W}) = B\big(\Gamma_{\mathrm{s}}(\mathcal{W})\big)$ *and* $\mathfrak{M}_{\mathrm{s}}(\{0\}) = \mathbb{C}\mathbb{1}$;
(4) $\mathfrak{M}_{\mathrm{s}}(\vee_{i\in I}\mathcal{V}_i) = \vee_{i\in I}\mathfrak{M}_{\mathrm{s}}(\mathcal{V}_i)$;
(5) $\mathfrak{M}_{\mathrm{s}}(\cap_{i\in I}\mathcal{V}_i) = \cap_{i\in I}\mathfrak{M}_{\mathrm{s}}(\mathcal{V}_i)$;
(6) $\mathfrak{M}_{\mathrm{s}}(\mathcal{V})' = \mathfrak{M}_{\mathrm{s}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})$;
(7) $\mathfrak{M}_{\mathrm{s}}(\mathcal{V})$ *is a factor iff* $\mathcal{V} \cap \mathrm{i}\mathcal{V}^{\mathrm{perp}} = \{0\}$.

Proof To prove (1), let $\mathcal{V}_1$, $\mathcal{V}_2$ be two distinct closed subspaces. We may assume that $\mathcal{V}_2 \not\subset \mathcal{V}_1$, and hence $\mathrm{i}\mathcal{V}_1^{\mathrm{perp}} \not\subset \mathrm{i}\mathcal{V}_2^{\mathrm{perp}}$. For $w \in \mathrm{i}\mathcal{V}_1^{\mathrm{perp}} \backslash \mathrm{i}\mathcal{V}_2^{\mathrm{perp}}$, we have $W(w) \in \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1)' \backslash \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_2)'$. This implies that $\mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1)' \neq \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_2)'$, which proves (1).

(2) and (3) are immediate, as are the $\supset$ part of (4) and the $\subset$ part of (5). The $\subset$ part of (4) follows again from the strong continuity of $w \mapsto W(w)$. If we know (6), then the $\supset$ part of (5) follows from the $\subset$ part of (4). (7) follows from (1), (5) and (6).

Thus it remains to prove (6). Assume first that $\mathcal{V}$ is in generic position in $\mathcal{W}$. Then, using Thm. 17.49 and identifying $\epsilon\mathcal{Z}$ with $\overline{\mathcal{Z}}$, we obtain a decomposition $\mathcal{W} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ and a positive operator ρ on $\mathcal{Z}$ such that

$$\Big\{\Big((\mathbb{1}+\rho)^{\frac{1}{2}}z, \overline{\rho}^{\frac{1}{2}}\overline{z}\Big) \ : \ z \in \mathcal{Z}\Big\} = \mathcal{V}.$$

This implies that $\mathfrak{M}_{\mathrm{s}}(\mathcal{V})$ is the left Araki–Woods algebra $\mathrm{CCR}_{\gamma,\mathrm{l}}$. By Thm. 17.24, we know that the commutant of $\mathrm{CCR}_{\gamma,\mathrm{l}}$ is $\mathrm{CCR}_{\gamma,\mathrm{r}}$. But, again by Thm. 17.49,

$$\Big\{\Big(\rho^{\frac{1}{2}}z + (\mathbb{1}+\overline{\rho})^{\frac{1}{2}}\overline{z}\Big) \ : \ z \in \mathcal{Z}\Big\} = \mathrm{i}\mathcal{V}^{\mathrm{perp}}.$$

Therefore, $\mathrm{CCR}_{\gamma,\mathrm{r}}$ coincides with $\mathfrak{M}_{\mathrm{s}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})$. This ends the proof of (6), if $\mathcal{V}$ is in generic position.

For an arbitrary real subspace $\mathcal{V}$, we write as in Prop. 17.53:

$$\begin{aligned}\mathcal{W} &= \mathcal{W}_+ \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_-,\\ \mathcal{V} &= \mathcal{W}_+ \oplus \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{0\},\\ \mathrm{i}\mathcal{V}^{\mathrm{perp}} &= \{0\} \oplus \mathrm{i}\mathcal{V}_0^{\mathrm{perp}} \oplus \mathcal{V}_1 \oplus \mathcal{W}_-,\end{aligned}$$

where $\mathcal{V}_0$ is in generic position and $\mathcal{W}_1 = \mathbb{C}\mathcal{V}_1$. Using the exponential law, we have the unitary identifications

$$\begin{aligned}B\big(\Gamma_{\mathrm{s}}(\mathcal{W})\big) &\simeq B\big(\Gamma_{\mathrm{s}}(\mathcal{W}_+)\big) \otimes B\big(\Gamma_{\mathrm{s}}(\mathcal{W}_0)\big) \otimes B\big(\Gamma_{\mathrm{s}}(\mathcal{W}_1)\big) \otimes B\big(\Gamma_{\mathrm{s}}(\mathcal{W}_-)\big),\\ \mathfrak{M}_{\mathrm{s}}(\mathcal{V}) &\simeq B\big(\Gamma_{\mathrm{s}}(\mathcal{W}_+)\big) \otimes \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_0) \otimes \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1) \otimes \mathbb{1},\\ \mathfrak{M}_{\mathrm{s}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}}) &\simeq \mathbb{1} \otimes \mathfrak{M}_{\mathrm{s}}(\mathrm{i}\mathcal{V}_0^{\mathrm{perp}}) \otimes \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1) \otimes B\big(\Gamma_{\mathrm{s}}(\mathcal{W}_-)\big).\end{aligned}$$

Since $\mathcal{V}_0$ is in generic position in $\mathcal{W}_0$, $\mathfrak{M}_{\mathrm{s}}(\mathcal{V}_0)' = \mathfrak{M}_{\mathrm{s}}(\mathrm{i}\mathcal{V}_0^{\mathrm{perp}})$. Since $\mathcal{W}_1 = \mathbb{C}\mathcal{V}_1$, using the real-wave representation of Sect. 9.3, we see that $\mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1)' = \mathfrak{M}_{\mathrm{s}}(\mathcal{V}_1)$. Therefore, $\mathfrak{M}_{\mathrm{s}}(\mathcal{V})' = \mathfrak{M}_{\mathrm{s}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})$, which completes the proof (6). □

We can interpret Thm. 17.59 as the fact that the map $\mathcal{V} \mapsto \mathfrak{M}_{\mathrm{s}}(\mathcal{V})$ is an order preserving isomorphism between the complete lattice of closed real subspaces of $\mathcal{W}$ and the complete lattice of von Neumann algebras $\mathfrak{M}_{\mathrm{s}}(\mathcal{V}) \subset B\big(\Gamma_{\mathrm{s}}(\mathcal{W})\big)$, preserving the operations $\wedge$, $\vee$, and the complementations given respectively by the symplectic complement and the commutant.

17.3.5 Lattice of von Neumann algebras on a fermionic Fock space

In this subsection we consider the fermionic analog of Thm. 17.59. Again let $\mathcal{W}$ be a complex Hilbert space, and let us identify $\mathcal{W}$ with $\mathrm{Re}(\mathcal{W} \oplus \overline{\mathcal{W}})$ using $w \mapsto (w, \overline{w})$; see (1.29). Consider the Hilbert space $\Gamma_{\mathrm{a}}(\mathcal{W})$ and the corresponding Fock representation $\mathcal{W} \ni w \mapsto \phi(w) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{W})\big)$.

Definition 17.60 *For a real subspace $\mathcal{V} \subset \mathcal{W}$, we define the von Neumann algebra*

$$\mathfrak{M}_{\mathrm{a}}(\mathcal{V}) := \{\phi(w) \ : \ w \in \mathcal{V}\}'' \subset B\big(\Gamma_{\mathrm{a}}(\mathcal{W})\big).$$

As usual, set $\Lambda = (-1)^{N(N-\mathbb{1})/2}$.

Note first that, by the norm continuity of $\mathcal{W} \ni w \mapsto \phi(w)$, we have $\mathfrak{M}_{\mathrm{a}}(\mathcal{V}) = \mathfrak{M}_{\mathrm{a}}(\mathcal{V}^{\mathrm{cl}})$. Therefore, in the sequel it suffices to consider closed real subspaces of $\mathcal{W}$.

Theorem 17.61 (1) $\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1) = \mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2)$ *iff* $\mathcal{V}_1 = \mathcal{V}_2$,
(2) $\mathcal{V}_1 \subset \mathcal{V}_2$ *implies* $\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1) \subset \mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2)$,
(3) $\mathfrak{M}_{\mathrm{a}}(\mathcal{W}) = B\big(\Gamma_{\mathrm{s}}(\mathcal{W})\big)$ *and* $\mathfrak{M}_{\mathrm{a}}(\{0\}) = \mathbb{C}\mathbb{1}$,
(4) $\mathfrak{M}_{\mathrm{a}}(\vee_{i\in I}\mathcal{V}_i) = \vee_{i\in I}\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_i)$,

(5) $\mathfrak{M}_{\mathrm{a}}(\cap_{i\in I}\mathcal{V}_i)=\cap_{i\in I}\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_i)$,
(6) $\mathfrak{M}_{\mathrm{a}}(\mathcal{V})'=\Lambda\mathfrak{M}_{\mathrm{a}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})\Lambda$.

The proof of Thm. 17.61 is very similar to the proof of Thm. 17.59. The main additional difficulty is the behavior of the fermionic fields under the tensor product, which is studied in the following theorem.

Theorem 17.62 *Let $\mathcal{W}_i$, $i=1,2$, be two Hilbert spaces and $\mathcal{W}=\mathcal{W}_1\oplus\mathcal{W}_2$. Let us unitarily identify $\Gamma_{\mathrm{a}}(\mathcal{W})$ with $\Gamma_{\mathrm{a}}(\mathcal{W}_1)\otimes\Gamma_{\mathrm{a}}(\mathcal{W}_2)$ by the exponential law (see Subsect. 3.3.7). Let $\mathcal{V}_i\subset\mathcal{W}_i$ be closed real subspaces. Then*

$$\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1\oplus\mathcal{V}_2)\simeq\big(\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1)\otimes 1\!\!1+(-1)^{N_1\otimes N_2}1\!\!1\otimes\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2)(-1)^{N_1\otimes N_2}\big)''; \tag{17.78}$$
$$\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1\oplus\{0\})\simeq\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1)\otimes 1\!\!1; \tag{17.79}$$
$$\mathfrak{M}_{\mathrm{a}}(\mathcal{W}_1\oplus\mathcal{V}_2)\simeq B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_1)\big)\otimes\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2); \tag{17.80}$$
$$\Lambda\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1\oplus\mathcal{W}_2)\Lambda\simeq\Lambda_1\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1)\Lambda_1\otimes B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_2)\big); \tag{17.81}$$
$$\Lambda\mathfrak{M}_{\mathrm{a}}(\{0\}\oplus\mathcal{V}_2)\Lambda\simeq 1\!\!1\otimes\Lambda_2\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2)\Lambda_2. \tag{17.82}$$

Proof Clearly, for $v_1\in\mathcal{V}_1$,

$$\phi(v_1,0)\simeq\phi(v_1)\otimes 1\!\!1.$$

Therefore, (17.79) holds. By Thm. 3.56, for $v_2\in\mathcal{V}_2$, we have

$$\phi(0,v_2)\simeq(-1)^{N_1}\otimes\phi(v_2)=(-1)^{N_1\otimes N_2}1\!\!1\otimes\phi(v_2)(-1)^{N_1\otimes N_2}.$$

Therefore,

$$\mathfrak{M}_{\mathrm{a}}(\{0\}\oplus\mathcal{V}_2)\simeq(-1)^{N_1\otimes N_2}1\!\!1\otimes\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2)(-1)^{N_1\otimes N_2}. \tag{17.83}$$

Now (17.79) and (17.83) imply (17.78), which implies

$$\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1\otimes\mathcal{W}_2)\simeq(-1)^{N_1\otimes N_2}\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1)\otimes B\big(\Gamma_{\mathrm{a}}(\mathcal{V}_2)\big)(-1)^{N_1\otimes N_2}. \tag{17.84}$$

Noting that $\Lambda\simeq(-1)^{N_1\otimes N_2}\Lambda_1\otimes\Lambda_2$, (17.83) implies (17.82), and (17.84) implies (17.81). □

Proof of Thm. 17.61. To prove (1), let $\mathcal{V}_1$, $\mathcal{V}_2$ be two distinct closed subspaces. We may assume that $\mathcal{V}_2\not\subset\mathcal{V}_1$, and hence $\mathrm{i}\mathcal{V}_1^{\mathrm{perp}}\not\subset\mathrm{i}\mathcal{V}_2^{\mathrm{perp}}$. For $w\in\mathrm{i}\mathcal{V}_1^{\mathrm{perp}}\backslash\mathrm{i}\mathcal{V}_2^{\mathrm{perp}}$, using (3.30), we have $\Lambda\phi(w)\Lambda\in\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1)'\backslash\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2)'$. This implies that $\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_1)'\neq\mathfrak{M}_{\mathrm{a}}(\mathcal{V}_2)'$, which implies (1). (2) and (3) are immediate. The proof of (4), (5) are similar to the bosonic case, given (6).

It remains to prove (6). Assume first that $\mathcal{V}$ is in generic position in $\mathcal{W}$. By Prop. 17.53, we can write

$$\mathcal{W}=\mathcal{W}_0\oplus\mathcal{W}_1,$$
$$\mathcal{V}=\mathcal{V}_0\oplus\mathcal{V}_1,$$
$$\mathrm{i}\mathcal{V}^{\mathrm{perp}}=\mathrm{i}\mathcal{V}_0^{\mathrm{perp}}\oplus\mathcal{V}_1,$$

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where $\mathcal{V}_0$ is in generic position in $\mathcal{W}_0$ and $\mathrm{i}\mathcal{V}_1^{\mathrm{perp}} = \mathbb{C}\mathcal{V}_1$, where the orthogonal complement is taken inside $\mathcal{W}$. Again using Thm. 17.49, we obtain a decomposition $\mathcal{W}_0 = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ together with a self-adjoint operator $0 \leq \chi \leq \frac{1}{2}\mathbb{1}$ such that $\operatorname{Ker}\chi = \operatorname{Ker}(\chi - \frac{1}{2}\mathbb{1}) = \{0\}$ and

$$\begin{aligned} \big\{(\mathbb{1}-\chi)^{\frac{1}{2}} z + \overline{\chi}^{\frac{1}{2}}\overline{z} \ : \ z \in \mathcal{Z}\big\} \oplus \mathcal{V}_1 &= \mathcal{V}, \\ \big\{\chi^{\frac{1}{2}} z + (\mathbb{1}-\overline{\chi})^{\frac{1}{2}}\overline{z} \ : \ z \in \mathcal{Z}\big\} \oplus \mathcal{V}_1 &= \mathrm{i}\mathcal{V}^{\mathrm{perp}}. \end{aligned}$$

Then we are in the framework of Thm. 17.42, which implies that $\mathfrak{M}_{\mathrm{a}}(\mathcal{V})' = \Lambda\mathfrak{M}_{\mathrm{a}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})\Lambda$.

For an arbitrary $\mathcal{V}$, we write

$$\begin{aligned} \mathcal{W} &= \mathcal{W}_+ \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_-, \\ \mathcal{V} &= \mathcal{W}_+ \oplus \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{0\}, \\ \mathrm{i}\mathcal{V}^{\mathrm{perp}} &= \{0\} \oplus \mathrm{i}(\mathcal{V}_0 \oplus \mathcal{V}_1)^{\mathrm{perp}} \oplus \mathcal{W}_-, \end{aligned}$$

where $\mathcal{V}_0$, $\mathcal{V}_1$ are as above. Using Thm. 17.62, we have the unitary identifications

$$\begin{aligned} B\big(\Gamma_{\mathrm{a}}(\mathcal{W})\big) &\simeq B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_+)\big) \otimes B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_0 \oplus \mathbb{C}\mathcal{V}_1)\big) \otimes B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_-)\big), \\ \mathfrak{M}_{\mathrm{a}}(\mathcal{V}) &\simeq B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_+)\big) \otimes \mathfrak{M}_{\mathrm{a}}(\mathcal{V}_0 \oplus \mathcal{V}_1) \otimes \mathbb{1}. \end{aligned}$$

Let N_{01}, resp. N_{01-} be the number operator on $\Gamma_{\mathrm{a}}(\mathcal{W}_0 \oplus \mathcal{W}_1)$, resp $\Gamma_{\mathrm{a}}(\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_-)$. We define Λ_{01}, resp. Λ_{01-} in the obvious way. The commutant of $\mathfrak{M}_{\mathrm{a}}(\mathcal{V})$ is

$$\begin{aligned} \mathfrak{M}_{\mathrm{a}}(\mathcal{V})' &\simeq \mathbb{1} \otimes \mathfrak{M}_{\mathrm{a}}(\mathcal{V}_0 \oplus \mathcal{V}_1)' \otimes B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_-)\big) \\ &= \mathbb{1} \otimes \Lambda_{01}\mathfrak{M}_{\mathrm{a}}(\mathrm{i}(\mathcal{V}_0 \oplus \mathcal{V}_1)^{\mathrm{perp}})\Lambda_{01} \otimes B\big(\Gamma_{\mathrm{a}}(\mathcal{W}_-)\big) \\ &\simeq \mathbb{1} \otimes \Lambda_{01-}\mathfrak{M}_{\mathrm{a}}(\mathrm{i}(\mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{0\})^{\mathrm{perp}})\Lambda_{01-} \\ &\simeq \Lambda\mathfrak{M}_{\mathrm{a}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})\Lambda, \end{aligned}$$

again using Thm. 17.62. □

17.3.6 Even fermionic von Neumann algebras

We continue within the framework of the previous subsection.

Definition 17.63 *For a real subspace $\mathcal{V}$ of $\mathcal{W}$, we introduce the even part of the fermionic von Neumann algebra $\mathfrak{M}_{\mathrm{a}}(\mathcal{V})$:*

$$\mathfrak{M}_{\mathrm{a},0}(\mathcal{V}) := \big\{A \in \mathfrak{M}_{\mathrm{a}}(\mathcal{V}) \ : \ IAI = A\big\} = \mathfrak{M}_{\mathrm{a}}(\mathcal{V}) \cap \{I\}'.$$

Recall that we described the commutant of $\mathfrak{M}_{\mathrm{a}}(\mathcal{V})$ in terms of the symplectic complement: $\mathfrak{M}_{\mathrm{a}}(\mathcal{V})' = \Lambda\mathfrak{M}_{\mathrm{a}}(\mathrm{i}\mathcal{V}^{\mathrm{perp}})\Lambda$. If we are interested just in the even part of the commutant, the role of the symplectic complement can be to some extent taken by the real orthogonal complement:

Proposition 17.64 *We have $\mathfrak{M}_{\mathrm{a}}(\mathcal{V})' \cap \{I\}' = \mathfrak{M}_{\mathrm{a},0}(\mathcal{V}^{\mathrm{perp}})$.*

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Proof Write

$$\begin{aligned}\mathfrak{M}_{\rm a}(\mathcal{V})' \cap \{I\}' &= \Lambda \mathfrak{M}_{\rm a}(\mathrm{i}\mathcal{V}^{\rm perp})\Lambda \cap \{I\}' \\ &= \Lambda\big(\mathfrak{M}_{\rm a}(\mathrm{i}\mathcal{V}^{\rm perp}) \cap \{I\}'\big)\Lambda = \Lambda \mathfrak{M}_{\rm a,0}(\mathrm{i}\mathcal{V}^{\rm perp})\Lambda.\end{aligned}$$

Every element of $\mathfrak{M}_{\rm a,0}(\mathrm{i}\mathcal{V}^{\rm perp})$ is the strong limit of even polynomials in $\phi(v)$, where $v \in \mathrm{i}\mathcal{V}^{\rm perp}$. Since

$$\Lambda\phi(\mathrm{i}v_1)\phi(\mathrm{i}v_2)\Lambda = \phi(v_1)\phi(v_2), \quad v_1, v_2 \in \mathcal{V},$$

we have

$$\Lambda \mathfrak{M}_{\rm a,0}(\mathrm{i}\mathcal{V}^{\rm perp})\Lambda = \mathfrak{M}_{\rm a,0}(\mathcal{V}^{\rm perp}).$$

□

17.4 Notes

In the physics literature, quasi-free states go back to the early days of quantum theory. The Planck law and the Fermi–Dirac distribution belong to the oldest formulas of quantum physics – in the terminology of this chapter they describe the density of a thermal state for the free Bose, resp. Fermi gas.

In the mathematical literature, quasi-free states were first identified by Robinson (1965) and Shale–Stinespring (1964). Quasi-free representations were extensively studied, especially by Araki (1964, 1970, 1971), Araki–Shiraishi (1971), Araki–Yamagami (1982), Powers–Stoermer (1970) and van Daele (1971). Applications of quasi-free states to quantum field theory on curved space-times were studied in Kay–Wald (1991), where a result essentially equivalent to Thm. 17.12 was proven.

Araki–Woods representations first appeared in Araki–Woods (1963). Araki–Wyss representations go back to Araki–Wyss (1964).

It is instructive to use the Araki–Woods and Araki–Wyss representations as illustrations for the Tomita–Takesaki theory and for the so-called standard form of a W^*-algebra as in Haagerup (1975); see also Araki (1970), Connes (1974), Bratteli–Robinson (1987), Stratila (1981) and Dereziński–Jakšić–Pillet (2003). They are quite often used in recent works on quantum statistical physics; see e.g. Jakšić–Pillet (2002) and Dereziński–Jakšić (2003).

The relative position of two subspaces in a Hilbert space was first investigated by Dixmier (1948) and Halmos (1969). The study of a position of a real subspace in a complex Hilbert space is an important ingredient of the version of the Tomita–Takesaki theory presented by Rieffel–van Daele (1977).

The theorem about the lattice of real subspaces of a Hilbert space and the corresponding von Neumann algebras on a bosonic Fock space were first proven by Araki (1963); see also Eckmann–Osterwalder (1973). The analogous theorem about von Neumann algebras on a fermionic Fock space was apparently first given in a review article by Dereziński (2006).

Most of the chapter closely follows Dereziński (2006). The proof of the factoriality of algebras $\mathrm{CAR}_{\gamma,1}$ is due to Araki (1970).

The use of Araki–Woods and Araki–Wyss representations in the description of quantum systems at positive temperatures was advocated in papers of Jakšić–Pillet (1996, 2002); see also Dereziński–Jakšić (2001).