ON EQUI-CARDINAL RESTRICTIONS OF A GRAPH

J. C. Beatty and R. E. Miller

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1. Introduction. A graph G is an ordered pair (V, E) where V is a set of objects called vertices, and E is a set of unordered pairs of vertices (v, v') in which each such pair can occur at most once in E, and if $(v, v') \in E$ then $v \neq v'$. The order of G is the cardinality of the set V, and the degree $\delta(v)$ of an element $v \in V$ is the number of elements of E which contain v. G is said to be regular of degree d if $\delta(v) = d$ for each $v \in V$. G is a complete graph if E contains every pair of elements of V. A graph $H = (V', E')$ is a partial graph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. H is a restriction of G if H is a partial graph of G in which $V' = V$. Let $S = \{e_1, \ldots, e_\ell\}$ be a subset of E such that $e_j = \{v_{j-1}, v_j\}$ for $1 \leq j \leq \ell$. Then S is called an arc of G of length ℓ (from v_0 to v_ℓ) in case the vertices $v_0, v_1, \ldots, v_\ell$ are all distinct. The two vertices v_0 and v_ℓ are said to be connected if there exists an arc from v_0 to v_ℓ. In case $\ell + 1$ is the order of G and S is an arc of length ℓ, then it is called a Hamilton arc of G. In case v_0 and v_ℓ are the only two identical vertices of the above arc and G has order ℓ, then S is called a Hamilton circuit of G. G is connected if every pair $\{v_0, v_\ell\}$ of its vertices is connected.

The connectedness relation of vertices in G is readily seen to be an equivalence relation, so that it partitions G into a set $\{G_c\}$ of connected graphs. Each such G_c is called a component of G. A k-equi-cardinal restriction of G

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(designated as a ker of G) is a restriction of G in which each component of the restriction is of order k. For a graph G to have a ker, obviously the order of G must be some multiple of k. Also, only $k \geq 2$ are of interest.

The problem we consider here is to find the minimum degree d such that every regular graph of order $n = mk$ and degree $\geq d$ has a k-equi-cardinal restriction.

The concept of a ker of G is related to that of a (k-1)-factor of G discussed by Tutte [1] and others. In particular, when $k = 2$ a ker of G is identical to a 1-factor of G, but this relationship does not carry over for general k.

2. The Problem. As stated previously, we wish to determine a minimum degree d such that every regular graph of order $n = mk$ and degree $\geq d$ has a ker. The following properties will be useful.

Property 1: A connected graph either has a Hamilton circuit or its maximal arcs have length ℓ satisfying $\ell \geq \delta(v_0) + \delta(v_\ell)$ where v_0 and v_ℓ are vertices connected by such an arc. (Theorem 3.4.3, p. 55 of Ore [2].)

Property 2: If the order of G is a multiple of k and G contains a Hamilton arc, then G has a ker.

Proof: Let the G_c components of order k be subgraphs consisting of successive vertices and edges along the Hamilton arc.

Property 3: If G is a regular graph of degree $d \geq \frac{n-1}{2}$, where n is the order of G, then G contains a Hamilton arc.

Proof: Suppose G was not connected; then the largest possible degree for a regular graph would be obtained by having G consist of two complete subgraphs, each containing $\frac{n}{2}$ vertices. In this case $d = \frac{n}{2} - 1 = \frac{n-2}{2}$. Thus G is connected if $d \geq \frac{n-1}{2}$. Finally, from property 1 G has a Hamilton circuit,

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and thereby a Hamilton arc, or else an arc of length $\ell \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1$ which is also a Hamilton arc.

For our problem, properties 2 and 3 determine that every regular graph of degree $d \geq \frac{n-1}{2}$, where $n = mk$, has a ker. Thus we need consider only the cases with $d < \frac{n-1}{2}$.

Case 1: m even

Here $n = mk$, so that $\frac{n}{2} = \frac{m}{2}k$ is divisible by k. Thus $\frac{n}{2} - 1$ and $\frac{n}{2} + 1$ are not divisible by k. Let G consist of two components G_1 and G_2, where G_1 is the complete graph on $\frac{n}{2} - 1$ vertices and G_2 is obtained from the complete graph on $\frac{n}{2} + 1$ vertices by deleting the edges of one Hamilton circuit. Then G is regular of degree $d = \frac{n}{2} - 2$. Obviously G does not contain a ker, since the orders of G_1 and G_2 are not divisible by k. Thus the minimum degree d for our problem is $\frac{n}{2} - 2 < d \leq \frac{n}{2}$ (since for even m, $\frac{n-1}{2}$ is not an integer). We now consider the remaining case here for which $d = \frac{n}{2} - 1$. We shall show that G must contain a ker.

Suppose G has degree $\frac{n}{2} - 1$ but does not contain a ker. If G is not connected, then for regularity G must consist of two complete subgraphs, each of order $\frac{n}{2}$, but this graph obviously contains a ker $(\frac{n}{2} = \frac{m}{2}k)$; thus G is connected. Now by property 1, G either has a Hamilton circuit (and this is impossible by property 2 under the assumption that G does not contain a ker) or else its maximal arcs are of length $\ell \geq (\frac{n}{2} - 1) + (\frac{n}{2} - 1) = n - 2$. If $\ell = n - 1$ then G has a ker so we must have $\ell = n - 2$. Let $v_1, \ldots, v_{n-1}$ be the successive vertices along such an arc and let v_n be the only remaining vertex of G.

Suppose G has an edge (v_j, v_n) where $j = qk + r$, q and r are integers and $1 \leq r < k$. Then a ker of G can be formed as follows: The first q segments $(v_1, \ldots, v_k)$, $(v_{k+1}, \ldots, v_{2k})$, ..., $(v_{(q-1)k+1}, \ldots, v_{qk})$ of the $n - 2$ length path can form components of order k; $(v_{qk+1}, \ldots, v_{(q+1)k-1}, v_n)$ can form a component; and successive components, starting with vertex $v_{(q+1)k}$ up to v_{n-1} can be formed along the path of length $n - 2$, giving the desired ker. Thus each edge (v_j, v_n) of G must be such that j is a multiple of k; $j = qk$. There are $\frac{n}{k} - 1$ such vertices from $v_1, \ldots, v_{n-1}$, thus $\delta(v_n) = \frac{n}{2} - 1$ can be attained only for $k = 2$, and only when for each $j = 2q$, $1 \leq q \leq \frac{n}{2} - 1$, (v_j, v_n) is an edge of G. It remains to show that for $k = 2$, $d = \frac{n}{2} - 1$, the graph G contains a ker. Now G cannot contain an edge (v_i, v_j), $1 \leq i, j \leq n - 1$, in which both i and j are odd; for if so (v_i, v_j) could form a component of a ker with the other components as follows: $(v_1, v_2), \ldots, (v_{i-2}, v_{i-1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-3}, v_{j-2}), (v_{j-1}, v_n), (v_{j+1}, v_{j+2}), \ldots, (v_{n-2}, v_{n-1})$, where we have assumed that $i < j$ with no loss in generality. Thus, in order to satisfy $\delta(v_i) = \frac{n}{2} - 1$, each odd numbered vertex v_i, $1 \leq i \leq n - 1$ must have an edge to each even numbered vertex. Under this assumption, however, $\delta(v_p) \geq \frac{n}{2} + 1$, where v_p is any even numbered vertex since there are $\frac{n}{2}$ odd numbered vertices and v_n has an edge connected to each such v_p. By assumption of regularity of G, however, $\delta(v_p) = \frac{n}{2} - 1$, proving that every regular G of degree $\frac{n}{2} - 1$ contains a ker. Thus for m even, the minimum d for our problem is $d = \frac{n}{2} - 1$.

Case 2: m odd

Subcase 2a: k even. Here $\frac{n}{2}$ is not divisible by k, so one can obtain a regular graph of degree $\frac{n}{2} - 1$ which contains no ker by taking two replicas of the complete graph on $\frac{n}{2}$ vertices. Thus $d = \frac{n}{2}$ is the solution to our problem.

Subcase 2b: k odd.

(i) $\frac{n+1}{2}$ even. Here one can obtain a regular graph of degree $\frac{n-3}{2}$ with two components by taking the complete graph on $\frac{n-1}{2}$ vertices together with the graph found by deleting the alternate edges of a Hamilton circuit from the complete graph on $\frac{n+1}{2}$ vertices. Since not both $\frac{n-1}{2}$ and $\frac{n+1}{2}$ are divisible by k, this graph does not contain a ker. Thus $\frac{n-3}{2} < d \leq \frac{n-1}{2}$ and $d = \frac{n-1}{2}$.

(ii) $\frac{n+1}{2}$ odd. Here one obtains a regular graph containing no ker of degree $\frac{n-5}{2}$ by deleting a Hamilton circuit from the complete graph on $\frac{n+1}{2}$ vertices and deleting alternate edges of a Hamilton circuit from the complete graph on $\frac{n-1}{2}$ vertices. Thus $\frac{n-5}{2} < d \leq \frac{n-1}{2}$. Since n is odd, there is no regular graph on n nodes of odd degree, in particular of degree $\frac{n-3}{2}$. Thus $d = \frac{n-3}{2}$.

The following table summarizes the solution d to our problem where the order of G is $n = mk$.

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m			d
even			$\frac{n}{2} - 1$
odd	k even		$\frac{n}{2}$
	k odd	$\frac{n+1}{2}$ even	$\frac{n-1}{2}$
		$\frac{n+1}{2}$ odd	$\frac{n-3}{2}$

Table 1: Minimum d Such That All Regular Graphs of Degree $\geq$ d Contain a Ker.

In the proofs many of the regular graphs not containing a ker were non-connected graphs. Little is known about the problem if one adds the hypothesis that the graph be connected. Regular connected graphs which do not contain kers, for $k = 2$, can be formed for a degree which is somewhat less than $\frac{n}{3}$ using a structure having one central vertex. A simple example with $d = 3$, $n = 16$ is shown below, but it is not known whether or not $\frac{n}{3}$ is near the minimum degree for which every regular connected graph contains a 2-equi-cardinal restriction.

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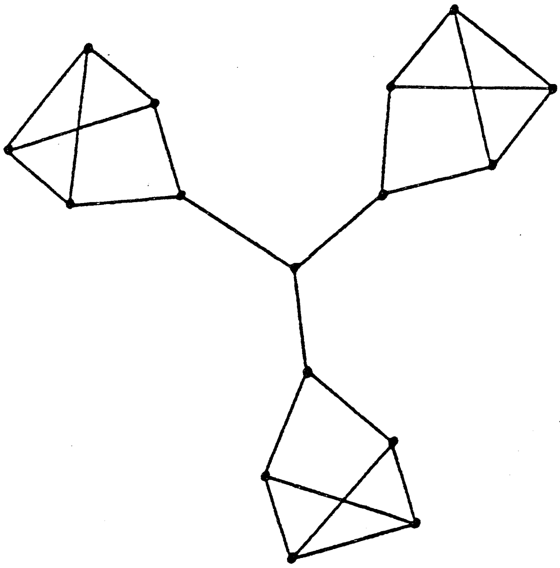

Figure 1

A Connected Regular Graph Having
No 2-Equi-Cardinal Restriction

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I. B. M., Yorktown Heights, N. Y.

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