A Method of Evaluating Certain Determinants

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1. Let $[a_{st}]$ $(s, t = 0, 1, \ldots, n)$ be a square matrix of order $n+1$ and determinant $|a_{st}|$ and suppose that by repeated "isolation" of the variables the corresponding bilinear form has been expressed as

$$\sum_{s=0}^{n} \sum_{t=0}^{n} a_{st} X_s Y_t = \sum_{r=0}^{n} c_r \left\{ \sum_{s=r}^{n} p_{rs} X_s \right\} \left\{ \sum_{t=r}^{n} q_{rt} Y_t \right\} \tag{1}$$

where, for all r,

$$p_{rr} = q_{rr} = 1. \tag{2}$$

Then

$$|a_{st}| = \prod_{r=0}^{n} c_r. \tag{3}$$

Now (1) implies, and is implied by, the identities

$$a_{st} = \sum_{r=0}^{\min(s,t)} c_r p_{rs} q_{rt} \quad (s, t = 0, 1, \ldots, n). \tag{4}$$

Thus, from any known identity of the form (4), subject to the condition (2), we may at once infer, using (3), the value of the corresponding determinant $|a_{rs}|$.

I have elsewhere[1] applied this method, without explicit formulation, to evaluate the determinants in which

(i) $a_{st} = F(a, -s-t; c; x)$,

(ii) $a_{st} = \binom{s+t+2\lambda-1}{s+t}^{-1} P_{s+t}^{\lambda}(x)$,

where F is the hypergeometric function and P_n^{λ} the ultra-spherical polynomial. I collect here some further instances, in part I believe new, in part furnishing alternative proofs of results given previously by myself and others[2].

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[1] (1) §6.

[2] (2) and (3) for the formulae of §§2, 3.

2. Let

$$H_n(x) = \exp(\tfrac{1}{2}x^2)(-d/dx)^n \exp(-\tfrac{1}{2}x^2).$$

be the Hermitian polynomial of degree n. Then[1]

$$H_{s+t}(x) = \sum_{r=0}^{\min(s,t)} (-1)^r r! \binom{s}{r}\binom{t}{r} H_{s-r}(x)\, H_{t-r}(x)$$

which we identity with (4) on letting

$$c_r = (-1)^r r!, \; p_{rs} = \binom{s}{r} H_{s-r}(x), \; q_{rt} = \binom{t}{r} H_{t-r}(x).$$

Since (2) is clearly satisfied we have at once

$$|H_{s+t}(x)| = (-1)^{\frac{1}{2}n(n+1)} \prod_{r=1}^{n} (r!). \tag{5}$$

3. Let $P_n(x)$ be Legendre's polynomial, let $2u = x+1$, $2v = x-1$ and $k_p{}^q$ be the coefficient of z^{2p-q} in the expansion of

$$\{(z+u)(z+v)\}^p.$$

Evidently $k_p{}^0 = 1$ for all p.

Now[2]

$$P_{s+t}(x) = k_s{}^s k_t{}^t + 2 \sum_{r=1}^{\min(s,t)} (uv)^r k_s^{s-r} k_t^{t-r} = \sum_{r=0}^{\min(s,t)} c_r p_{rs} q_{rt}$$

with $$c_0 = 1, \; c_r = 2(uv)^r \quad (r > 1).$$

Hence

$$|P_{s+t}(x)| = 2^n (uv)^{\frac{1}{2}n(n+1)} = 2^{-n^2}(x^2-1)^{\frac{1}{2}n(n+1)}. \tag{6}$$

4. Let

$$F^{(1)}(s,t) = \sum_{m=0}^{s} \sum_{n=0}^{t} \frac{(a)_{m+n}(-s)_m(-t)_n}{m!\; n!\; (c)_{m+n}} x^m y^n$$

$$F^{(2)}(s,t) = \sum_{m=0}^{s} \sum_{n=0}^{t} \frac{(a)_{m+n}(-s)_m(-t)_n}{m!\; n!\; (c)_m (c')_n} x^m y^n$$

$$F^{(3)}(s,t) = \sum_{m=0}^{s} \sum_{n=0}^{t} \frac{(a)_m (a')_n (-s)_m(-t)_n}{m!\; n!\; (c)_{m+n}} x^m y^n,$$

where $$(k)_r = k(k+1)\dots(k+r-1).$$

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[1] (4), 10 (5), with a change of notation.

[2] (2), 230 (17).

These are, of course, instances of Appell's hypergeometric functions of two variables in which a pair of numerator parameters have been replaced by negative integers and, in consequence, the functions reduce to polynomials in two variables of total degree $s+t$.

If now we adapt to these special parameters known expansions[1] of Appell's functions we have

$$F^{(1)}(s, t) = \sum_{r=0}^{\min(s,t)} \frac{r!\,(a)_r\,(c-a)_r}{(c+r-1)_r\,(c)_{2r}} (xy)^r \binom{s}{r}\binom{t}{r} F\left[\begin{matrix} a+r, & -s+r, \\ & c+2r, \end{matrix}\; x\right] \times F\left[\begin{matrix} a+r, & -t+r, \\ & c+2r, \end{matrix}\; y\right] \quad (6)$$

$$F^{(2)}(s, t) = \sum_{r=0}^{\min(s,t)} \frac{r!\,(a)_r}{(c)_r\,(c')_r} (xy)^r \binom{s}{r}\binom{t}{r} F\left[\begin{matrix} a+r, & -s+r, \\ & c+r, \end{matrix}\; x\right] \times F\left[\begin{matrix} a+r, & -t+r, \\ & c'+r, \end{matrix}\; y\right] \quad (7)$$

and

$$F^{(3)}(s, t) = \sum_{r=0}^{\min(s,t)} \frac{(-1)^r\,(a)_r\,(a')_r}{(c+r-1)_r\,(c)_{2r}} (xy)^r \binom{s}{r}\binom{t}{r} F\left[\begin{matrix} a+r, & -s+r, \\ & c+2r, \end{matrix}\; x\right] \times F\left[\begin{matrix} a'+r, & -t+r, \\ & c+2r, \end{matrix}\; y\right], \quad (8)$$

where F is in every case the ordinary hypergeometric function.

Each of these formulae is of the form (4); the condition (2) is satisfied and we deduce

$$|F^{(1)}(s, t)| = (xy)^{\frac{1}{2}n(n+1)} \prod_{r=1}^{n} \left[\frac{r!\,(a)_r\,(c-a)_r}{(c+r-1)_r\,(c)_{2r}}\right] \quad (9)$$

$$|F^{(2)}(s, t)| = (xy)^{\frac{1}{2}n(n+1)} \prod_{r=1}^{n} \left[\frac{r!\,(a)_r}{(c)_r\,(c')_r}\right] \quad (10)$$

and

$$|F^{(3)}(s, t) = (-xy)^{\frac{1}{2}n(n+1)} \prod_{r=1}^{n} \left[\frac{r!\,(a)_r\,(a')_r}{(c+r-1)_r\,(c)_{2r}}\right]. \quad (11)$$

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[1] (5), pp. 253–4 (30), (26), (28).

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If we recall that, when $y = x$, $F^{(1)}$ reduces to an ordinary hypergeometric function [1] we see that (9) includes as a special case

$$\left| F\begin{bmatrix} a, & -s-t, & x \\ & c, & \end{bmatrix} \right| = x^{n(n+1)} \prod_{r=1}^{n} \left[\frac{r!\,(a)_r\,(c-a)_r}{(c+r-1)_r\,(c)_{2r}} \right], \tag{12}$$

which I have given elsewhere [2].

Formulae of more apparent complexity may be obtained by increasing the number of the non-integral parameters involved and employing for example the formulae (11)–(13) of (6). Alternatively we may consider degenerate hypergeometric functions and from [3]

$$\begin{aligned} &{}_1F_1[-s-t;\ c;\ x] \\ &= \sum_{r=0}^{\min(s,t)} \frac{(-1)^r\, r!\,(x)^{2r}}{(c+r-1)_r\,(c)_{2r}} \binom{s}{r}\binom{t}{r} {}_1F_1[-s+r;\, c+2r;\, x]\, {}_1F_1[-t+r;\, c+2r;\, x] \end{aligned}$$

obtain the result

$$\left| {}_1F_1[-s-t;\ c;\ x] \right| = (-x^2)^{\frac{1}{2}n(n+1)} \prod_{r=1}^{n} \left[\frac{r!}{(c+r-1)_r\,(c)_{2r}} \right]. \tag{13}$$

5. Inspection of the formulae (9)–(11) shows that the right-hand side of each arises from the terms of highest degree only in each element of the determinant on the left. If we take these terms only and recall that

$$\left| (-1)^{s+t} a_{st} \right| = \left| a_{st} \right|,$$

we find that

$$\left| \frac{(a)_{s+t}}{(c)_{s+t}} \right| = \prod_{r=1}^{n} \left[\frac{r!\,(a)_r\,(c-a)_r}{(c+r-1)_r\,(c)_{2r}} \right], \tag{14}$$

$$\left| \frac{(a)_{s+t}}{(c)_s\,(c')_t} \right| = \prod_{r=1}^{n} \left[\frac{r!\,(a)_r}{(c)_r\,(c')_r} \right], \tag{15}$$

$$\left| \frac{(a)_s\,(a')_t}{(c)_{s+t}} \right| = (-1)^{\frac{1}{2}n(n+1)} \prod_{r=1}^{n} \left[\frac{r!\,(a)_r\,(a')_r}{(c+r-1)_r\,(c)_{2r}} \right]. \tag{16}$$

Of these I have given (14) elsewhere [4]: (15) and (16) are more simply written as

$$\left| (a)_{s+t} \right| = \prod_{r=1}^{n} \left[r!\,(a)_r \right],$$

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[1] (7), 23 (25).

[2] (1) (9).

[3] (6), 126 (71).

[4] (1) (11).

which is elementary, and the more recondite

$$\left|\frac{1}{(c)_{s+t}}\right| = (-1)^{\frac{1}{2}n(n+1)} \prod_{r=1}^{n} \left[\frac{r!}{(c+r-1)_r\,(c)_{2r}}\right].$$

REFERENCES.

1. J. L. Burchnall, " Some determinants with hypergeometric elements ", *Quart. J. of Math.* (Oxford) (2), 3(1952), 151-157.
2. ————, " An algebraic property of the classical polynomials ", *Proc. London Math. Soc.* (3), I (1951), 232–240.
3. J. Geronimus, " On some persymmetric determinants formed by the polynomials of M. Appell ", *Journal London Math. Soc.*, 6 (1931), 55–59.
4. J. L. Burchnall, " A note on the polynomials of Hermite ", *Quart. J. of Math.* (Oxford), 12 (1941), 9–11.
5. J. L. Burchnall and T. W. Chaundy, " Expansions of Appell's double hypergeometric functions ", *Quart. J. of Math.* (Oxford), 11 (1940), 249–270.
6. ————, " Expansions of Appell's double hypergeometric functions, II ", *Quart. J. of Math.* (Oxford), 12 (1941), 112–128.
7. P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques* (Paris, 1926).

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