A GENERALIZATION OF AN EARLY MULTIVARIATE INTEGRAL

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Earlier this year Professor W. L. Edge drew my attention to the paper (**1**). Since Black probably wrote his paper about 1890, the integral he considered must have been one of the earliest n-variate integrals to be evaluated. The present paper generalizes Black's result from a column vector as variate to a rectangular matrix—his integral is the case $p = m = k = 1$ of the integral J below.

According to M. J. M. Hill, Black's integral was part of an unpublished manuscript on the theory of evolution. His paper was of a mathematical-statistical nature and the generalized integral J is the conditional mean value of the quantity raised to the power p, with the function defined by the integral of the lemma as the marginal probability density function (**2**, page 269).

The integral to be evaluated is

$$J = \int_{\mathscr{Y}} \left\{ \sigma(X', Y') \begin{pmatrix} P_1, & Q_1' \\ Q_1, & R_1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right\}^p \operatorname{etr} \left\{ -(X', Y') \begin{pmatrix} P, & Q' \\ Q, & R \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right\} dY,$$

where X and Y are real matrices of orders $m \times k$ and $n \times k$ respectively; $\begin{pmatrix} P_1, & Q_1' \\ Q_1, & R_1 \end{pmatrix}$ and $\begin{pmatrix} P, & Q' \\ Q, & R \end{pmatrix}$ are real symmetric matrices with R_1 and R positive definite and P_1, Q_1, R_1 of orders $m \times m$, $n \times m$, $n \times n$ respectively as also are P, Q, R; $\sigma(M)$ is the trace of the matrix M and $\operatorname{etr}(M) = e^{\sigma(M)}$; $dY = \Pi dy_{ij}$, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$; $\mathscr{Y}$ is the set of all real $n \times k$ matrices and p is a non-negative integer.

First we consider the special case $p = 0$ as a

Lemma. *Let* $\begin{pmatrix} A, & H' \\ H, & B \end{pmatrix}$ *be partitioned like* $\begin{pmatrix} P, & Q' \\ Q, & R \end{pmatrix}$ *and let* B *be positive definite, then*

$$\int_{\mathscr{Y}} \operatorname{etr} \left\{ -(X', Y') \begin{pmatrix} A, & H' \\ H, & B \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right\} dY = \frac{\pi^{\frac{1}{2}nk}}{|B|^{\frac{1}{2}k}} \operatorname{etr} \{ -X'(A - H'B^{-1}H)X \}.$$

Proof. Let $Y = U - B^{-1}HX$ and the integral becomes

$$\operatorname{etr} \{ -X'(A - H'B^{-1}H)X \} \int_{\mathscr{Y}} \operatorname{etr}(-U'BU) dU.$$

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One of the ways of reducing a quadratic form to a sum of squares (**3**, page 136) is equivalent to the statement that if B is positive definite then there is a triangular matrix T such that $B = T'T$. Put $W = TU$ and the Jacobian of this change of variable is (**4**, page 87) $\dfrac{\partial(W)}{\partial(U)} = |T|^k = |B|^{\frac{1}{2}k}$, so

$$\int_{\mathscr{Y}} \operatorname{etr}(-U'BU)dU = \frac{1}{|B|^{\frac{1}{2}k}} \int_{\mathscr{Y}} \operatorname{etr}(-W'W)dW.$$

This completes the proof since the integral on the right is the product of nk integrals each of the type $\int_{-\infty}^{\infty} e^{-w^2}dw.$

To evaluate the integral J, we notice that its value is the pth derivative w.r.t. λ, evaluated at $\lambda = 0$, of

$$J_\lambda = \int_{\mathscr{Y}} \operatorname{etr}\left\{-(X', Y')\begin{pmatrix} P-\lambda P_1, & Q'-\lambda Q'_1 \\ Q-\lambda Q_1, & R-\lambda R_1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\right\} dY,$$

where λ is in a sufficiently small neighbourhood of zero, in order that the matrix $R-\lambda R_1$ be positive definite.

To justify differentiating under the integral sign one remarks that the integrand of the differentiated integral is a polynomial times an exponential term and that the exponential term is dominant. As in the proof of the lemma above, the part of the exponential depending on the new variable of integration U, is $\operatorname{etr}\{-U'(R-\lambda R_1)U\}$. Since both R and R_1 are positive definite they can be reduced simultaneously to diagonal form (**6**, page 106). This done,

$$\operatorname{etr}\{-U'(R-\lambda R_1)U\}$$

is the product of nk terms each of the type $\exp\{-(r-\lambda r_1)u^2\}$ with $r>0$, $r_1>0$, and for $\lambda \leqq \frac{1}{2}r/r_1$ this is bounded above by $\exp\{-\frac{1}{2}ru^2\}$. Thus if λ is less than the smallest root of $\frac{1}{2}RR_1^{-1}$, the differentiated integral is uniformly convergent and so the differentiation is valid.

Using the lemma, we have

$$J_\lambda = \frac{\pi^{\frac{1}{2}nk}}{|R-\lambda R_1|^{\frac{1}{2}k}} \operatorname{etr}\{-X'((P-\lambda P_1)-(Q'-\lambda Q'_1)(R-\lambda R_1)^{-1}(Q-\lambda Q_1))X\}$$

and it remains to find the pth derivative of this expression.

To do this let ρ_i, $1 \leqq i \leqq n$, be the roots of $R^{-1}R_1$ then (**6**, page 160) for $|\lambda| < \min\left(\frac{1}{\rho_i}\right)$

$$(R-\lambda R_1)^{-1} = (R(I-\lambda R^{-1}R_1))^{-1} = (I-\lambda R^{-1}R_1)^{-1}R^{-1} = \sum_{r=0}^{\infty} \lambda^r L_r,$$

where $L_r = (R^{-1}R_1)^r R^{-1}$. For $r \geqq 0$ let

$$M_r = (Q', Q'_1)\begin{pmatrix} L_r, & -L_{r-1} \\ -L_{r-1}, & L_{r-2} \end{pmatrix}\begin{pmatrix} Q \\ Q_1 \end{pmatrix}$$

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with $L_{-1} = O = L_{-2}$ and we have

$$(Q'-\lambda Q_1')(R-\lambda R_1)^{-1}(Q-\lambda Q_1)$$
$$= Q'R^{-1}Q+\lambda(Q'R^{-1}R_1R^{-1}Q-Q_1'R^{-1}Q-Q'R^{-1}Q_1)+\sum_{r=2}^{\infty}\lambda^r M_r.$$

Now let $S = R^{-1}Q-R_1^{-1}Q_1$ and $N_r = R_1(R^{-1}R_1)^r$ and it follows that

$$(P-\lambda P_1)-(Q'-\lambda Q_1')(R-\lambda R_1)^{-1}(Q-\lambda Q_1)$$
$$= (P-Q'R^{-1}Q)-\lambda(P_1-Q_1'R_1^{-1}Q_1+S'N_1S)-\sum_{r=2}^{\infty}\lambda^r S'N_rS.$$

Since $|R-\lambda R_1|^{-\frac{1}{2}k} = |R|^{-\frac{1}{2}k}d(\lambda)$, where $d(\lambda) = \prod_{i=1}^{n}(1-\lambda\rho_i)^{-\frac{1}{2}k}$, we have for the same choice of λ, being inside the circle of convergence of the logarithmic series,

$$\log d(\lambda) = -\tfrac{1}{2}k\sum_{i=1}^{n}\log(1-\lambda\rho_i) = \tfrac{1}{2}k\sum_{r=1}^{\infty}\sum_{i=1}^{n}\frac{\lambda^r\rho_i^r}{r} = \tfrac{1}{2}k\sum_{r=1}^{\infty}\frac{\lambda^r s_r}{r},$$

where s_r is the power sum $\rho_1^r+\rho_2^r+\ldots+\rho_n^r$.

Thus finally,

$$J_\lambda = \frac{\pi^{\frac{1}{2}nk}}{|R|^{\frac{1}{2}k}}\operatorname{etr}\{-X'(P-Q'R^{-1}Q)X\}\exp\left\{\sum_{r=1}^{\infty}a_r\lambda^r\right\},$$

where $a_1 = \sigma\{X'(P_1-Q_1'R_1^{-1}Q_1+S'N_1S)X\}+\frac{1}{2}ks_1$ and for $r \geqq 2$,

$$a_r = \sigma\{X'S'N_rSX\}+\frac{1}{2r}ks_r.$$

Now the pth derivative at $\lambda = 0$ of $\exp\left\{\sum_{r=1}^{\infty}a_r\lambda^r\right\}$ is the same as the pth derivative at $\lambda = 0$ of $\exp(a_1\lambda+a_2\lambda^2+\ldots+a_p\lambda^p)$ and so is $p!$ times the coefficient of λ^p in the product

$$\left(\sum_{\alpha_1=1}^{\infty}\frac{a_1^{\alpha_1}\lambda^{\alpha_1}}{\alpha_1!}\right)\left(\sum_{\alpha_2=1}^{\infty}\frac{a_2^{\alpha_2}\lambda^{2\alpha_2}}{\alpha_2!}\right)\cdots\left(\sum_{\alpha_p=1}^{\infty}\frac{a_p^{\alpha_p}\lambda^{p\alpha_p}}{\alpha_p!}\right);$$

that is, the required derivative is

$$A_p = p!\sum_{(\alpha)}\frac{a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_p^{\alpha_p}}{\alpha_1!a_2!\ldots\alpha_p!}$$

the sum being taken over all sets of non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that $\alpha_1+2\alpha_2+\ldots+p\alpha_p = p$.

Although the formula for the pth derivative A_p is quite explicit, the A_p could be calculated in terms of the $a_1, a_2, \ldots, a_p$ by using an operator Δ which acts on the $a_1, a_2, a_2, \ldots, a_p$. The required properties of Δ are that

$$\Delta a_r = (r+1)a_{r+1},\ r \geqq 1$$

and $A_p = (a_1+\Delta)A_{p-1}$ for $p \geqq 2$ with $A_1 = a_1$; this operator is similar to one considered by the author in (5, page 11).

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To see this let $f(\lambda)=\sum_{r=1}^{\infty} a_r\lambda^r$ and $g(\lambda)=e^{f(\lambda)}$. Then differentiating w.r.t. λ, we have $g'(\lambda)=g(\lambda)f'(\lambda)=g(\lambda)\left(f'(\lambda)+\frac{d}{d\lambda}\right)(1)$. It is now easy to prove by induction that $g^{(p)}(\lambda)=g(\lambda)\left(f'(\lambda)+\frac{d}{d\lambda}\right)^p(1)$. Now put $\lambda=0$ and since $g^{(p)}(0)=A_p$ and $f'(0)=a_1$ this last equation may be written as

$$A_p=(a_1+\Delta)^p(1)=(a_1+\Delta)A_{p-1}.$$

Since $f^{(r)}(0)=r!a_r$ and $f^{(r+1)}(0)=(r+1)!a_{r+1}$, it follows that

$$\Delta(r!a_r)=(r+1)!a_{r+1},$$

as is required.

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