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On the Relationship between Interpolation of Banach Algebras and Interpolation of Bilinear Operators

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Abstract. We show that if the general real method $(\cdot\,,\,\cdot)_\Gamma$ preserves the Banach-algebra structure, then a bilinear interpolation theorem holds for $(\cdot\,,\,\cdot)_\Gamma$.

1 Introduction

Let $\bar{A} = (A_0, A_1)$ be a Banach couple, that is, two Banach spaces A_j, $j = 0, 1$, which are continuously embedded in some Hausdorff topological vector space. As is well known, applying any interpolation method $\mathfrak{F}$ to $\bar{A}$ one obtains a Banach space $\mathfrak{F}(\bar{A})$ such that the following continuous inclusions hold:

$$A_0 \cap A_1 \hookrightarrow \mathfrak{F}(\bar{A}) \hookrightarrow A_0 + A_1.$$

In addition $\mathfrak{F}(\bar{A})$ has the interpolation property for linear operators (see, for example, [1] or [13]). Classical interpolation methods are the real method $\mathfrak{F} = (\cdot\,,\,\cdot)_{\theta,q}$ and the complex method $\mathfrak{F} = (\cdot\,,\,\cdot)_{[\theta]}$. For the special case of the couple (L_1, L_∞) it turns out that $(L_1, L_\infty)_{\theta,q} = L_{p,q}$ and $(L_1, L_\infty)_{[\theta]} = L_p$, where $1/p = 1 - \theta$, $0 < \theta < 1$, and $1 \le q \le \infty$.

We shall mainly work with the general real method $\mathfrak{F} = (\cdot\,,\,\cdot)_\Gamma$, which is defined similarly to $(\cdot\,,\,\cdot)_{\theta,q}$ but replacing the usual weighted ℓ_q norm by a more general lattice norm Γ. This method was introduced by Peetre in [12]. One of its distinguishing features is that any interpolation space with respect to the couple (L_1, L_∞) can be obtained by applying the general real method for a suitable choice of the lattice Γ (see [4] or [11]).

Freedom for the choice of Γ is very useful when working with Banach algebras. So Martínez and the present authors have shown in [7] that a necessary and sufficient condition for $(\cdot\,,\,\cdot)_\Gamma$ to preserve the Banach-algebra structure is that Γ be a Banach algebra with multiplication defined as convolution. In particular, this yields that the real method $(\cdot\,,\,\cdot)_{\theta,q}$ preserves the Banach-algebra structure only if $q = 1$.

Previous results on interpolation of Banach algebras are due to Bishop [2], A. P. Calderón [5], Zafran [14], Kaijser [9], and Blanco, Kaijser, and Ransford [3], among others.

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The approach of Calderón in [5] was to prove first a bilinear (in fact, multilinear) interpolation theorem for the complex method and then, as a direct consequence, to derive that the complex method interpolates Banach algebras. However, for the case of the general real method, preservation of Banach-algebra structure was obtained by direct arguments. See [7] and also [3]. Concerning $(\cdot\,,\,\cdot)_{\theta,1}$ and related methods, the arguments in [2], [14], and [9] are also direct.

In this paper we investigate the relationship between real interpolation of Banach algebras and interpolation of bilinear operators. We show that if $(\cdot\,,\,\cdot)_\Gamma$ interpolates Banach algebras, then a bilinear interpolation theorem holds between J- and K-realizations of $(\cdot\,,\,\cdot)_\Gamma$. In fact, these two conditions are equivalent. We also show that boundedness of the Calderón transform on Γ and validity of a J_Γ-bilinear interpolation theorem is another equivalent statement to the previous conditions. As an application we get that adding certain logarithmic weights into the definition of $(\cdot\,,\,\cdot)_{\theta,q}$, the resulting method interpolates bilinear operators.

As for the organization of the paper, we start by recalling in Section 2 some basic facts concerning the general real method. Next, in Section 3, we establish the bilinear interpolation results.

2 The General Real Method

Let Γ be a Banach lattice of real valued sequences with $\mathbb{Z}$ as index set, that is, whenever $|\xi_m| \le |\mu_m|$ for each $m \in \mathbb{Z}$ and $\{\mu_m\} \in \Gamma$, then $\{\xi_m\} \in \Gamma$ and $\|\{\xi_m\}\|_\Gamma \le \|\{\mu_m\}\|_\Gamma$. We assume

$$\ell_\infty(\max(1, 2^{-m})) \subseteq \Gamma \subseteq \ell_1(\min(1, 2^{-m})). \tag{2.1}$$

Here, given any sequence $\{\omega_m\}$ of positive numbers and $1 \le q \le \infty$, we put $\ell_q(\omega_m) = \{\xi = \{\xi_m\} : \{\omega_m \xi_m\} \in \ell_q\}$.

Condition (2.1) is equivalent to

$$\{\min(1, 2^m)\} \in \Gamma \text{ and } \sup\Big\{ \sum_{m=-\infty}^{\infty} \min(1, 2^{-m})|\xi_m| : \|\xi\|_\Gamma \le 1 \Big\} < \infty \tag{2.2}$$

(see [10]).

Let e_m be the sequence which is zero at all co-ordinates but the m-th co-ordinate where it is one. We also suppose that

$$\xi = \lim_{n\to\infty} \sum_{j=-n}^{n} \xi_j e_j \quad \text{in } \Gamma \text{ for any} \quad \xi = \{\xi_m\} \in \Gamma. \tag{2.3}$$

Given any Banach couple $\bar{A} = (A_0, A_1)$, Peetre's K- and J-functionals are defined by

$$K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, \quad a \in A_0 + A_1,$$

and

$$J(t,a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \quad a \in A_0 \cap A_1.$$

The general real interpolation space, realized as a K-space in discrete way, $\bar{A}_{\Gamma;K} = (A_0, A_1)_{\Gamma;K}$, is formed by all $a \in A_0 + A_1$ such that $\{K(2^m, a)\} \in \Gamma$. The norm of $\bar{A}_{\Gamma;K}$ is $\|a\|_{\bar{A}_{\Gamma;K}} = \|\{K(2^m, a)\}\|_\Gamma$. The general J-space $\bar{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J}$ is defined as the collection of all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in A_0+A_1), with $\{u_m\} \subseteq A_0 \cap A_1$ and $\{J(2^m, u_m)\} \in \Gamma$. We put

$$\|a\|_{\bar{A}_{\Gamma;J}} = \inf\Big\{ \|\{J(2^m, u_m)\}\|_\Gamma : a = \sum_{m=-\infty}^{\infty} u_m \Big\}$$

where the infimum is taken over all representations of a as above.

Using (2.3) it is not hard to check that $A_0 \cap A_1$ is dense in $\bar{A}_{\Gamma;J}$. The spaces $\bar{A}_{\Gamma;J}$ and $\bar{A}_{\Gamma;K}$ are Banach spaces.

When $\Gamma = \ell_q(2^{-\theta m})$, the space ℓ_q with the weight $\{2^{-\theta m}\}$, K- and J-spaces coincide with the real interpolation space

$$(A_0, A_1)_{\theta,q} = (A_0, A_1)_{\ell_q(2^{-\theta m});K} = (A_0, A_1)_{\ell_q(2^{-\theta m});J}$$

(see [1], [4], or [13]). Here $1 \le q \le \infty$ and $0 < \theta < 1$.

In general K- and J-spaces do not coincide, but we have the continuous inclusion $\bar{A}_{\Gamma;K} \hookrightarrow \bar{A}_{\Gamma;J}$ for any Banach couple $\bar{A}$. This is a consequence of the so-called fundamental lemma of interpolation theory (see [1, Lemma 3.3.2]). However, as can be seen in [10, Lemma 2.5], if the Calderón transform

$$\Omega\{\xi_m\} = \Big\{ \sum_{k=-\infty}^{\infty} \min(1, 2^{m-k})\xi_k \Big\}_{m\in\mathbb{Z}}$$

is a bounded operator in Γ, then we get the equality $\bar{A}_{\Gamma;K} = \bar{A}_{\Gamma;J}$ with equivalence of norms. When we have equality, we denote any of these two spaces simply by $\bar{A}_\Gamma = (A_0, A_1)_\Gamma$. By $\|\cdot\|_{\bar{A}_\Gamma}$ we mean any of the equivalent norms $\|\cdot\|_{\bar{A}_{\Gamma;K}}$, $\|\cdot\|_{\bar{A}_{\Gamma;J}}$.

For $k \in \mathbb{Z}$, the shift operator τ_k is defined by

$$\tau_k \xi = \{\xi_{m+k}\}_{m\in\mathbb{Z}} \quad \text{for} \quad \xi = \{\xi_m\} \in \Gamma.$$

The following assumption is useful to compute with the norms of K- and J-spaces

$$\text{(2.4)} \qquad \lim_{n\to\infty} 2^{-n}\|\tau_n\|_{\Gamma,\Gamma} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\tau_{-n}\|_{\Gamma,\Gamma} = 0.$$

(see [6], [7], or [8]). For example, it is shown in [7, Lemma 2.2] that if Ω is bounded in Γ and (2.4) holds then the norm $\|\cdot\|_{\bar{A}_{\Gamma;J}}$ is equivalent on $A_0 \cap A_1$ to

$$\text{(2.5)} \qquad \|a\|^\star_{\bar{A}_{\Gamma;J}} = \inf\Big\{ \|\{J(2^m, u_m)\}\|_\Gamma : a = \textstyle\sum_{m=-\infty}^{\infty} u_m \text{ and only a finite number of } u_m \neq 0 \Big\}.$$

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3 Bilinear Operators

First we fix the terminology with a definition.

Definition 3.1 Let Γ be a lattice satisfying (2.1) and (2.3).

We say that the bilinear interpolation theorem $J_\Gamma \times J_\Gamma \to K_\Gamma$ holds if the following condition is satisfied. For any Banach couples $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{C} = (C_0, C_1)$ and for any bilinear operator T defined in $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values in $C_0 \cap C_1$ and such that

$$\|T(a,b)\|_{C_j} \le M_j \|a\|_{A_j} \|b\|_{B_j},\ j = 0, 1, \text{ for any } a \in A_0 \cap A_1, b \in B_0 \cap B_1,$$

there exists a constant $M = M(T)$ such that

$$\|T(a,b)\|_{\bar{C}_{\Gamma;K}} \le M \|a\|_{\bar{A}_{\Gamma;J}} \|b\|_{\bar{B}_{\Gamma;J}} \text{ for any } a \in A_0 \cap A_1, b \in B_0 \cap B_1.$$

If a similar condition holds when replacing $\bar{C}_{\Gamma;K}$ by $\bar{C}_{\Gamma;J}$, that is, if for all $\bar{A}$, $\bar{B}$, $\bar{C}$, and T as before, there is a constant $M' = M'(T)$ such that

$$\|T(a,b)\|_{\bar{C}_{\Gamma;J}} \le M' \|a\|_{\bar{A}_{\Gamma;J}} \|b\|_{\bar{B}_{\Gamma;J}} \text{ for any } a \in A_0 \cap A_1, b \in B_0 \cap B_1,$$

then we say that the bilinear interpolation theorem $J_\Gamma \times J_\Gamma \to J_\Gamma$ is fulfilled.

Remark 3.2 Since $A_0 \cap A_1$ is dense in $\bar{A}_{\Gamma;J}$ and $B_0 \cap B_1$ is dense in $\bar{B}_{\Gamma;J}$, if the theorem $J_\Gamma \times J_\Gamma \to K_\Gamma$ holds then T can be uniquely extended to a continuous bilinear mapping from $\bar{A}_{\Gamma;J} \times \bar{B}_{\Gamma;J}$ to $\bar{C}_{\Gamma;K}$. Similarly, in the case of the theorem $J_\Gamma \times J_\Gamma \to J_\Gamma$ the extension is from $\bar{A}_{\Gamma;J} \times \bar{B}_{\Gamma;J}$ to $\bar{C}_{\Gamma;J}$.

Note also that if the bilinear interpolation theorem $J_\Gamma \times J_\Gamma \to K_\Gamma$ holds, then the theorem $J_\Gamma \times J_\Gamma \to J_\Gamma$ is satisfied because $\bar{C}_{\Gamma;K} \hookrightarrow \bar{C}_{\Gamma;J}$.

Following [7], we say that the interpolation method $(\cdot,\cdot)_\Gamma$ preserves the Banach-algebra structure if for any Banach couple $\bar{A} = (A_0, A_1)$ formed by Banach algebras A_j with the property that multiplications in A_0 and A_1 coincide in $A_0 \cap A_1$, there exists a constant $c = c(\bar{A}_\Gamma)$ such that

$$\|ab\|_{\bar{A}_\Gamma} \le c \|a\|_{\bar{A}_\Gamma} \|b\|_{\bar{A}_\Gamma} \text{ for all } a, b \in A_0 \cap A_1.$$

Next we proceed to state and prove the result announced in the introduction.

Theorem 3.3 *Let Γ be a lattice satisfying* (2.1) *and* (2.3). *Assume also that shift operators in Γ fulfil* (2.4). *Then the following are equivalent.*

(i) *The bilinear interpolation theorem $J_\Gamma \times J_\Gamma \to K_\Gamma$ holds.*
(ii) *Γ is a Banach algebra with multiplication defined as convolution.*
(iii) *The Calderón transform Ω is bounded in Γ, and the theorem $J_\Gamma \times J_\Gamma \to J_\Gamma$ is satisfied.*

Proof (i) ⇒ (ii). Take $\bar{A} = \bar{B} = \bar{C} = (\ell_1, \ell_1(2^{-m}))$ and choose T as convolution

$$T(\xi, \eta) = \xi * \eta = \Big\{ \sum_{k=-\infty}^{\infty} \xi_k \eta_{m-k} \Big\}_{m\in\mathbb{Z}}, \quad \xi = \{\xi_m\}, \eta = \{\eta_m\}.$$

Since the theorem $J_\Gamma \times J_\Gamma \to K_\Gamma$ holds, there is a constant M such that

$$\|\xi * \eta\|_{(\ell_1,\ell_1(2^{-m}))_{\Gamma;K}} \le M\|\xi\|_{(\ell_1,\ell_1(2^{-m}))_{\Gamma;J}} \|\eta\|_{(\ell_1,\ell_1(2^{-m}))_{\Gamma;J}}$$

for all $\xi, \eta \in \ell_1 \cap \ell_1(2^{-m})$. On the other hand, according to [10, p. 295] or [7, p. 639], we have

$$(3.1) \qquad (\ell_1, \ell_1(2^{-m}))_{\Gamma;K} \hookrightarrow \Gamma \hookrightarrow (\ell_1, \ell_1(2^{-m}))_{\Gamma;J},$$

and the norms of these embeddings are less than or equal to 1. Therefore, we obtain that

$$\|\xi * \eta\|_\Gamma \le M\|\xi\|_\Gamma \|\eta\|_\Gamma \text{ for all } \xi, \eta \in \Gamma.$$

In other words, Γ is a Banach algebra with convolution.

(ii) ⇒ (iii). Put $\sigma = \{\min(1, 2^m)\}$. Then, by (2.2), σ belongs to Γ. The Calderón transform can be expressed in terms of σ as

$$\Omega\xi = \Big\{ \sum_{k=-\infty}^{\infty} \min(1, 2^{m-k})\xi_k \Big\}_{m\in\mathbb{Z}} = \xi * \sigma, \xi \in \Gamma.$$

Then the boundedness of Ω in Γ follows from the fact that $(\Gamma, *)$ is a Banach algebra.

In order to check that the theorem $J_\Gamma \times J_\Gamma \to J_\Gamma$ holds, take any Banach couples $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{C} = (C_0, C_1)$ and take any bilinear operator

$$T\colon (A_0 \cap A_1) \times (B_0 \cap B_1) \longrightarrow (C_0 \cap C_1)$$

such that for any $a \in A_0 \cap A_1$, $b \in B_0 \cap B_1$,

$$\|T(a, b)\|_{C_j} \le M_j \|a\|_{A_j} \|b\|_{B_j}, \; j = 0, 1.$$

Given any $a \in A_0 \cap A_1$, $b \in B_0 \cap B_1$ and any J-representations $a = \sum_{m=-\infty}^{\infty} u_m$, $b = \sum_{k=-\infty}^{\infty} v_k$ with only a finite number of terms u_m, v_m distinct from zero, we have

$$T(a, b) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} T(u_m, v_k) = \sum_{m=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} T(u_k, v_{m-k}) \Big).$$

Put $w_m = \sum_{k=-\infty}^{\infty} T(u_k, v_{m-k})$. Then $w_m \in C_0 \cap C_1$ and

$$\begin{aligned} J(2^m, w_m) &\le \sum_{k=-\infty}^{\infty} \max(M_0\|u_k\|_{A_0}\|v_{m-k}\|_{B_0}, 2^m M_1\|u_k\|_{A_1}\|v_{m-k}\|_{B_1}) \\ &\le \max(M_0, M_1) \sum_{k=-\infty}^{\infty} J(2^k, u_k) J(2^{m-k}, v_{m-k}). \end{aligned}$$

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Using that $(\Gamma, *)$ is a Banach algebra, we obtain

$$\begin{aligned}\|T(a,b)\|_{\bar{C}_{\Gamma;J}} &\le \|\{J(2^m, w_m)\}\|_\Gamma \\ &\le \max(M_0, M_1)\|\{J(2^m, u_m)\} * \{J(2^m, v_m)\}\|_\Gamma \\ &\le M\|\{J(2^m, u_m)\}\|_\Gamma \|\{J(2^m, v_m)\}\|_\Gamma .\end{aligned}$$

By (2.5) we conclude that there is a constant M' such that for any $a \in A_0 \cap A_1$, $b \in B_0 \cap B_1$ we have

$$\|T(a,b)\|_{\bar{C}_{\Gamma;J}} \le M'\|a\|_{\bar{A}_{\Gamma;J}} \|b\|_{\bar{B}_{\Gamma;J}} .$$

This establishes the bilinear theorem $J_\Gamma \times J_\Gamma \to J_\Gamma$.

(iii) $\Rightarrow$ (i). The boundedness of Ω in Γ yields that $\bar{C}_{\Gamma;K} = \bar{C}_{\Gamma;J}$ for any Banach couple $\bar{C}$. Hence, theorem $J_\Gamma \times J_\Gamma \to K_\Gamma$ follows from theorem $J_\Gamma \times J_\Gamma \to J_\Gamma$.

This completes the proof. ■

Remark 3.4 As we have seen in the course of the proof, if Γ is a Banach algebra with multiplication defined as convolution, then Ω is bounded in Γ, and so (3.1) yields

$$(\ell_1, \ell_1(2^{-m}))_{\Gamma;K} = (\ell_1, \ell_1(2^{-m}))_{\Gamma;J} = \Gamma.$$

Note that (iii) implies that $(\cdot\,,\,\cdot)_\Gamma$ preserves the Banach-algebra structure. On the other hand, it was shown in [7, Theorem 3.7] that if $(\cdot\,,\,\cdot)_\Gamma$ preserves the Banach-algebra structure, then $(\Gamma, *)$ is a Banach algebra. In other words, (ii) is satisfied. Therefore, we get another equivalent condition.

Corollary 3.5 *Let Γ be a lattice satisfying* (2.1), (2.3), *and* (2.4). *Then any of the three conditions stated in Theorem 3.3 is equivalent to*

(iv) *Ω is bounded in Γ and $(\cdot\,,\,\cdot)_\Gamma$ preserves the Banach-algebra structure.*

Let $1 < q < \infty$ and let $\Gamma = \ell_q\big(2^{-\theta m}\big)$. As one can see, for example, in [7, Corollary 3.8], the real method $(\cdot\,,\,\cdot)_{\theta,q}$ does not satisfy the bilinear interpolation theorem $J_{\theta,q} \times J_{\theta,q} \to K_{\theta,q}$ (which is, in this case, equivalent to the theorem $J_{\theta,q} \times J_{\theta,q} \to J_{\theta,q}$). But if we take $\gamma > (q-1)/q$ and we add the logarithmic terms $\{(1+|m|)^\gamma\}$ in the weight then the resulting space is $\ell_q\big(2^{-\theta m}(1+|m|)^\gamma\big)$ which is a Banach algebra with multiplication defined as convolution (see, for example, [3, Proposition 2.3] or [7, Corollary 3.9]) and also satisfies conditions (2.1), (2.3), and (2.4) (see [7, Example 2.4]). Consequently, as a direct application of Theorem 3.3 we obtain the following.

Corollary 3.6 *Let $\Gamma = \ell_q\big(2^{-\theta m}(1+|m|)^\gamma\big)$ where $0 < \theta < 1$, $1 < q < \infty$, and $\gamma > (q-1)/q$. Then the bilinear interpolation theorem $J_\Gamma \times J_\Gamma \to K_\Gamma$ holds.*

Note added in proof Related results can be found in S. V. Astashkin, *Interpolation of bilinear operators by the real method.* Math. Notes **52**(1992), 641–648.

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