TWISTED GROUP RINGS WHICH ARE SEMI-PRIME GOLDIE RINGS

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(Received 6 October, 1973; revised 13 March, 1974)

In this paper we examine when a twisted group ring, $R^\gamma(G)$, has a semi-simple, artinian quotient ring. In §1 we assemble results and definitions concerning quotient rings, Ore sets and Goldie rings and then, in §2, we define $R^\gamma(G)$. We prove a useful theorem for constructing a twisted group ring of a factor group and establish an analogue of a theorem of Passman. Twisted polynomial rings are discussed in §3 and I am indebted to the referee for informing me of the existence of [**4**]. These are used as a tool in proving results in §4.

A group G is a *poly-* (*torsion-free abelian or finite*) group if G has a series of subgroups $\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_n = G$ such that H_i/H_{i-1} is either torsion-free abelian or finite ($i = 1, 2, \ldots, n$). These groups are considered here and we prove (Theorem 4.5) that if such a group G has only a finite set S of periodic elements with $|S|$ regular in R and R is semi-prime, left Goldie, then $R^\gamma(G)$ is semi-prime, left Goldie.

In §5 we define a class of groups $\mathscr{C}$ such that if G is a torsion-free element of $\mathscr{C}$ and D is a division ring then $D^\gamma(G)$ is an Ore domain. We call these groups Ore groups and prove a theorem similar to Theorem 4.5 for this class of groups.

Throughout, R will denote a ring with identity element 1 and G a multiplicative group with identity e. By artinian and noetherian we mean left artinian and left noetherian.

1. Goldie rings.

We restate the following definitions which appear in [**2**, pp. 228, 229].

An element of a ring R is *regular* if it is neither a left nor a right zero divisor. A set T of regular elements of R which is multiplicatively closed is a *left Ore set* if, whenever $a \in R$, $c \in T$, there exist $a' \in R$, $c' \in T$ such that $c'a = a'c$.

A ring Q is a *left quotient ring* of *R with respect to a set T* of regular elements of R if

(i) $Q \supseteq R$,
(ii) the elements of T are units in Q,
(iii) the elements of Q have the form $c^{-1}a$ where $c \in T$, $a \in R$.

If such a ring Q exists, it will be denoted by R_T. When T is the set of all regular elements of R we say that Q is the *left quotient ring of R*.

THEOREM 1.1. *Let T be a set of regular elements of R. Then R_T exists if and only if T is a left Ore set in R.*

† This work was supported by the Science Research Council and forms part of the author's Ph.D. thesis (Aberdeen). I wish to thank Professor D. A. R. Wallace for his invaluable encouragement and advice and the referee for his helpful suggestions. In particular I am indebted to the referee for pointing out the method of proof of Theorem 2.7. My original, longer proof followed the lines of [**6**, Appendices 2, 3].

Proof. [**3**, p. 170].

A ring R has finite *left Goldie rank* if it contains no infinite direct sum of non-zero left ideals. Let S be a non-empty subset of R; then $\ell(S)$, the *left annihilator of* S, is the left ideal $\{a \in R: as = 0 \text{ for all } s \in S\}$. A ring R is a *left Goldie ring* if (i) R has finite left Goldie rank and (ii) R has ascending chain condition on left annihilators.

GOLDIE'S THEOREM [**2**, Theorem 1.37]. *A ring R has a semi-simple artinian left quotient ring if and only if R is a semi-prime left Goldie ring.*

LEMMA 1.2 [**11**, Corollary 2.5]. *Let Q be an artinian ring with subring R such that every element of Q has the form $c^{-1}a$, where $c, a \in R$. Then Q is the left quotient ring of R.*

For convenience, we formulate the following straightforward lemmas.

LEMMA 1.3. *Let R be a ring and let $T \subseteq R$ be a left Ore set.*

(i) *Let L be a left ideal and let $L_T = R_T L$, the left ideal in R_T generated by L. Then $L_T = \{c^{-1}r : c \in T, r \in L\}$.*

(ii) *Let L and J be left ideals in R. Then $L_T \cap J_T = (L \cap J)_T$.*

(iii) *If L is a left annihilator in R, then L_T is a left annihilator in R_T and $L_T \cap R = L$.*

(iv) *If R_T is a left Goldie ring, then R is a left Goldie ring.*

LEMMA 1.4. *Let $R_1, R_2, \ldots, R_n$ be a finite number of left Goldie rings. Then $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ is also a left Goldie ring.*

2. Twisted group rings.

DEFINITION. Let G be a group with identity element e, R a ring with identity 1, R^* the group of central units of R and $\gamma: G \times G \to R^*$ a 2-cocycle. [That is, $\gamma(g, h)\gamma(gh, k) = \gamma(g, hk)\gamma(h, k)$, $g, h, k \in G$]. Let $R^\gamma(G)$ be the free left R-module with basis $\{\bar{g} : g \in G\}$. Define multiplication in $R^\gamma(G)$ by

$$\bar{g}\,\bar{h} = \gamma(g, h)\overline{gh} \quad (g, h \in G)$$

extending this, by linearity, to the whole of $R^\gamma(G)$. Then $R^\gamma(G)$ is an associative ring with identity element $\gamma(e, e)^{-1}\bar{e}$. We call $R^\gamma(G)$ the *twisted group ring* of G over R with *twist* γ.

We shall identify an element $r \in R$ with its image $r\gamma(e, e)^{-1}\bar{e}$ in $R^\gamma(G)$.

In this section we prove some results about $R^\gamma(G)$ that we shall require later.

THEOREM 2.1. *Let G be a group with a central normal subgroup Z and $R^\gamma(G)$ a twisted group ring such that $\gamma(g, z) = \gamma(z, g)$ for all $g \in G$ and $z \in Z$. Then there exists a twisted group ring of G/Z over $R^\gamma(Z)$ with twist δ such that*

$$R^\gamma(G) \cong [R^\gamma(Z)]^\delta(G/Z).$$

https://doi.org/10.1017/S0017089500002433 Published online by Cambridge University Press

Proof. Let T be a set of coset representatives for Z in G. Then every element of G is uniquely represented in the form tz for some $t \in T$, $z \in Z$. Thus given $t_1, t_2 \in T$ there are a unique $\tau(t_1, t_2) \in T$ and $z \in Z$ such that $t_1 t_2 = \tau(t_1, t_2)z$. Then, in $R^\gamma(G)$,

$$\bar{t}_1\bar{t}_2 = \gamma(t_1, t_2)\,\overline{\tau(t_1, t_2)z} = \gamma(t_1, t_2)\gamma(z, \tau(t_1, t_2))^{-1}\,\bar{z}\,\overline{\tau(t_1, t_2)}.$$

Thus

$$\bar{t}_1\bar{t}_2(\overline{\tau(t_1, t_2)})^{-1} = \gamma(t_1, t_2)\gamma(z, \tau(t_1, t_2))^{-1}\,\bar{z} \in \text{central units of } R^\gamma(Z).$$

Let $F = G/Z$. Then for each $f \in F$ there is a unique $t \in T$ such that $f = tZ$. Define $\delta\colon F \times F \to (R^\gamma(Z))^*$ by

$$\delta(f_1, f_2) = \bar{t}_1\bar{t}_2(\overline{\tau(t_1, t_2)})^{-1}, \quad \text{where} \quad f_1 = t_1 Z, f_2 = t_2 Z, t_1, t_2 \in T.$$

Given f_1, f_2, then t_1, t_2 and $\tau(t_1, t_2)$ are uniquely determined. Thus δ is well-defined and it is readily verified that δ is a 2-cocycle.

Hence we have defined $[R^\gamma(Z)]^\delta(F)$. We shall denote by f^* the image in $[R^\gamma(Z)]^\delta(F)$ of an element $f \in F$.

Now we construct an isomorphism between $R^\gamma(G)$ and $[R^\gamma(Z)]^\delta(F)$. As remarked earlier, given $g \in G$ there are a unique $t \in T$ and $z \in Z$ with $g = tz = zt$. Then $\bar{g} = \gamma(z, t)^{-1}\bar{z}\bar{t}$ in $R^\gamma(G)$. Define $\theta\colon R^\gamma(G) \to [R^\gamma(Z)]^\delta(F)$ to be the R-homomorphism defined by

$$\theta\colon \bar{g} = \gamma(z, t)^{-1}\bar{z}\bar{t} \mapsto \gamma(z, t)^{-1}\bar{z}(tZ)^*.$$

We show that θ is also a ring homomorphism. To do this, it is sufficient to show that $\theta(\bar{g}_1\bar{g}_2) = \theta(\bar{g}_1)\theta(\bar{g}_2)$ $(g_1, g_2 \in G)$. Let $g_1 = z_1 t_1$, $g_2 = z_2 t_2$, where $z_1, z_2 \in Z$, $t_1, t_2 \in T$. Then

$$\begin{aligned}\bar{g}_1\bar{g}_2 &= \gamma(z_1, t_1)^{-1}\bar{z}_1\bar{t}_1\gamma(z_2, t_2)^{-1}\bar{z}_2\bar{t}_2\\ &= \gamma(z_1, t_1)^{-1}\gamma(z_2, t_2)^{-1}\gamma(z_1, z_2)\gamma(t_1, t_2)\,\overline{z_1 z_2}\,\overline{t_1 t_2}\\ &= \gamma(z_1, t_1)^{-1}\gamma(z_2, t_2)^{-1}\gamma(z_1, z_2)\gamma(z_1 z_2, z_3)\gamma(z_3, t_3)^{-1}\gamma(t_1, t_2)\,\overline{z_1 z_2 z_3}\,\bar{t}_3\end{aligned}$$

(where $t_1 t_2 = z_3 t_3$, $z_3 \in Z$, $t_3 \in T$). Thus

$$\theta(\bar{g}_1\bar{g}_2) = \gamma(z_1, t_1)^{-1}\gamma(z_2, t_2)^{-1}\gamma(z_1, z_2)\gamma(z_1 z_2, z_3)\gamma(z_3, t_3)^{-1}\gamma(t_1, t_2)\,\overline{z_1 z_2 z_3}\,(t_3 Z)^*.$$

Also

$$\begin{aligned}\theta(\bar{g}_1)\theta(\bar{g}_2) &= \gamma(z_1, t_1)^{-1}\,\bar{z}_1\,(t_1 Z)^*\gamma(z_2, t_2)^{-1}\,\bar{z}_2\,(t_2 Z)^*\\ &= \gamma(z_1, t_1)^{-1}\gamma(z_2, t_2)^{-1}\gamma(z_1, z_2)\,\overline{z_1 z_2}\,\delta(t_1 Z, t_2 Z)(t_3 Z)^*.\end{aligned}$$

Thus, recalling that

$$\delta(t_1 Z, t_2 Z) = \bar{t}_1\bar{t}_2(\bar{t}_3)^{-1} = \gamma(t_1, t_2)\gamma(z_3, t_3)^{-1}\bar{z}_3,$$

it follows that $\theta(\bar{g}_1\bar{g}_2) = \theta(\bar{g}_1)\theta(\bar{g}_2)$.

Hence θ is a ring homomorphism and, since θ is clearly both one-one and onto, the required isomorphism is established.

COROLLARY 2.2. *Let G be a group, Z a central normal subgroup of G and R a ring. Then there exists a twisted group ring of G/Z over $R(Z)$ with twist δ, such that*

$$R(G) \cong R(Z)^{\delta}(G/Z).$$

Thus twisted group rings occur in a fairly natural way and we have a useful method of expressing a group ring in terms of a subgroup and a factor group.

For Lemma 2.5 we shall require the following result. We denote the set of positive integers by $\mathbf{P}$.

LEMMA 2.3. *Let R be a semi-simple, artinian ring and let $n \in \mathbf{P}$. Let $W = \{w \in R^*: w^n = 1\}$. Then W is finite.*

Proof. Let S be the centre of R. Then, since R is semi-simple artinian, there exist fields $F_1, F_2, \ldots, F_r$ (say) such that $S = F_1 \oplus F_2 \oplus \ldots \oplus F_r$. For $w \in W$, let $(w_1, w_2, \ldots, w_r)$ be the image of w in $F_1 \oplus F_2 \oplus \ldots \oplus F_r$. Then $w^n = 1$ implies that $w_i^n = 1$ $(i = 1, 2, \ldots, r)$. Hence $W = W_1 \oplus W_2 \oplus \ldots \oplus W_r$, where W_i is the set of nth roots of unity in F_i. But the set of nth roots of unity in a field is finite. Hence W is finite.

COROLLARY 2.4. *Let R be a semi-prime left Goldie ring and let $n \in \mathbf{P}$. Let $W = \{w \in R^*: w^n = 1\}$. Then W is finite.*

Proof. Let Q be the semi-simple, artinian quotient ring of R. Then $W \subseteq \{w \in Q^*: w^n = 1\}$ which, by the lemma, is finite.

DEFINITION. Let $R^{\gamma}(G)$ be a twisted group ring and let $H \leqq G$. Define

$$\bar{C}_G(H) = \{g \in G: \bar{g}\bar{h} = \bar{h}\bar{g} \quad \text{for all} \quad h \in H\}$$
$$= \{g \in C_G(H): \gamma(g, h) = \gamma(h, g) \quad \text{for all} \quad h \in H\}.$$

It is readily verified that $\bar{C}_G(H)$ is a subgroup of G.

LEMMA 2.5. *Let R be a semi-prime left Goldie ring and let $R^{\gamma}(G)$ be a twisted group ring. Let H be a subgroup of G. Then (i) $\bar{C}_G(H) \lhd C_G(H)$ and (ii) if, further, $|H| < \infty$, then $|C_G(H): \bar{C}_G(H)| < \infty$.*

Proof. Let $g_1, g_2 \in C_G(H), h \in H$. Then

$$\frac{\gamma(g_1, h)\gamma(g_2, h)}{\gamma(h, g_1)\gamma(h, g_2)} = \frac{\gamma(g_1, h)\gamma(g_2, h)\gamma(hg_1, g_2)}{\gamma(h, g_1)\gamma(h, g_2)\gamma(hg_1, g_2)}$$

https://doi.org/10.1017/S0017089500002433 Published online by Cambridge University Press

$$= \frac{\gamma(g_1, hg_2)\gamma(h, g_2)\gamma(g_2, h)}{\gamma(h, g_1g_2)\gamma(g_1, g_2)\gamma(h, g_2)}$$

$$= \frac{\gamma(g_1, g_2)\gamma(g_1g_2, h)}{\gamma(h, g_1g_2)\gamma(g_1, g_2)}$$

$$= \gamma(g_1g_2, h)\gamma(h, g_1g_2)^{-1}.$$

Now define $\theta_h: C_G(H) \to R^*$ by

$$\theta_h(g) = \gamma(g, h)\gamma(h, g)^{-1} \qquad (g \in C_G(H)).$$

Then, by the above argument, θ_h is a group homomorphism, Ker $\theta_h = \{g \in C_G(H): \gamma(g, h) = \gamma(h, g)\}$ and hence

$$\bar{C}_G(H) = \bigcap_{h \in H} \text{Ker } \theta_h.$$

It follows that $\bar{C}_G(H) \lhd C_G(H)$.

Now suppose that $|H| = n$ and let $h \in H$, $g \in C_G(H)$. Then $(\bar{h}\bar{g})^n = a\bar{h}^n\bar{g}^n$ for some $a \in R$. But $h^n = e$ therefore $\bar{h}^n \in R$ and so $(\bar{h}\bar{g})^n = b\bar{g}^n$ for some $b \in R$. Thus

$$(\bar{h}\,\bar{g})^n = \bar{g}(\bar{h}\bar{g})^n\bar{g}^{-1} = (\bar{g}\,\bar{h})^n = [\gamma(g, h)\gamma(h, g)^{-1}\,\bar{h}\,\bar{g}]^n.$$

Therefore $[\gamma(g, h)\gamma(h, g)^{-1}]^n = 1$ and so

$$C_G(H)/\text{Ker } \theta_h \cong \text{subgroup of group of } n\text{th roots of unity in } R^*.$$

Hence, by Corollary 2.4, $|C_G(H): \text{Ker } \theta_h| < \infty$. Further, since $|H| < \infty$, $|C_G(H): \bar{C}_G(H)| < \infty$ and the result is proved.

We now give a lemma concerning rings of quotients.

LEMMA 2.6. (i) *Let $H \lhd G$ such that $R^\gamma(G)$ has a left quotient ring and let T be the set of regular elements in $R^\gamma(H)$. Then T is a left Ore set in $R^\gamma(G)$.*

(ii) *If R has a left quotient ring Q, then $Q^\gamma(G)$ is well-defined and is the left quotient ring of $R^\gamma(G)$ with respect to the set of regular elements of R.*

Proof. (i) Adapt [**12,** Lemma 2.6].

(ii) This is clear.

We shall wish to know when $R^\gamma(G)$ is semi-prime. We denote by $PR^\gamma(G)$ the prime radical of $R^\gamma(G)$. In the 'untwisted' situation we have the following theorem due to D. Passman [**6,** p. 162, see also **7**] and I. Connell [**6,** Appendices 2 and 3].

THEOREM A. *The group ring $R(G)$ is semi-prime if and only if R is semi-prime and the order of each finite normal subgroup of G is regular in R.*

In [**8,** Theorem 3.7] Passman proves the following extension of this.

THEOREM B. *Let K be an algebraically closed field of characteristic $p > 0$ and $K^{\gamma}(G)$ a twisted group ring. Then $K^{\gamma}(G)$ is semi-prime if and only if G has no finite normal subgroups of order divisible by p.*

Let K be any field of characteristic $p > 0$, F its algebraic closure and $K^{\gamma}(G)$ a twisted group ring. Then $F^{\gamma}(G)$ is well-defined and, arguing as in [**1**, Proposition 9], it can be shown that

$$PK^{\gamma}(G = K^{\gamma}(G) \cap PF^{\gamma}(G).$$

It is immediate from this and Theorem B that, if G has no finite normal subgroups of order divisible by p then, $K^{\gamma}(G)$ is semi-prime and we generalise this below in Theorem 2.7. The converse of this, however, is not true. We recall a counter example discussed in [**9**]. Let K be a field over which the polynomials $x^{p^n} - a$ are irreducible for some $a \in K$ and where $p =$ char K. Let $G = \mathbf{Z}p^{\infty}$. Then we may construct a twisted group ring $K^{\gamma}(G)$ which is a field and hence semi-prime. The orders of finite normal subgroups of G, however, are powers of p.

THEOREM 2.7. *Let R be a semi-prime ring and one of the following: (i) commutative, (ii) a semi-direct product of simple rings, (iii) left Goldie. Let G be a group such that the order of each finite normal subgroup is regular in R and let $R^{\gamma}(G)$ be a twisted group ring. Then $R^{\gamma}(G)$ is semi-prime.*

Proof. (i) As in [**1**, proof of Theorem 5, p. 668].

(ii) As in [**1**, proof of Proposition 10, pp. 669 and 670].

(iii) Let Q be the semi-simple artinian left quotient ring of R. Then, by (ii), $Q^{\gamma}(G)$ is semi-prime and hence $R^{\gamma}(G)$ is semi-prime.

3. Twisted polynomial rings.

DEFINITION. Let R be a ring and $\theta: R \to R$ an automorphism of R. Let $\langle x \rangle$ be an infinite cyclic group. We define $R_{\theta}(x)$ to be the free left R-module with basis $\langle x \rangle$ and, for $r \in R$, we define multiplication on $R_{\theta}(x)$ by

$$xr = \theta(r)x$$
$$x^{-1}r = \theta^{-1}(r)x^{-1},$$

extending by linearity to the whole of $R_{\theta}(x)$. With this definition of multiplication $R_{\theta}(x)$ is an associative ring.

Thus $R_{\theta}(x)$ is a ring of polynomials in x and x^{-1} with coefficients from R. The subring of $R_{\theta}(x)$ containing only the polynomials in non-negative powers of x, denoted by $R_{\theta}[x]$, is called a *twisted polynomial ring*.

A. Horn in [**4**, §2] has proved the following.

THEOREM 3.1. *Let R be a noetherian ring. Then $R_{\theta}[x]$ has an artinian left quotient ring if and only if R has an artinian left quotient ring.*

From this we may deduce the following corollary.

COROLLARY 3.2. *Let R have an artinian left quotient ring. Then $R_\theta(x)$ has an artinian left quotient ring.*

Proof. Let Q be the left quotient ring of R. Then, by the theorem, $Q_\theta[x]$ has an artinian left quotient ring $\bar{Q}$. Since x^i is regular in $Q_\theta[x]$, $x^{-i} \in \bar{Q}$ ($i \in \mathbf{P}$) and hence

$$R_\theta(x) \subseteq Q_\theta(x) \subset \bar{Q}.$$

It is now clear from Lemma 1.2 that $\bar{Q}$ is the artinian left quotient ring of $R_\theta(x)$.

4. Quotient rings of $R^\gamma(G)$. In this section we obtain sufficient conditions for $R^\gamma(G)$ to have a semi-simple artinian quotient ring, similar to but less stringent than those obtained by P. Smith in [**12,** Theorem 2.18] for $R(G)$. By Goldie's Theorem, if $R^\gamma(G)$ is to have a semi-simple artinian left quotient ring, then it must itself be a semi-prime left Goldie ring and therefore must have both a.c.c. on left annihilators and finite left Goldie rank.

LEMMA 4.1. *Let $R^\gamma(G)$ be semi-prime and let $H \lhd G$ be such that (i) $|G: H| < \infty$ and (ii) $R^\gamma(H)$ is semi-prime left Goldie. Then $R^\gamma(G)$ is semi-prime left Goldie.*

Proof. By Lemma 2.6, the set T of regular elements of $R^\gamma(H)$ is a left Ore set in $R^\gamma(G)$. Let $S = [R^\gamma(G)]_T$. Then S is semi-prime and $S = \sum_{c \in C} Q\bar{c}$, where Q is the left quotient ring of $R^\gamma(H)$ and C is a set of coset representatives for H in G. But C is finite; therefore S is an artinian Q-module and hence an artinian ring. It follows from Lemma 1.2 that S is the left quotient ring of $R^\gamma(G)$ and so, by Goldie's Theorem, $R^\gamma(G)$ is a semi-prime left Goldie ring.

LEMMA 4.2. *Let $R^\gamma(G)$ have a left quotient ring and let $H \lhd G$ be such that*

(i) *$R^\gamma(H)$ is semi-prime left Goldie, and*
(ii) *G/H is ordered.*

Then $R^\gamma(G)$ is semi-prime left Goldie.

Proof. We prove that every essential left ideal in $R^\gamma(G)$ contains a regular element. Let E be an essential left ideal in $R^\gamma(G)$ and let

$$E_0 = \{a \in R^\gamma(H): \bar{g}_0a+\bar{g}_1a_1+\ldots+\bar{g}_na_n \in E \text{ for some } n \text{ and } a_i \in R^\gamma(H) \text{ and where } g_0H < g_1H < \ldots < g_nH \text{ in } G/H\}.$$

Then E_0 is a left ideal in $R^\gamma(H)$. Let $a \in R^\gamma(H)$, $a \neq 0$. Then there exists $\alpha = \bar{k}_1b_1+\bar{k}_2b_2+\ldots+\bar{k}_mb_m \in R^\gamma(G)$, $b_i \in R^\gamma(H)$, $k_1H < k_2H < \ldots < k_mH$ in G/H, such that $\alpha a \neq 0$ and $\alpha a \in E$. Therefore $b_ia \neq 0$ and $b_ia \in E_0$ for some $1 \leqq i \leqq m$ and it follows that E_0 is essential in $R^\gamma(H)$. But $R^\gamma(H)$ is semi-prime left Goldie; therefore E_0 contains a regular element of

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$R^\gamma(H)$. That is, there exists $x \in E$ with $x = \bar{g}_0 c + \bar{g}_1 c_1 + \ldots + \bar{g}_n c_n$, where $c_i \in R^\gamma(H)$), c is regular in $R^\gamma(H)$ and $g_0 H < g_1 H < \ldots < g_n H$ in G/H. It is readily verified that x is regular in $R^\gamma(G)$.

Now since every essential left ideal of $R^\gamma(G)$ contains a regular element, Q, the left quotient ring of $R^\gamma(G)$, contains no proper essential left ideals and is therefore a semi-simple artinian ring [**2**, p. 234 and p. 219].

COROLLARY 4.3. *Let $R^\gamma(G)$ be a twisted group ring and $H \lhd G$ be such that G/H is infinite cyclic and $R^\gamma(H)$ is semi-prime left Goldie. Then $R^\gamma(G)$ is semi-prime left Goldie.*

Proof. $G/H = \langle gH \rangle$ for some $g \in G \backslash H$. Define $\theta: R^\gamma(H) \to R^\gamma(H)$ by $\theta(\alpha) = \bar{g}\alpha\bar{g}^{-1}$ ($\alpha \in R^\gamma(H)$). Then, since $H \lhd G$, θ is an automorphism of $R^\gamma(H)$ and, in the notation of §3, with $\bar{g} = x$, $R^\gamma(G) = R^\gamma(H)_\theta(\bar{g})$. Now it follows from Corollary 3.2 that $R^\gamma(G)$ has an artinian left quotient ring and so, G/H being an ordered group, $R^\gamma(G)$ is semi-prime left Goldie.

LEMMA 4.4. *Let $R^\gamma(G)$ be a twisted group ring and let $H \lhd G$ be such that (i) $R^\gamma(H)$ is semi-prime left Goldie, and (ii) G/H is torsion-free abelian. Then $R^\gamma(G)$ is semi-prime left Goldie.*

Proof. G/H is an ordered group. Thus, from Lemma 4.2, it will be sufficient to prove that $R^\gamma(G)$ has a left quotient ring. To do so it is enough to show that $R^\gamma(G_1)$ has a left quotient ring for every subgroup G_1 such that G_1/T is finitely generated. But G_1/H is a direct sum of a finite number of infinite cyclic groups and the required result follows by induction from Corollary 4.3.

THEOREM 4.5. *Let G be a poly- (torsion-free abelian or finite) group and let S be the set of all periodic elements of G. Let R be semi-prime left Goldie and let S be finite with $|S|$ regular in R. Then $R^\gamma(G)$ is semi-prime left Goldie.*

Proof. By Theorem 2.7, $R^\gamma(G)$ is semi-prime and so the result follows by induction from Lemmas 4.1, 4.4.

EXAMPLES of poly- (torsion-free abelian or finite) groups.

(i) Nilpotent groups with finite set of periodic elements. (A torsion-free nilpotent group has central series with factors all torsion-free abelian [**5**, Theorem 1.2].)

(ii) Soluble groups with derived series whose factors have only a finite number of periodic elements.

(iii) FC-soluble groups [**10**, pp. 121, 129] with series

$$\{e\} = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$$

such that H_{i+1}/H_i is an FC-group whose torsion subgroup [**10**, p. 121, Theorem 4.32] is finite ($i = 0, 1, \ldots, n-1$).

((i) and (ii) are particular examples of (iii).)

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5. Ore groups.

DEFINITION. A ring R is called a *left Ore domain* if

(i) R contains no proper zero divisors, and
(ii) R satisfies the left Ore condition.

We shall be interested in the class of groups such that, given G torsion-free and an Ore domain R, then $R^{\gamma}(G)$ is an Ore domain. We therefore make the following definition.

DEFINITION. Let $\mathscr{C}$ be the class of groups such that

(i) $G \in \mathscr{C}$, $H \leqq G \Rightarrow H \in \mathscr{C}$,
(ii) $G \in \mathscr{C}$, $H \lhd G$, $|H| < \infty \Rightarrow G/H \in \mathscr{C}$,
(iii) if $G \in \mathscr{C}$ is torsion-free, D is a division ring and $D^{\gamma}(G)$ a twisted group ring, then $D^{\gamma}(G)$ is an Ore domain.

If $G \in \mathscr{C}$ we call G an *Ore group*. Every periodic group is an Ore group. Also abelian groups, nilpotent groups and FC-groups are Ore groups.

THEOREM 5.1. *Let G be a group such that any twisted group ring $D^{\delta}(G)$, where D is a division ring, is semi-prime left Goldie. Let R be a semi-prime left Goldie ring. Then $R^{\gamma}(G)$ is semi-prime left Goldie.*

Proof. Let Q be the semi-simple artinian quotient ring of R. By Lemmas 2.6 and 1.3, (iv), it is sufficient to prove that $Q^{\gamma}(G)$ is semi-prime left Goldie. Then

$$Q = M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \ldots \oplus M_{n_r}(D_r)$$

for some integers $n_1, \ldots, n_r$ and division rings $D_1, D_2, \ldots, D_r$. Also there exist orthogonal central idempotents $e_1, e_2, \ldots, e_r \in Q$ such that $M_{n_i}(D_i) = Qe_i$ $(i = 1, 2, \ldots, r)$. Let $g, h \in G$. Since $\gamma(g, h)$ is a central unit of R, $\gamma(g, h)e_i$ is a central unit of D_i $(i = 1, 2, \ldots, r)$ and thus, defining $\gamma_i(g, h) = \gamma(g, h)e_i$, we have defined twisted group rings $D_i^{\gamma_i}(G)$ $(i = 1, 2, \ldots, r)$. It follows that

$$Q^{\gamma}(G) = M_{n_1}(D_1^{\gamma_1}(G)) \oplus M_{n_2}(D_2^{\gamma_2}(G)) \oplus \ldots \oplus M_{n_r}(D_r^{\gamma_r}(G)).$$

Hence it is sufficient to prove that each $M_{n_i}(D_i^{\gamma_i}(G))$ is semi-prime left Goldie. But $D_i^{\gamma_i}(G)$ has a semi-simple artinian quotient ring Q_i, by the hypotheses of the theorem; hence [**11**, Theorem 3.1] $M_{n_i}(Q_i)$ is the semi-simple artinian quotient ring of $M_{n_i}(D^{\gamma_i}(G))$.

COROLLARY 5.2. *Let R be a semi-prime left Goldie ring and G a torsion-free Ore group. Then $R^{\gamma}(G)$ is semi-prime left Goldie.*

Before the main theorem of this section we require the following lemma, the proof of which is routine.

LEMMA 5.3. *Let G be a group and let S be the set of all periodic elements of G. Then*

(i) $C_G(S) \lhd G$,
(ii) $|S| < \infty \Rightarrow S \lhd G$,
(iii) $|S| < \infty \Rightarrow |G : C_G(S)| < \infty$.

THEOREM 5.4. *Let R be a semi-prime left Goldie ring and let G be an Ore group such that the set S of all periodic elements of G is finite with $|S|$ regular in R. Then $R^\gamma(G)$ is semi-prime left Goldie.*

Proof. Let $\bar{C}_G(S) = \{g \in C_G(S)\colon \gamma(g, s) = \gamma(s, g) \text{ for all } s \in S\}$. By Lemma 2.5, $|C_G(S)\colon \bar{C}_G(S)| < \infty$. Hence, since $|G\colon C_G(S)| < \infty$, $|G\colon \bar{C}_G(S)| < \infty$. Also, by Theorem 2.7, $PR^\gamma(G) = 0$ and so, by Lemma 4.1, it is sufficient to prove that $R^\gamma(\bar{C}_G(S))$ is semi-prime left Goldie. Let $C = \bar{C}_G(S) \cap S$. Then C is a central subgroup of $\bar{C}_G(S)$ and, since $C \subseteq S$, $\bar{g}\,\bar{c} = \bar{c}\bar{g}$ for all $g \in \bar{C}_G(S)$, $c \in C$. Therefore, by Theorem 2.1, we may construct a twisted group ring of $\bar{C}_G(S)/C$ over $R^\gamma(C)$ with twist δ (say) such that

$$R^\gamma(\bar{C}_G(S)) \cong [R^\gamma(C)]^\delta(\bar{C}_G(S)/C).$$

But, since $|C| < \infty$ and $|C|$ is regular in R, $R^\gamma(C)$ is semi-prime left Goldie (Lemma 4.1). Also, since G is an Ore group, $\bar{C}_G(S)$ is an Ore group. Then, since C is the set of periodic elements of $\bar{C}_G(S)$ and C is finite, $\bar{C}_G(S)/C$ is a torsion-free Ore group. It now follows from Corollary 5.2 that $[R^\gamma(C)]^\delta(\bar{C}_G(S)/C)$ is a semi-prime left Goldie ring. That is, $R^\gamma(\bar{C}_G(S))$ is semi-prime left Goldie and hence $R^\gamma(G)$ is also semi-prime left Goldie.

DEFINITIONS. If $\mathfrak{X}$ is a class of groups, $L\mathfrak{X}$ is the class of *locally $\mathfrak{X}$-groups* consisting of all groups G such that every finite subset of G is contained in a $\mathfrak{X}$-subgroup.

$\mathfrak{X}$ is called a *local class* if $L\mathfrak{X} = \mathfrak{X}$. [**10**, part 1 p. 5, part 2 p. 93].

THEOREM 5.5. *The class $\mathscr{C}$ of Ore groups is a local class.*

Proof. Let $G \in L\mathscr{C}$. Let S be a finite subset of G and let $H = \langle S \rangle$. Since $G \in L\mathscr{C}$, there exists $K \in \mathscr{C}$ such that $S \subseteq K$. Then $H \leqq K$ and so $H \in \mathscr{C}$. From this it is clear that $L\mathscr{C}$ satisfies (i) and (ii) of the definition of an Ore group. We must now prove that if $G \in L\mathscr{C}$ is torsion-free and D is a division ring then $D^\gamma(G)$ is an Ore domain. To prove this we show that

(a) $xy = 0$ if and only if $x = 0$ or $y = 0$ $(x, y \in D^\gamma(G))$;
(b) given $x, y \in D^\gamma(G)$, there exist $x', y' \in D^\gamma(G)$ such that $x'x = y'y$.

Let $x, y \in D^\gamma(G)$; then there exists a finitely generated subgroup H such that $x, y \in D^\gamma(H)$. Then $H \in \mathscr{C}$ so that H is a torsion-free Ore group and $D^\gamma(H)$ is an Ore domain. Now, since $x, y \in D^\gamma(H)$, they satisfy conditions (a) and (b). Hence $D^\gamma(G)$ is an Ore domain. We have shown that $L\mathscr{C}$ satisfies (i), (ii) and (iii) of the definition of $\mathscr{C}$. Hence $L\mathscr{C} \subseteq \mathscr{C}$ and so $L\mathscr{C} = \mathscr{C}$.

COROLLARY 5.6. *Let G be a locally nilpotent group (locally FC group); then G is an Ore group.*

THEOREM 5.7. *Let G be a locally nilpotent (locally FC) group. Then $R(G)$ is semi-prime left Goldie if and only if*

(i) *R is semi-prime left Goldie, and*
(ii) *the subgroup S of all periodic elements of G is finite with $|S|$ regular in R.*

Proof. That (i) and (ii) are sufficient for $R(G)$ to be semi-prime left Goldie follows from Theorem 5.4.

Conversely, let $R(G)$ be semi-prime left Goldie. It is not hard to show that R must be a left Goldie ring. Then, by Theorem A and the fact that the set of periodic elements of a locally nilpotent (locally FC) group is a locally finite subgroup, it follows that (i) and (ii) hold true.

THEOREM 5.8. *Let G be a group and let $H \lhd G$ be such that H is periodic and G/H is an Ore group. Then G is an Ore group.*

Proof. Let $\mathfrak{X} = \{G: G$ has a periodic normal subgroup H with G/H an Ore group$\}$. Clearly $\mathscr{C} \subseteq \mathfrak{X}$. We shall prove that $\mathfrak{X}$ satisfies the definition of $\mathscr{C}$ and hence that $\mathfrak{X} = \mathscr{C}$.

Let $G \in \mathfrak{X}$ with $H \lhd G$ such that H is periodic and $G/H \in \mathscr{C}$.

(i) If $K \leqq G$, then $K \cap H$ is a periodic normal subgroup of K. Also $K/(K \cap H) \cong KH/H \leqq G/H \in \mathscr{C}$. Hence $K/(K \cap H) \in \mathscr{C}$ and it follows that $K \in \mathfrak{X}$.

(ii) Let $K \lhd G$, $|K| < \infty$. Now $HK/K \cong H/(H \cap K)$ is a periodic normal subgroup of G/K. Also $(G/K)/(HK/K) \cong (G/H)/(HK/H)$ which belongs to $\mathscr{C}$, since $G/H \in \mathscr{C}$ and $HK/H \cong K/(H \cap K)$ is a finite normal subgroup of G/H. Hence $G/K \in \mathfrak{X}$.

(iii) If G is torsion-free, then H is trivial and hence $G \in \mathscr{C}$.

We have shown that $\mathfrak{X}$ satisfies conditions (i), (ii) and (iii) of the definition of $\mathscr{C}$. Hence $\mathfrak{X} = \mathscr{C}$.

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https://doi.org/10.1017/S0017089500002433 Published online by Cambridge University Press