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Symmetric S-unimodal mappings and positive Liapunov exponents

TOMASZ NOWICKI
IZMiS, SGGW-AR, Nowoursynowska 166, *Warsaw, Poland*

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Abstract. Symmetric S-unimodal functions with positive Liapunov exponent of the critical value have an invariant measure absolutely continuous with respect to Lebesgue measure.

Introduction

An important branch in one-dimensional dynamics is the research of the invariant measure absolutely continuous with respect to Lebesgue measure.

Collet and Eckmann [3] proved the existence of such a measure for S-unimodal mappings which (apart from some weak regularity assumptions) satisfy the two following conditions:

There are two constants $\lambda > 1$ and $K > 0$ such that for all $n \in \mathbb{N}$

(C1) $|(d(f^n)/dx)(f(c))| \geq K\lambda^n$;

(C2) if $f^n(z) = z$ then $|(d(f^n)/dx)(z)| \geq K\lambda^n$;

where $f^n = f \circ \cdots \circ f$ n times and c is the unique critical point of f.

Moreover, Collet [1] proved that this measure is unique and ergodic.

In this paper we shall deal with the function $f \in C^3, f: [0, 1] \to [0, 1]$, which satisfies the following assumptions:

(A0) f is S-unimodal; that means that there exists a unique $c \in (0, 1)$ such that f is increasing on $(0, c)$ and decreasing on $(c, 1)$ and $Sf \leq 0$ where $Sf = f'''/f' - 3/2(f''/f')^2$;

(A1) $f''(c) \neq 0$;

(A2) $f(0) = f(1) = 0$;

(A3) $f(x) = f(1-x)$ for $x \in (0, 1)$.

The regularity assumptions in [3] are a weaker form of (A0)–(A2). In fact, we choose the stronger form in order to omit some technical lemmas. Instead of (A2) one can require the existence of a restrictive central point. The only significantly new assumption is (A3), the symmetry of f, needed in the proof of lemma 9. (A3) is satisfied by usually considered families of functions such as the example $f(x) = 4\alpha x(1-x)$.

The result of this paper is proposition 13 which states that under (A0)–(A3), (C1) implies (C2). From the results of [1] and [3] we have:

THEOREM. *If f satisfies* (A0)–(A3) and (C1) *then f has an invariant measure absolutely continuous with respect to Lebesgue measure. This measure is unique and ergodic.*

One would expect this theorem to be helpful in computer experiments, as (C1) is much easier to check than (C2).

Preliminaries

In this section we quote without proof some useful lemmas.

We shall use the following notation: $f^1 = f$ and for $n \geq 1$, $f^{n+1} = f^n \circ f$, $x_n = f^n(x)$, $Df^n = d(f^n)/dx$.

LEMMA 1 (see [2, II.4]). *If $Sf \leq 0$ and $Sg \leq 0$ then $S(f \circ g) \leq 0$ and $Sf^n \leq 0$. $Sf \leq 0$ implies that $|f'|$ has no positive local minima.*

We call the interval I a *sink* for f if there is an n such that $f^n(I) \subset I$ and $Df^n|_I \neq 0$.

LEMMA 2 (see [2, II.4]). *If f satisfies* (A0) *and* (C1) *then $f|_{(f^2(c), f(c))}$ has no sinks and no attractive periodic orbits.*

Remark 3. By lemma 2 we may assume later on that $|f'(0)| > 1$.

LEMMA 4 (see [3], lemma 2.2]). *There are two constants $m > 0$ and $M > 0$ such that*

$$m|x - c| \leq |f'(x)| \leq M|x - c|$$

and

$$\frac{m}{2}|x - c|^2 \leq |x_1 - c_1| \leq \frac{M}{2}|x - c|^2.$$

We define x' by $f(x') = f(x)$ and $x' \neq x$ if $x \neq c$ and $c' = c$. We shall use (a, b) to denote the interval with the endpoints a and b independent of their order, that means not necessarily $a < b$.

LEMMA 5 (see [3], Lemma II.5.6.]). *Let*

$$\underline{K}^n = \{x \colon x_i \notin (x, x') \text{ for } i = 1, \ldots, n-1 \text{ and } x_n \in (x, x')\}.$$

Every connected component of $\underline{K}^n$ is of the form (p, q') with $p_n = p$ and $q_n = q$. Moreover $Df^n|_{\underline{K}^n} \neq 0$.

LEMMA 6 (see [1, lemma 2.6]). *If $Sf \leq 0$ and $Df^n|_{(x,y)} \neq 0$ then*

$$|x_n - y_n| \geq (Df^n(x) Df^n(y))^{\frac{1}{2}} |x - y|.$$

Estimates

In this section we assume tacitly in all lemmas that f satisfies (A0)-(A3) and (C1).

First we prove a technical lemma. It will be useful at the very end of the paper, but we put it here in order to avoid interruptions during the estimations.

LEMMA 7. *Let $g \in C^2(u, v)$; $g'(u) = g'(v) = 0$ and there is a unique $w \in (u, v)$ with $g''(w) = 0$. Define for a fixed $x \in (u, v)$ a function $h(t)$; $t \in (u, v)$ by*

$$h(t) = \frac{g(t) - g(x)}{t - x} \qquad \text{for } t \neq x \text{ and } h(x) = g'(x).$$

Then $|h(t)|$ has only one local extremum in (u, v) and it is a maximum. Hence for $a \leq t \leq b$, $|h(t)| \geq \min(|h(a)|, |h(b)|)$.

Proof. Let us first consider the case $g' > 0$. Assume $x \in (u, w)$. The situation with $x \in (w, v)$ can be handled similarly. $g' > 0$ implies $h > 0$ and $g''|_{(u,w)} > 0$ and $g''|_{(w,v)} < 0$. We have for $t \neq x$

$$h'(t) = \frac{d}{dt}\left(\frac{g(t) - g(x)}{t - x}\right) = \frac{g'(t) - h(t)}{t - x}$$

Thus $h'(u) = -h(u)/(u - x) > 0$ and $h'(v) = -h(v)/(v - x) < 0$. We conclude that h has at least one local maximum in (u, v). Now we shall prove the uniqueness of this extremum.

By the mean value theorem we have for some $t_1 \in (t, x)$ and $t_2 \in (t, t_1)$

$$h'(t) = \frac{g'(t) - g'(t_1)}{t - x} = \frac{g''(t_2)(t - t_1)}{t - x}.$$

From $t_1 \in (t, x)$ it follows that $\operatorname{sgn} h'(t) = \operatorname{sgn} g''(t_2)$. Therefore $h(t)$ is increasing for $t \in (u, w)$ as then $t_2 \in (t, x) \subset (u, w)$. Thus if $h'(t_0) = 0$ then $t_0 \in (w, v)$. Then

$$h''(t_0) = \frac{g''(t_0) - 2h'(t_0)}{t_0 - x} = \frac{g''(t_0)}{t_0 - x}.$$

Since by our assumption we have $x < w < t_0$ and $g''(t_0) < 0$, $h''(t_0)$ is negative and h has a local maximum at t_0. The point t_0 is unique since if there was another maximum at say $t' \neq t_0$, h would have a minimum between t' and t_0 which by previous considerations is impossible.

If $x = w$ then h is increasing on (u, w) and decreasing on (w, v) and has maximum at $t = w$.

This proves the case $g' > 0$. In order to complete the proof let us consider the case $g' < 0$. Then

$$|h(t)| = -h(t) = \frac{(-g)(t) - (-g)(x)}{t - x}.$$

The function $(-g)$ satisfies the assumptions of lemma 7 and $(-g)' > 0$. By the previous part of the proof we can state that $|h(t)|$ has only one local maximum in (u, v). □

Definition 1. We say that the interval (c, b) satisfies $*(n)$ if $b_n = c$ and $Df^n|_{(c,b)} \neq 0$.

We say that the interval (a, b) satisfies $**(n)$ if $b_n = c$, and $Df^n|_{(a,b)} \neq 0$ and $Df^n(a) = 0$.

Remark 8. If (a, b) satisfies $**(n)$ then for some $r < n$, $a_r = c$ and (a_r, b_r) satisfies $*(n - r)$. This follows from the fact that $Df^n(x) = Df^{n-r}(x_r) \cdot Df^r(x)$. We omit the details.

LEMMA 9. *For every n and every interval (c, b), if (c, b) satisfies $*(n)$ then $|c_n - b_n| > |c - b|$.*

Proof. Suppose the contrary. Then there exists an n and an interval (c, b) satisfying $*(n)$ such that $|c_n - b_n| \leq |c - b|$. By symmetry we have $|c_n - (b')_n| \leq |c - b'|$ and (c, b') satisfies $*(n)$. Since $b_n = (b')_n = c$ we have either $f^n(c, b) \subset (c, b)$ or $f^n(c, b') \subset (c, b')$. Hence f has a sink which contradicts lemma 2. □

Definition 2. We say that p is the *central point* for f^n if $f^n(p)=p$ and $Df^n|_{(p,c)}>0$.

LEMMA 10. *Let* (c, b) *satisfy* $*(n)$. *Then*

(i) f^n *has a central point* p;

(ii) $(c, b)\subset(p, p')$;

(iii) *there exists* $q\in(b, b')$ *such that* $q_n=q$;

(iv) (q', p) *and* (q, p') *are connected components of* $\underline{K}^n$ ($\underline{K}^n$ *was defined in* lemma 5).

Proof. For definiteness take $c<b$. We may assume that f^n is decreasing on (c, b), otherwise we take f^n on (b', c). Therefore we have $c_n>b_n=c$ and $b_n=c<b$. Hence there exists a $q\in(c, b)$ such that $q_n=q$. f^n is decreasing on (c, b) so it has no other fixed point in this interval.

We claim that n is the prime period of q. Suppose the contrary. Let $k<n$ be the prime period of q. Then $q_k=q$ and $q_n=q$ we have $n=ks$, for some $s<n$. f^n is decreasing on (c, b) implies f^k decreasing on (c, b) so s is odd and $s\geq 3$. Let us consider $f^{2k}(c)$. Clearly $c_{2k}\neq c$. f^{2k} is increasing on (c, b) and by lemma 2 has no fixed point in this interval other than q.

If $c_{2k}>c$ then by lemma 1 $Df^{2k}(q)<1$ which contradicts lemma 2.

If $c_{2k}<c$ then as $q_{2k}=q>c$ there is a $z\in(c, q)$ with $z_{2k}=c$. This contradicts $Df^n|_{(c,b)}\neq 0$. Both contradictions complete the proof that n is the prime period of q.

Since f^n has no other fixed points in (b, b'), this implies that $q_i\notin(q, q')$ for $1\leq i\leq n-1$. Thus in some neighbourhood of q there are points of $\underline{K}^n$ and by lemma 5 q is one endpoint of some component of $\underline{K}^n$. Let p' be the other endpoint of the same component. By lemma 5 $p'=p_n$ and it is easy to check that p' is the required central point. □

PROPOSITION 11. *If f satisfies* (A0)-(A3) and (C1) *then there are two constants* $K_1>0$ *and* $\lambda_1>1$ *such that for every* (c, b) *satisfying* $*(n)$,

$$|c_n-b_n|\geq K_1\lambda_1^n|c-b|.$$

Proof. Let us consider f^n on $f(c, b)=(c_1, b_1)$. By lemma 1 there is a $q\in(c, b)$ such that $q_1\in(c_1, b_1)$ and $f^n(q)=q_1$. By symmetry we may assume that $q_n=q$. By lemmas 2, 4, 6 and (C1) we have

$$\text{(a)}\qquad |c_{n+1}-q_{n+1}|=|f^n(c_1)-f^n(q_1)|\geq(Df^n(c_1)Df^n(q_1))^{\frac{1}{2}}|c_1-q_1|$$

$$\geq K\lambda^{n/2}\frac{m}{2}|c-q|^2.$$

By lemmas 10 and 5 we have $|q_n-b_n|\geq|q-b|$ and hence

$$|b-c|=|b-q|+|q-c|\leq|q_n-b_n|+|q-c|=2|q-c|.$$

Therefore we have by the inequality (a)

$$\text{(b)}\qquad |c_{n+1}-q_{n+1}|\geq\frac{Km}{8}\lambda^{n/2}|b-c|^2.$$

On the other hand, again by lemma 4 we can estimate

$$\text{(c)} \qquad |c_{n+1} - q_{n+1}| = |f(c_n) - f(q_n)| \le |f(c_n) - f(b_n)|$$
$$= |f(c_n) - f(c)| \le \frac{M}{2} |c_n - b_n|^2$$

From the inequalities (b) and (c) we have

$$\frac{M}{2} |c_n - b_n|^2 \ge \frac{Km}{8} \lambda^{n/2} |b - c|^2.$$

The assertion follows for $\lambda_1 = \lambda^{n/4}$ and $K_1 = (Km/4M)^{\frac{1}{2}}$. □

Remark 12. Under the assumptions of proposition 11 there is a $\lambda_0 > 1$ such that for every n and every (c, b) satisfying $*(n)$ we have

$$\left| \frac{c_n - b_n}{c - b} \right| \ge \lambda_0^n.$$

Proof. By lemma 9 and proposition 11 we have

$$\lambda_0 = \inf_n \left| \frac{c_n - b_n}{c - b} \right|^{1/n} > 1.$$

PROPOSITION 13. *We assume that f satisfies* (A0)-(A3) *and* (C1). *Then for every n and every* (a, b) *satisfying* $**(n)$ *with* $b_n = c$ *we have*

$$\left| \frac{a_n - b_n}{a - b} \right| > \lambda_T^n; \qquad \text{A(n)}$$

and

$$|Df^n(b)| > \lambda_T^n; \qquad \text{B(n)}$$

where $\lambda_T = \min(\lambda_0, |f'(0)|^{\frac{1}{2}}) > 1$.

Proof. We prove $A(n)$ and $B(n)$ simultaneously by induction on n. For $n = 1$ $**(1)$ is equivalent to $*(1)$ and $A(1)$ is true by remark 12. Let (b, c) satisfy $*(1)$, for definiteness let $b < c$. We have by lemma 1 either

$$|Df(b)| \ge |Df(x)| \qquad \text{for } x \in (b, c), \text{ and then}$$
$$|Df(b)| \ge \left| \frac{b_1 - c_1}{b - c} \right| \ge \lambda_T \qquad \text{by} \qquad A(1);$$

or

$$|Df(b)| \ge |Df(x)| \qquad \text{for } x \in (0, b), \text{ and then}$$
$$|Df(b)| \ge |Df(0)| \ge \lambda_T.$$

Hence B(1) is true.

Inductive step: Now we assume that $A(k)$ and $B(k)$ are true for $k < n$. We shall first prove $A(n)$ for all (a, b) satisfying $**(n)$ and then we shall use $A(n)$ to prove $B(n)$.

Let (a, b) satisfy $**(n)$ with $a_r = b_n = c$ for some $r < n$. Then

$$\left| \frac{a_n - b_n}{a - b} \right| = \left| \frac{f^{n-r}(a_r) - f^{n-r}(b_r)}{a_r - b_r} \right| \cdot \left| \frac{a_r - b_r}{a - b} \right|.$$

By Remark 8 (a_r, b_r) satisfies $*(n-r)$ and by remark 12 we can estimate the first quotient by λ_0^{n-r}.

We have to estimate the second quotient. Let (u, v) be the maximal interval containing (a, b) such that $Df^r|_{(u,v)} \neq 0$. We have two possibilities:

(1^0) $\{u, v\} \cap \{0, 1\} = \emptyset$. Thus $Df^r(u) = Df^r(v) = 0$. Then both (u, a) and (a, v) satisfy $**(r)$ and by A(r) we have

$$\left|\frac{u_r - a_r}{u - a}\right| \geq \lambda_T^r \quad \text{and} \quad \left|\frac{v_r - a_r}{v - a}\right| \geq \lambda_T^r.$$

(2^0) $\{u, v\} \cap \{0, 1\} \neq \emptyset$. For definiteness let $u = 0$. Then (a, v) satisfies $**(r)$ and by A(r) and B(r) we have

$$\left|\frac{v_r - a_r}{v - a}\right| \geq \lambda_T^r \quad \text{and} \quad |Df^r(a)| \geq \lambda_T^r,$$

hence by lemma 6

$$\left|\frac{u_r - a_r}{u - a}\right| \geq (Df^r(0)Df^r(a))^{\frac{1}{2}} \geq \lambda_T^r \qquad \text{as } |Df^r(0)| = |Df(0)|^r \geq \lambda_T^r.$$

We are ready to use lemma 7 with $g = f^r$, $x = a$, $t = b$. We have

$$\left|\frac{a_r - b_r}{a - b}\right| = |h(b)| \geq \min(|h(u)|, |h(v)|) \geq \lambda_T^r.$$

This completes the estimate of the second quotient and the proof of A(n). We can now prove B(n).

Let $b \in (a, d)$ where (a, d) is the maximal interval with $Df^n|_{(a,d)} \neq 0$. We again use lemma 7 with $g = f^n$, $(u, v) = (a, d)$ and $x = b$. We have

$$|Df^n(b)| = |h(b)| \geq \min(|h(a)|, |h(d)|).$$

By A(n) for (a, b) we have $|h(a)| \geq \lambda_T^n$. If $d \notin \{0, 1\}$ then also by A(n) for (b, d), $|h(d)| \geq \lambda_T^n$. If $d \in \{0, 1\}$ say $d = 0$ then by lemma 6 we have

$$|h(d)| \geq (Df^n(0)Df^n(b))^{\frac{1}{2}}$$

Observe that $|Df^n(b)| > 1$ as otherwise by lemma 1 $|Df^n|_{(a,b)}| \leq 1$ which contradicts A(n). Thus in all cases $|h(d)| \geq \lambda_T^n$ and we can conclude that $|Df^n(b)| \geq \lambda_T^n$. This proves B$(n)$ and completes the proof of proposition 13. □

In order to return to the condition (C2) let z be such that $z_n = c$. We can find a point a such that (z, a) satisfies $**(n)$. Now (C2) follows from B(n).

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