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ON WEAKLY SI-MODULES

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In this note we characterise finitely generated self-projective R-modules M satisfying the property that every non-zero M-singular R-module contains a non-zero M-injective submodule.

1. INTRODUCTION

Rings characterised by the property that every singular right R-module is injective, briefly SI-rings, have been introduced and investigated by Goodearl [2]. Later Rizvi, Yousif [5] and Sanh [4] have studied the class of rings for which every singular right R-module is continuous (briefly, SC-rings). A right R-module is called an SI-module [1] (respectively SC-module) if every M-singular right R-module is M-injective (respectively continuous). In this note we study a class of rings characterised by the property that every non-zero singular right R-module contains a non-zero injective submodule. We call them weakly SI-rings (briefly, WSI-rings). Similarly, an R-module M is called a WSI-module if every M-singular R-module contains a non-zero M-injective submodule. Clearly every SI-module is a WSI-module. We present here some characterisations of finitely generated self-projective WSI-modules.

2. RESULTS

Throughout this note R is an associative ring with identity and Mod-R the category of unitary right R-modules. For $M \in$ Mod-R, we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules (see Wisbauer [7]). A module M is called self-projective if it is M-projective. $Soc(M)$, $Rad(M)$ and $Z(M)$ denote the socle, radical and singular submodule of the module M, respectively.

Let M and N be R-modules. Then N is called *singular in* $\sigma[M]$ or *M-singular* if there exists a module L in $\sigma[M]$ containing an essential submodule K such that $N \simeq L/K$ (see [6]).

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By definition, every M-singular right R-module belongs to $\sigma[M]$. For $M = R$ the notion of R-singular is identical to the usual definition of singular R-modules (see [2]).

The class of all M-singular modules is closed under submodules, homomorphic images and direct sums (see Wisbauer [7, 17.3 and 17.4]). Hence every module $N \in \sigma[M]$ contains a largest M-singular submodule which we denote by $Z_M(N)$. The following properties of M-singular modules are shown in [6, 1.1] and [8, 2.4].

Lemma 1. *Let M be an R-module.*

(1) *A simple R-module E is M-singular or M-projective.*
(2) *If $Soc(M) = 0$, then every simple module in $\sigma[M]$ is M-singular.*
(3) *If M is self-projective and $Z_M(M) = 0$, then the M-singular modules form a hereditary torsion class in $\sigma[M]$.*

A ring R (respectively a module M) is called a right V-ring (respectively V-module) if every simple right R-module is injective (respectively M-injective). By using an argument similar to that given in [3], we have:

Lemma 2. *Let M be a finitely generated right R-module with $Z_M(M) = 0$. Then the following conditions are equivalent:*

(1) *Every simple M-singular module is M-injective;*
(2) *$Rad(N) = 0$ for every M-singular module N;*
(3) *Every proper essential submodule N of M is an intersection of maximal submodules of M.*

Proof: (1) $\Rightarrow$ (2). Let N be a M-singular right R-module. If $0 \neq x \in N$, then by Zorn's Lemma there is a submodule Y of N which is maximal among the submodules X of N with $x \notin X$. Let D denote the intersection of all submodules S of N with $S \supset Y$ but $S \neq Y$. Then $x \in D$ and D/Y is simple. Since D/Y is also M-singular, it is M-injective. Therefore $N/Y = D/Y \oplus K/Y$, where K is a submodule of N containing Y. Since x cannot be contained in K, it follows that Y is a maximal submodule of N. Hence $Rad(N) = 0$ because for every $x \in N$ there is a maximal submodule Y of N such that $x \notin Y$.

(2) $\Rightarrow$ (3). Since for every proper essential submodule N of M, M/N is M-singular, we have $Rad(M/N) = 0$ by (2). This shows that the intersection of all maximal submodules containing N equals N, proving (3).

(3) $\Rightarrow$ (1). Now let S be a simple M-singular right R-module and $\rho : X \to M$ be a monomorphism and $\alpha \in \mathrm{Hom}_R(X, S)$. Without loss of generality we may assume that α is nonzero, $\rho(X) = X \subset M$ and X is essential in M. If $Y = ker(\alpha)$, then, since $Z_M(M) = 0$ and S is M-singular, Y must be essential in X and therefore by (3) there is a maximal submodule Q of M such that $Q \supset Y$, and $Q \not\supset X$. Since X/Y

is a simple R-module, $Q \cap X = Y$. Therefore

$$M/Y = (Q+X)/Y = Q/Y \oplus X/Y.$$

Thus α can be extended to an R-module homomorphism $\widetilde{\alpha} \in \mathrm{Hom}_R(M,S)$. Hence S is M-injective. This completes the proof of the Lemma. □

From this Lemma we have:

Corollary 3. *Let R be a right non-singular ring. Then the following conditions are equivalent:*

(1) *Every simple singular right R-module is injective;*
(2) *$Rad(M) = 0$ for every singular right R-module M;*
(3) *Every proper essential right ideal of R is an intersection of maximal right ideals of R.*

Proposition 4. *Let M be a finitely generated, self-projective* WSI-*module. Then*

(1) $Z_M(M) = 0$;
(2) *Every simple M-singular right R-module is M-injective;*
(3) *$Rad(N) = 0$ for every M-singular right R-module N;*
(4) *Every proper essential submodule of M is an intersection of maximal submodules of M;*
(5) *Every simple right R-module is M-injective or M-projective;*
(6) *$Soc(M)$ is M-projective;*
(7) *$Rad(M) \subset Soc(M)$.*

Proof: (1) If $Z_M(M) \neq 0$ then $Z_M(M)$ contains a non-zero M-injective submodule which is then M-projective, a contradiction. Hence we must have $Z_M(M) = 0$, proving (1).

(2) Clearly, if N is simple and M-singular, then N is M-injective.

From Lemma 2 we have (3), (4) and from [**6**, Proposition 2.1] we have (5).

(6) Let S be a simple submodule of M. Since by (1), $Z_M(M) = 0$, then S is not M-singular, hence S is M-projective by Lemma 1. Therefore $Soc(M)$ is M-projective.

(7) For every essential submodule A of M, M/A is M-singular and hence $Rad(M/A) = 0$, by Lemma 2. This implies $Rad(M) \subset A$, that is, $Rad(M) \subset Soc(M)$, since $Soc(M)$ is the intersection of all essential submodules of M. □

Corollary 5. *Let R be a right* WSI-*ring. Then*

(1) $Z(R_R) = 0$;
(2) *Every simple singular right R-module is injective;*
(3) *$Rad(M) = 0$ for every singular right R-module M;*

(4) *Every proper essential right ideal of R is an intersection of maximal right ideals of R;*
(5) *Every simple right R-module is injective or projective;*
(6) $Soc(R_R)$ *is projective;*
(7) $Rad(R) \subset Soc(R_R)$*;*
(8) $(Rad(R))^2 = 0$*;*
(9) $I^2 = I$ *for every essential right ideal I of R.*

PROOF: The statements from (1) to (7) are clear by Proposition 4.

(8). It follows from (7) that $(Rad(R))^2 \subset [Soc(R_R)][Rad(R)] = 0$.

(9). Suppose on the contrary that for some essential right ideal I of R, there exists an $x \in I \setminus I^2$. First we see that if I and J are essential in R, then R/I and I/IJ are singular. Since R/IJ is an extension of I/IJ by R/I it must be singular, hence IJ is essential in R (see [**2**, Proposition 1.7]). In particular, I^2 is essential in R for every essential right ideal I of R. Then by (4) above, there exists a maximal right ideal M of R with $M \supset I^2$ but $x \notin M$. Observing that $M + xR = R$, we infer that $x \in Mx + xRx$. However, since $xRx \subset I^2 \subset M$, this leads to the contradiction that $x \in M$. ∎

COROLLARY 6. *If $R/Rad(R)$ is semisimple, then the following conditions are equivalent:*

(1) *R is a right* WSI-*ring;*
(2) *R is a right* SI-*ring;*
(3) *R is a left* SI-*ring;*
(4) *R is a left* WSI-*ring.*

PROOF: By Corollary 5, if R is right WSI, then R is right non-singular and $(Rad(R))^2 = 0$. Then [**2**, Proposition 3.5] applies. ∎

PROPOSITION 7. *Let M be a finitely generated* WSI-*module. Then $M/Soc(M)$ is a V-module.*

PROOF: We see from Lemma 2 and its proof that if M is a finitely generated WSI-module then every essential proper submodule of M is an intersection of maximal submodules of M. Therefore by [**9**, Lemma 4] we see that $M/Soc(M)$ is a V-module. ∎

THEOREM 8. *Let M be a finitely generated self-projective right R-module. Then the following conditions are equivalent:*

(1) *M is an* SI-*module;*
(2) *M is an* SC-*module with $Z_M(M) = 0$;*
(3) *$M/Soc(M)$ is a V-module, $Z_M(M) = 0$ and for every essential proper submodule K of M, M/K is finitely cogenerated;*

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(4) *M is a WSI-module and for every essential submodule K of M, M/K has finite uniform dimension.*

PROOF: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) from [4, Proposition 6 and Theorem 3].

(1) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). Let K be an essential submodule of M. Then by (4), M/K has finite uniform dimension, say m. From this there exist finitely many independent uniform submodules, say $\overline{U}_1, \dots, \overline{U}_m$, of M/K such that $\overline{U}_1 \oplus \cdots \oplus \overline{U}_m$ is essential in M/K. Since M is WSI and M/K is M-singular, we easily see that $\overline{U}_1 \oplus \cdots \oplus \overline{U}_m$ is semisimple and M-injective. It follows that M/K is semisimple. This shows that M/K is semisimple for every essential submodule K of M, therefore M is an SI-module by [1, Proposition 1.3], since we have $Z_M(M) = 0$ by Proposition 4. ∎

From this Theorem we obtain the following Corollary:

COROLLARY 9. *Let R be a ring, the following conditions are equivalent:*

(1) *R is a right SI-ring;*
(2) *R is a right SC-ring with $Z_r(R) = 0$;*
(3) *$R/Soc(R_R)$ is a right V-ring, $Z_M(M) = 0$ and for every essential proper right ideal K of R, R/K is finitely cogenerated;*
(4) *R is a right WSI-ring and for every essential right ideal K of R, R/K has finite uniform dimension.*

QUESTION. Is every right WSI-ring necessarily right SI?

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