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THE BAKER–PYM THEOREM AND MULTIPLIERS*

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Dedicated to Professor Junzo Wada and Professor Emeritus Masahiro Nakamura

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A new interpretation of the Baker–Pym theorem is given in terms of operators and applies to a characterization of multipliers on a Banach algebra.

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Introduction

In this note we give, in terms of operators, a new interpretation of the well-known Baker–Pym theorem [1], from which a commutativity condition for Banach algebras was derived. In fact we show, under some conditions, that if ϕ is a bilinear mapping of $A \times X$ into Y such that $\|\phi(a,x)\| \leqq \beta \|ax\|$ for some positive constant β and for all $a \in A$, $x \in X$, then there exists a bounded linear operator T of X into Y such that $\phi(a,x) = T(ax)$ for all $a \in A$, $x \in X$ (Theorem 4). Here A, X and Y denote a Banach algebra, an essential left Banach A-module and a Banach space, respectively. We also show that if the above assertion is true for $Y = \mathbb{C}$, the complex numbers, then this assertion is true for all Banach spaces Y (Theorem 3).

We further obtain a characterization of multipliers on a Banach algebra applying the results obtained by this new interpretation (Corollaries 5 and 6).

The Baker–Pym theorem

The Baker–Pym theorem is stated as follows: Let A be a (complex) normed algebra with bounded approximate identity $\{e_\lambda\}$, X an essential left normed A-module and Y a normed space. If ϕ is a continuous bilinear mapping of $A \times X$ into Y such that $\|\phi(a,x)\| \leqq \beta \|ax\|$ for some positive constant β and for all $a \in A$, $x \in X$, then $\phi(a,x) = \lim_\lambda \phi(e_\lambda, ax)$ for all $a \in A$, $x \in X$, and conversely.

In their theorem, we can assume, without loss of generality, that all three spaces are complete by passing to the completion of the spaces involved. Then throughout this

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note, let A be a Banach algebra, X an essential left Banach A-module and Y a Banach space, unless it is explicitly stated otherwise.

Let ϕ be a continuous bilinear mapping of $A\times X$ into Y such that $\|\phi(a,x)\|\leqq\beta\|ax\|$ for some positive constant β and for all $a\in A$, $x\in X$. If A has a bounded approximate identity $\{e_\lambda\}$, then for each $x\in X$, $\{\phi(e_\lambda,x)\}$ becomes a Cauchy net in Y by the essentiality of X, and hence the operator T of X into Y defined by $T(x)=\lim_\lambda\phi(e_\lambda,x)$ $(x\in X)$ belongs to $B(X,Y)$, the Banach space of all bounded linear operators of X into Y. Thus the Baker–Pym theorem can be formulated as follows: If ϕ is a bilinear mapping of $A\times X$ into Y such that $\|\phi(a,x)\|\leqq\beta\|ax\|$ for some positive constant β and for all $a\in A$, $x\in X$, then there exists $T\in B(X,Y)$ such that $\phi(a,x)=T(ax)$ for all $a\in A$, $x\in X$ and conversely.

But this is still true under a weaker condition on A. In fact we have the following:

Theorem 1. *Assume that A possesses a left approximate identity and let ϕ be a bilinear mapping of $A\times X$ into Y such that $\|\phi(a,x)\|\leqq\beta\|ax\|$ for some positive constant β and for all $a\in A$, $x\in X$. Then there exists $T\in B(X,Y)$ such that $\phi(a,x)=T(ax)$ for all $a\in A$, $x\in X$.*

Proof. Let $\{e_\lambda\}$ be a left approximate identity and X_0 the linear span of $AX=\{ax:a\in A,x\in X\}$. If $x\in X_0$, then, by the assumption on ϕ, $\{\phi(e_\lambda,x)\}$ is a Cauchy net in Y, and so it has a limit point in Y, say T_0x. Then T_0 is a bounded linear operator of X_0 into Y and hence T_0 has a unique continuous linear extension T to X because X is essential. In this case, we have from the essentiality of X that

$$\|\phi(a,x)+Tx\|\leqq\beta\|ax+x\|$$

for all $a\in A$, $x\in X$. Let $A_1=A\oplus\mathbb{C}$ be the Banach algebra obtained from A by adjoining an identity and define

$$\psi(a+\alpha,x)=\phi(a,x)+\alpha Tx\qquad(a+\alpha\in A_1,\ x\in X).$$

Then ψ is a continuous bilinear mapping of $A_1\times X$ into Y. If $\alpha\neq0$, then for each $a\in A$, $x\in X$,

$$\begin{aligned}\|\psi(a+\alpha,x)\|&=|\alpha|\,\|\phi(\alpha^{-1}a,x)+Tx\|\\&\leqq|\alpha|\beta\|\alpha^{-1}ax+x\|\\&=\beta\|(a+\alpha)x\|,\end{aligned}$$

so that $\|\psi(a+\alpha,x)\|\leqq\beta\|(a+\alpha)x\|$. Of course this inequality holds for $\alpha=0$. Then the Baker–Pym theorem implies that $\psi(a+\alpha,x)=\psi(1,ax+\alpha x)$ and hence $\phi(a,x)=T(ax)$ for all $a\in A$, $x\in X$. □

Theorem 2. *Let ϕ be a continuous bilinear mapping of $A\times X$ into Y. Then the following are equivalent:*

(1) $\phi(ab,x)=\phi(a,bx)$ *for all* $a, b\in A$ *and* $x\in X$.
(2) $\|\phi(a,bx)\|\leqq\beta_x\|ab\|$ *for all* $a,b\in A$, $x\in X$ *and for some* $\beta_x>0$ *depending on* x.

Proof. (1) ⇒ (2). Obviously.
(2) ⇒ (1). We prove this statement by the standard method (cf. 3, p. 227]). Let A_1 be the Banach algebra obtained from A by adjoining an identity. Then X becomes a unital left Banach A_1-module. For a, b, $c\in A$, $x\in X$ and $f\in Y^*$, the dual space of Y, set

$$F(\lambda)=f(\phi(a\exp(-\lambda b),(\exp(\lambda b))cx)).$$

Then F is an entire function and $|F(\lambda)|\leqq\beta_x\|f\|\ \|ac\|$ by (2), so that by Liouville's theorem F is constant. Note also that the coefficient of λ in the power series expansion of F is $f(\phi(a,bcx)-\phi(ab,cx))$. Then we have $f(\phi(a,bcx))=f(\phi(ab,cx))$, so that $\phi(a,bcx)=\phi(ab,cx)$ since f is arbitrary. We thus obtain (1) from the essentiality of X and the bilinearity and the continuity of ϕ. □

Let $BP(A,X,Y)$ be the following assertion: if ϕ is an arbitrary bilinear mapping of $A\times X$ into Y such that $\|\phi(a,x)\|\leqq\beta\|ax\|$ for some positive constant β and for all $a\in A$, $x\in X$, then there exists $T\in B(X,Y)$ such that $\phi(a,x)=T(ax)$ for all $a\in A$, $x\in X$. Then the following result is a reduction of the Baker–Pym type theorem.

Theorem 3. *If $BP(A,X,\mathbb{C})$ is true, then $BP(A,X,Y)$ is true for every Banach space Y.*

Proof. Suppose that $BP(A,X,\mathbb{C})$ is true and let Y be an arbitrary Banach space. Let ϕ be a bilinear mapping of $A\times X$ into Y such that $\|\phi(a,x)\|\leqq\beta\|ax\|$ for some positive constant β and for all $a\in A$, $x\in X$. For each $g\in Y^*$, set

$$\phi_g(a,x)=g(\phi(a,x))\quad(a\in A,x\in X).$$

Then ϕ_g is a bilinear mapping of $A\times X$ into $\mathbb{C}$ such that $\|\phi_g(a,x)\|\leqq\beta\|g\|\ \|ax\|$ for all $a\in A$, $x\in X$. By the assumption $BP(A,X,\mathbb{C})$, there exists $F(g)\in X^*$ such that $\phi_g(a,x)=\langle ax,F(g)\rangle$ for all $a\in A$, $x\in X$. Such an $F(g)$ is unique from the essentiality of X. In this case, we can easily see that F is a bounded linear operator of Y^* into X^*. If also $\lim_{n\to\infty}\|g_n-g\|=0$ and $\lim_{n\to\infty}\|F(g_n)-f\|=0$ for $f\in X^*$ and $g_n\in Y^*(n=1,2,\ldots)$, then for $a\in A$ and $x\in X$, we have

$$\langle ax,f\rangle=\lim_{n\to\infty}\langle\phi(a,x),g_n\rangle=\langle ax,F(g)\rangle,$$

so that $f=F(g)$ from the essentiality of X. Then F is continuous on Y^* from the closed graph theorem.

Now let us consider the dual mapping F^* of F. If Z is a Banach space, we denote by π_z the canonical mapping of Z into Z^{**}. Then for any $a\in A$ and $x\in X$, $F^*(\pi_x(ax))\in\pi_y(Y)$. Actually if a net $\{g_\lambda\}$ in Y^* converges to $g\in Y^*$ in the weak*-topology, then we have

$$\lim_\lambda\langle ax, F(g_\lambda)\rangle = \lim_\lambda\langle \phi(a,x), g_\lambda\rangle$$
$$= \langle \phi(a,x), g\rangle$$
$$= \langle ax, F(g)\rangle,$$

so that $F^*(\pi_x(ax))$ is weak*-continuous. Therefore $F^*(\pi_x(ax))$ must belong to $\pi_y(Y)$. It follows from this observation that for each $x \in X_0$, the linear span of AX, there exists a unique element T_0x of Y such that $F^*(\pi_x(x)) = \pi_y(T_0x)$. In this case, it is easy to see that T_0 is a continuous linear operator X_0 into Y. Then T_0 has a unique continuous linear extension T to X and we can see that T is the desired operator. In fact let $a \in A$ and $x \in X$. Then we have

$$\langle T(ax), g\rangle = \langle g, \pi_y(T(ax))\rangle = \langle g, F^*(\pi_x(ax))\rangle$$
$$= \langle F(g), \pi_x(ax)\rangle = \langle ax, F(g)\rangle$$
$$= \langle \phi(a,x), g\rangle$$

for all $g \in Y^*$, and hence $\phi(a,x) = T(ax)$. Consequently $BP(A,X,Y)$ is also true. □

Now a multiplier from A to X is a bounded linear mapping from A to X which commutes with module multiplication. Denote by $M(A,X)$ the set of all multipliers from A to X. Note that X^* becomes a right Banach A-module under the module multiplication given by $\langle x, f \circ a\rangle = \langle ax, f\rangle$ $(a \in A, x \in X, f \in X^*)$. For each $f \in X^*$, define a mapping τ_f from A to X^* by $\tau_f(a) = f \circ a$ $(a \in A)$. Then $\{\tau_f : f \in X^*\} \subset M(A, X^*)$. In this setting we have the following:

Theorem 4. *Suppose $\{\tau_f : f \in X^*\} = M(A, X^*)$. If ϕ is a bilinear mapping of $A \times X$ into Y such that $\|\phi(a,x)\| \leqq \beta\|ax\|$ for some positive constant β and for all $a \in A$, $x \in X$, then there exists $T \in B(X,Y)$ such that $\phi(a,x) = T(ax)$ for all $a \in A$, $x \in X$.*

Proof. It is sufficient to prove the case of $Y = \mathbb{C}$ from the preceding theorem. Let T_ϕ be defined by $\langle x, T_\phi(a)\rangle = \phi(a,x)$ for all $a \in A$ and $x \in X$. Then T_ϕ is a bounded linear operator of A into X^*. Also given $a, b \in A$, we have

$$\langle x, T_\phi(ab)\rangle = \phi(ab, x)$$
$$= \phi(a, bx) \quad \text{(by Theorem 2)}$$
$$= \langle bx, T_\phi(a)\rangle$$
$$= \langle x, (T_\phi(a)) \circ b\rangle$$

for all $x \in X$. Then $T_\phi(ab) = (T_\phi(a)) \circ b$. In other words, T_ϕ is in $M(A, X^*)$, so that there exists $f \in X^*$ with $\tau_f = T_\phi$ by the assumption. Then we have

$$\phi(a,x) = \langle x, T_\phi(a) \rangle = \langle x, \tau_f(a) \rangle = \langle x, f \circ a \rangle = f(ax)$$

for all $a \in A$ and $x \in X$. □

If A has a bounded approximate identity, then $\{\tau_f : f \in X^*\} = M(A, X^*)$ as considered by C. V. Comisky [2]. Then we can regard the preceding theorem as a generalization of the Baker–Pym theorem.

Applications

Let $ZM(A)$ be the central double multiplier algebra of A and $QM(A)$ be the quasi-multiplier space of A (*cf.* [4, 5, 6]). Let λ be the natural embedding from $ZM(A)$ into $QM(A)$, i.e., $\lambda(T)(a,b) = T(ab)$, a, $b \in A$. Then we have the following:

Corollary 5. *If A has a left approximate identity $\{e_\lambda\}$, then $\lambda(ZM(A))$ equals exactly the set of all $\phi \in QM(A)$ such that $\|\phi(a,b)\| \leqq \beta \|ab\|$ for some positive constant β and for all a, $b \in A$.*

Proof. Let us take $X = Y = A$ in Theorem 1. If $\phi \in QM(A)$ is such that $\|\phi(a,b)\| \leqq \beta \|ab\|$ for some positive constant β and for all a, $b \in A$, then there exists a bounded linear operator T of A into itself such that $\phi(a,b) = T(ab)$ for all a, $b \in A$. Hence

$$T(abc) = \phi(ab, c) = a\phi(b,c) = aT(bc),$$

$$T(abc) = \phi(a, bc) = \phi(a,b)c = (T(ab))c$$

for all a, b, $c \in A$. Then

$$T(ab) = \lim_\lambda T(ae_\lambda b) = \lim_\lambda aT(e_\lambda b) = aT(b),$$

$$T(ab) = \lim_\lambda T(e_\lambda ab) = \lim_\lambda (T(e_\lambda a))b = (T(a))b$$

for all a, $b \in A$. In other words, $T \in ZM(A)$ and $\phi = \lambda(T)$. □

Corollary 6. *Assume that the right Banach A-module X^* is essential. If T is a continuous linear mapping of A into X such that $|\langle Ta, f \rangle| \leqq \beta \|f \circ a\|$ for some positive constant β and for all $a \in A$ and $f \in X^*$, then $T \in M(A, X)$, and conversely provided A has a bounded approximate identity.*

Proof. All the above arguments are true for the right module case. In Theorem 2 for such a case, replace X by X^* and take $Y=\mathbb{C}$, and set $\phi(a,f)=\langle Ta,f\rangle$ for each $a\in A$ and $f\in X^*$. Then the desired result follows immediately. □

We will close this note by proposing the following:

Problem. Is $BP(A,X,\mathbb{C})$ usually true?

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