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THE PROXIMAL SUBGRADIENT AND CONSTANCY

In memory of Hans Zassenhaus

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ABSTRACT. If f is a lower semicontinuous function mapping a connected open subset of $\mathbb{R}^n$ to $(-\infty, \infty]$, and if the proximal subgradient of f reduces to zero wherever it exists, then f is constant.

RÉSUMÉ. Soit f une fonction semicontinue inférieurement de U à $(-\infty, \infty]$, où U est un ensemble ouvert et connexe de $\mathbb{R}^n$. Si tout sousgradient proximal de f est nul, alors f est constante.

1. **Introduction.** Throughout this note U is a connected open subset of $\mathbb{R}^n$ and f is a lower semicontinuous function mapping U to $(-\infty, \infty]$. The scalar product of two vectors $u, v \in \mathbb{R}^n$ is written by juxtaposition, uv. At any given point $x \in U$ the proximal subgradient [1] is the set of vectors $v \in \mathbb{R}^n$ such that

$$\liminf_{y \to x} \frac{f(y) - f(x) - (y - x)v}{|y - x|^2} > -\infty.$$

If the set of such vectors v is empty, or if $f(x) = \infty$, it is said that the proximal subgradient at x does not exist. For simplicity we denote the proximal subgradient by $f'(x)$ remembering, however, that it is a set. At a point where $f \in C^2$, $f'(x)$ is a singleton whose sole element is grad $f(x)$. If $f = g + h$, where $g \in C^2$, then $f'(x)$ exists at any point where $h'(x)$ exists. Also, with a natural interpretation, $f'(x) = g'(x) + h'(x)$.

The following theorem was established by the first author in [2] and was there used in developing an existence theory for a broad class of variational problems:

THEOREM 1. *Suppose $f'(x) = \{0\}$ at every point $x \in U$ where it exists. Then f is constant on U.*

At a meeting of the Canadian Mathematical Society in June 1991 the first author asked whether there might be a simpler proof than that in [2]. Such a proof is given now.

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Let $f(x_0) < \infty$ at some point $x_0 \in U$; in the contrary case there is nothing to do. As in [2] let

$$g(x) = \frac{1}{\epsilon^2 - |x - x_0|^2}$$

where ϵ is a positive constant so small that f is bounded below in the sphere $|x - x_0| \leq \epsilon$. If $\delta > 0$ the function $h = f + \delta g$ attains a minimum at some interior point z of this sphere. At the minimum $0 \in h'(z)$. The equation $f = h - \delta g$ shows that $f'(z)$ exists, hence $f'(z) = \{0\}$, hence $g'(z) = \{0\}$. Therefore $z = x_0$. Since the minimum of h occurs at x_0 we conclude that $h(x_0) \leq h(x)$ for any x in the sphere. Letting $\delta \longrightarrow 0$ yields

$$f(x_0) \leq f(x), \quad |x - x_0| < \epsilon.$$

Thus the minimum of f over the sphere is attained at the center.

Suppose next that $|x_1 - x_0| < \epsilon/2$. At first, we do not even know that $f(x_1) < \infty$. Nevertheless, by the above argument, with (x_0, ϵ) replaced by $(x_1, \epsilon/2)$, the minimum of f over the sphere $|x - x_1| < \epsilon/2$ is attained at the center. Hence $f(x_1) \leq f(x_0)$ and equality holds.

With $c = f(x_0)$ let S be the set of values $x \in U$ at which $f(x) = c$. The above result shows that S is both open and closed relative to U. Namely, if $f(y) = c$ at some point $y \in U$ we have $f(x) = c$ in some sphere centered at y. On the other hand if $\{x_j\}$ is a sequence of points of U such that $f(x_j) = c$ and $x_j \longrightarrow z \in U$ our argument yields $f(z) = c$. Being both open and closed, S is either empty or is all of U. In the first case $f(x) = \infty$ throughout U, a case we excluded at the start. In the second case $f(x) = c$ throughout U. This completes the proof.

COROLLARY. *If there is a function $g \in C^2(U)$ for which $f'(x) \subseteq g'(x)$ for all $x \in U$, then $f - g$ is constant on U.*

REMARK 1. It was pointed out by Philip Loewen that the conclusion does not follow if we assume only $0 \in f'(x)$, rather than $f'(x) = \{0\}$, at all points where $f'(x)$ exists. Loewen's example is as follows: Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ satisfy $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for ≥ 0. Then f is lower semicontinuous and $f'(x) = \{0\}$ at every point except $x = 0$. Furthermore $0 \in f'(0)$. Yet f is not constant.

REMARK 2. Consider the following alternate way to characterize a proximal subgradient v of f at x:

$$\liminf_{y \longrightarrow x} \frac{f(y) - f(x) - (y - x)v}{|y - x|} \geq 0.$$

The proximal subgradient resulting from this definition contains the one invoked in the theorem. It follows that the theorem remains true if in its statement the phrase "proximal subgradient" is interpreted in this alternate sense, and it follows that the corollary remains valid when g is merely differentiable in U; the condition $g \in C^2$ is not needed.

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