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ON ESWARATHASAN–LEVINE AND BOYD'S CONJECTURES FOR HARMONIC NUMBERS

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Abstract

We provide numerical evidence towards three conjectures on harmonic numbers by Eswarathasan, Levine and Boyd. Let J_p denote the set of integers $n \geq 1$ such that the harmonic number H_n is divisible by a prime p. The conjectures state that: (i) J_p is always finite and of the order $O(p^2(\log\log p)^{2+\epsilon})$; (ii) the set of primes for which J_p is minimal (called harmonic primes) has density e^{-1} among all primes; (iii) no harmonic number is divisible by p^4. We prove parts (i) and (iii) for all $p \leq 16843$ with at most one exception, and enumerate harmonic primes up to 50×10^5, finding a proportion close to the expected density. Our work extends previous computations by Boyd by a factor of approximately 30 and 50, respectively.

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1. Introduction

The sequence of harmonic numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is a much studied one in the literature, mainly due to its connections with the Riemann zeta function and Bernoulli numbers. Over the centuries, many arithmetic properties of H_n have been discovered; a well-known example is Wolstenholme's theorem [19], which states that p^2 divides the numerator of H_{p-1} for every prime $p \geq 5$. More generally, the divisibility of H_n by a given prime p has attracted much interest [1, 2, 4, 5, 7, 10–16, 20].

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Our motivation for looking into the sequence H_n is threefold. First, harmonic numbers are related to p-adic L-functions [18], which are less well understood than the classical ones. A striking fact in this context is that we do not even know if p-adic zeta functions are always nonzero on the positive integers (see, for example, [3, 6]).

Second, the set of harmonic numbers divisible by a given prime p can be described by a probabilistic model, which allows one to make conjectures on what we should expect. This has been worked out in full detail by Boyd [5].

Third, specialised software is available to test the predictions made by the probabilistic model. As explained by Boyd in [5, Section 5], the 'naive' approach of computing H_n from H_{n-1} and then checking the divisibility is unfeasible for large values of n. Instead, a better-tailored p-adic method can reach much higher values.

1.1. The set J_p. The central object in this paper is the set

$$J_p := \{n \geq 1 : \ \nu_p(H_n) \geq 1\},$$

where p is a prime and $\nu_p(a)$ denotes the p-adic valuation of a. In other words, J_p contains those n such that p divides the numerator of H_n (when written in lowest terms). We aim to describe the two extreme cases of how small and how large the cardinality $|J_p|$ can be.

In 1991, Eswarathasan and Levine [10] initiated a study of J_p and computed the full set when $p = 3, 5, 7$. Based on the fact that these sets were all finite, they conjectured that this should always be the case (a rather ambitious conjecture, in view of the limited evidence).

CONJECTURE 1.1. The set J_p is finite for all primes p.

In the same paper, they showed that for all $p \geq 5$, the set J_p always contains $p-1$, p^2-p and p^2-1. They called *harmonic primes* those p for which $|J_p| = 3$ and suggested that they should occur infinitely often.

CONJECTURE 1.2. There are infinitely many harmonic primes.

To explore these conjectures, Eswarathasan and Levine devised an algorithm based on the decomposition

$$H_{pn+k} = H^*_{pn+k} + \frac{H_n}{p}, \tag{1.1}$$

where $k \in [0, p-1]$ and H^*_n denotes a sum as in H_n, but restricted to integers coprime to p. Since $H^*_{pn+k} \equiv H_k$ modulo p, it follows from (1.1) that

$$H_{pn+k} \equiv H_k + \frac{H_n}{p} \pmod{p}. \tag{1.2}$$

Therefore, $pn + k \in J_p$ if and only if $n \in J_p$ and $p^{-1}H_n \equiv -H_k$ modulo p (see [10, Theorem 3.1]). In particular, this suggests a search strategy as follows: after computing H_k modulo p for all $k = 1, \ldots, p-1$, determine the elements of $J_p \cap [p^m, p^{m+1}-1)$ and then use the above criterion to find $J_p \cap [p^{m+1}, p^{m+2}-1)$.

In 1994, Boyd [5] extended this method by exploiting a p-adically convergent series for $H_{pn} - p^{-1}H_n$ (see [5, Theorem 5.2]), which allowed him to essentially iterate the recursion in (1.2) and get back to computing only the initial interval $J_p \cap [1, p-1]$, but to a high p-adic precision. He managed to establish that J_p is finite for all primes $p \leq 547$ except possibly for $p \in \{83, 127, 397\}$.

Boyd also explained how the set J_p can be described in terms of a probabilistic Galton–Watson branching process. Such a random model suggests that, with probability one, the cardinality $|J_p|$ is indeed finite (in agreement with Conjecture 1.1) and of the order $O(p^2(\log\log p)^{2+\epsilon})$, with infinitely many primes satisfying $|J_p| \geq p^2(\log\log p)^2$. In addition, Boyd's model predicts that harmonic primes should have density e^{-1} among all primes, which gives a quantitative refinement of Conjecture 1.2. Finally, it predicts that H_n cannot be divisible by high powers of p [5, page 288].

CONJECTURE 1.3. There are no pairs (p, n) with $\nu_p(H_n) \geq 5$. The case $\nu_p(H_n) = 4$, if it occurs at all, should occur only finitely many times.

In contrast, it is very common that $\nu_p(H_n) \leq 2$. The case $\nu_p(H_n) = 3$ occurs too, although rarely, the first instance being when $p = 11$ and $n = 848$.

1.2. Main result. We extend Boyd's results in two directions. First, in a 'vertical direction', so to speak, we consider small primes and check how large $|J_p|$ can get. For a single prime, this can become very time-consuming and so we decided to stop at $p = 16843$ (the first Wolstenholme prime), extending Boyd's computations by a factor of approximately 30. In a 'horizontal direction', instead, we count harmonic primes up to some large bound. The computation for a single prime in this case is fast and we go up to 50×10^5, extending Boyd's computation by a factor of 50. Our findings are summarised in the following theorem.

THEOREM 1.4.

(i) *For all primes* $p \leq 16843$, *the set* J_p *is finite, with at most one exception, namely* $p = 1381$.
(ii) *There are* 128594 *harmonic primes in the interval* $[5, 50 \times 10^5]$, *corresponding to* $\approx 36.89812\%$ *of all primes in this range.*
(iii) *There are no pairs* (p, n) *with* $p \leq 16843$, $p \neq 1381$, *for which* $\nu_p(H_n) \geq 4$. *If any such pair exists when* $p = 1381$, *we must have* $n \geq 1381^{3801}$.

The first point of Theorem 1.4 confirms Conjecture 1.1 for all primes $p \leq 16843$, with the exception of 1381. We did not complete the full enumeration of J_{1381}, since we kept finding new elements all the way up to height 1381^{3800} (and in each of the last twenty p-adic intervals, there are more than 4000 elements, suggesting that we are far from completion). A more precise version of point (i) is stated in Observation 2.1, where we explain that $|J_p| \leq p^2$ for all the primes we examined with four exceptions that satisfy instead the inequality $|J_p| \geq p^2(\log\log p)^2$. This is in agreement with Boyd's quantitative version of Conjecture 1.1. In Observation 2.3, we also discuss the extinction time of J_p, namely the largest power of p needed to visit the whole set, and compare it with the predictions from the model (see [5, page 301]).

The second point in Theorem 1.4 (see Observation 2.2) is in agreement with the prediction that harmonic primes should have density $e^{-1} = 0.3678794411\ldots$ among all primes and hints at the correctness of Conjecture 1.2. Figure 2 shows the fluctuations around the value e^{-1}.

Finally, in the last point of Theorem 1.4, we confirm that we never observe a p-adic valuation larger than 3, in agreement with Conjecture 1.3. We found 21 elements with valuation 3 (see Observation 2.4 and Table 2).

Regarding progress towards a proof of Conjectures 1.1–1.3, Sanna [15, Theorem 1.1] proved that for any prime p and any $x \geq 1$,

$$|J_p \cap [1, x]| < 129\, p^{2/3}\, x^{0.765}.$$

Although not giving finiteness, this shows that J_p has density zero in the integers. Sanna's result has been improved by Wu and Chen [20, Theorem 1.1] to

$$|J_p \cap [1, x]| \leq 3\, x^{2/3+1/25 \log p}.$$

Bounds of this type have also been proved for harmonic numbers of exponent greater than one by Altuntaş [1, Theorem A]. As for the possibility of having large p-adic valuation, De Filpo and the first and third authors showed that if $p \nmid n$ and $\nu_p(H_n)$ equals 3 or 4 ([8, Theorems 2.5 and 2.6], respectively), then $\nu_p(H_{p^m n})$ grows linearly in m before going down again to something ≤ 2. If we believe Conjecture 1.3 is correct, then we should expect that the descent occurs immediately. Our data confirm this, as we can see from Table 2 where no two consecutive values of m appear.

All computations were made with pari/gp [17]; source code is available online [9].

2. Proof of Theorem 1.4

We wish to understand whether J_p is finite or not, what is the largest size J_p can reach as p varies, and what is the largest p-adic valuation of its elements. Our first step consists in splitting the integers in p-adic blocks and checking the divisibility of harmonic numbers in each block. Let $m \geq 1$ and define the mth p-adic block of J_p as

$$J_{p,m} := J_p \cap [p^{m-1}, p^m - 1].$$

Clearly, J_p is the union of the sets $J_{p,m}$ as m varies. Moreover, as explained by Boyd in [5, Section 3], J_p has a recursive structure, so that its mth block can be obtained from the previous one, provided we understand the latter sufficiently well. To see this, let $k \in [0, p-1]$ and set $H_0 = 0$. By [5, Lemma 3.1],

$$H_{pn+k} = H_{pn} + H_k + O(p) = \frac{H_n}{p} + H_k + O(p). \tag{2.1}$$

Here and in the rest of the paper, we use the convention that something is $O(p^s)$ if it is divisible by p^s. Therefore, if we know H_n up to an error $O(p^2)$ for all $n \in [p^{m-1}, p^m - 1]$, as well as the value of H_k up to an error $O(p)$ for all $k \in [0, p-1]$,

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we can determine if H_{pn+k} is a p-adic integer and if $\nu_p(H_{pn+k}) \geq 1$ for all integers $pn + k \in [p^m, p^{m+1} - 1]$. In particular, (2.1) implies that if $\nu_p(H_n) \leq 0$ for all integers n in a given p-adic block, then for all integers in the next block, we have again $\nu_p(H_n) \leq 0$. In turn, this shows that the finiteness of J_p is equivalent to showing that $J_{p,m} = \emptyset$ for some $m \geq 1$, that is, eventually there is an empty block.

By the above discussion, it follows that the elements of J_p can be arranged in a tree, where the nodes at level m are those n in the interval $[p^{m-1}, p^m - 1]$ with $\nu_p(H_n) \geq 1$ and for every integer $k \in [0, p - 1]$, there is an edge from n to $pn + k$ if and only if $H_n \equiv -pH_k \bmod p^2$. The set of residues $R = \{H_k \bmod p\}$ plays an important role in the structure of such a tree. Assuming that the elements in R are essentially randomly distributed, Boyd [5, Sections 3 and 6–7] gave a heuristic argument that suggests that for every fixed $\epsilon > 0$, we should have

$$|J_p| \ll_\epsilon p^2(\log\log p)^{2+\epsilon}$$

for all primes, although there should be infinitely many primes for which

$$|J_p| > p^2(\log\log p)^2. \tag{2.2}$$

Boyd computed J_p for all primes $p < 550$ and his results were consistent with these predictions. In particular, he found that $|J_{11}| = 638 > 11^2$, giving one instance of (2.2). When $p = 83, 127$ and 397, he could not determine the set J_p in full, but obtained lower bounds on $|J_p|$ by looking at p-adic blocks $J_{p,m}$ with $m \leq 100$ (see [5, page 288]). We complete the computation for these three primes and go further, up to the first Wolstenholme prime, obtaining the following observation.

Observation 2.1. For $5 \leq p \leq 16843$, we have $|J_p| \leq p^2$, unless $p = 11, 83, 397$ or 1381, in which case, we have

$$|J_{11}| = 638, \quad |J_{83}| = 43038, \quad |J_{397}| = 701533, \quad |J_{1381}| \geq 7521563. \tag{2.3}$$

The prime $p = 127$ completes with precision $m = 146$ and gives $|J_{127}| = 3515$. When $p = 1381$, we could not complete the determination of J_p (we reached precision 3800) and that is why we only have a lower bound in (2.3).

One can also look at how the cardinalities $|J_p|$ distribute as p varies (see Figure 1). They certainly do not distribute uniformly, but rather tend to favour small numbers. For instance, more than sixty percent of all primes $p \leq 16843$ have $|J_p| \leq 31$ and approximately 36 percent have $|J_p| = 3$. In Table 1, the distribution among the observed cardinalities up to 31 is given. Curiously, in this range, not all integers are observed. For instance, there is no prime $p \leq 16843$ with $|J_p| = 5$. Another visible feature is that most observed integers are odd. This can partly be explained by Boyd's probabilistic model: apart from the set $J_{p,1}$ which often contains the single element $p - 1$, the model predicts that at each successive level $J_{p,m}$, an even number of elements is generated [5, Section 6.2], making the total count odd. We indeed observe such a parity phenomenon in most levels. Nevertheless, there are some primes with $|J_p|$ even, too.

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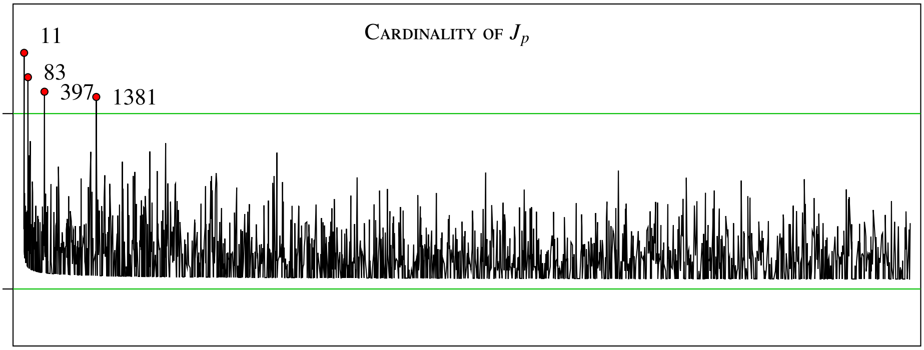

FIGURE 1. Cardinality of J_p on a logarithmic scale. On the horizontal axis, we have $5 \le p \le 16843$ and on the vertical axis, the quantity $\log |J_p|/\log p$. The profile on the bottom corresponds to the curve $\log 3/\log p$ associated with harmonic primes for which $|J_p| = 3$.

TABLE 1. For $3 \le N \le 31$, count of primes $5 \le p \le 16843$ with $|J_p| = N$ and corresponding percentage of the total (the last digit is rounded down). The values 18, 20, 24, 26 occur exactly once and are omitted. No other $N \le 30$ appears. Values above 31 appear less than 19 times each (less than 1% of the total) and are omitted.

DISTRIBUTION OF $\lvert J_p\rvert$													
3	7	9	11	13	15	17	19	21	23	25	27	29	31
706	99	57	48	44	36	38	35	31	39	25	24	26	33
36.35	5.09	2.93	2.47	2.26	1.85	1.95	1.80	1.59	2.00	1.28	1.23	1.33	1.69

The case $|J_p| = 3$ is special. For every $p \ge 5$, Eswarathasan and Levine [10] showed that J_p contains $p-1, p^2-1, p^2-p$ and therefore $|J_p| \ge 3$. As we explained in the introduction, they called those primes for which equality holds ‘harmonic primes’. Table 1 shows that out of 1942 primes in the interval [5, 16843], approximately 36.35 percent are harmonic. Boyd’s model predicts that harmonic primes should have density $e^{-1} = 0.36787944\ldots$ among all primes [5, Section 4]. He computed harmonic primes up to 10^5, which agreed with such a prediction, although he writes that ‘the number of harmonic primes in a given interval is perhaps somewhat higher than expected’.

We extend Boyd’s computation to primes up to 50×10^5 and in Figure 2, we plot the ratio of harmonic primes in fifty intervals of size 10^4 (top) and of size 10^5 (bottom) over all primes in the same interval. There are fluctuations around the value e^{-1}, but it seems very convincing that this should be the correct density. For instance, in the interval [490000, 500000], the fit is so accurate that we find 284 harmonic primes out of 772 primes, for a ratio of

$$\frac{284}{772} = 0.367875647\ldots$$

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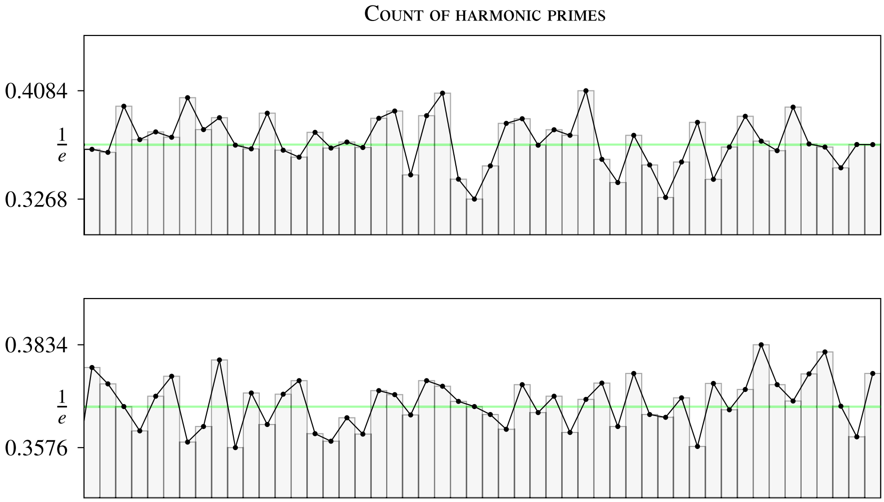

FIGURE 2. Count of harmonic primes in 50 intervals of size 10^4 (top) and of size 10^5 (bottom). In the top part, the first 10 columns correspond to [5, Table 1]. The value $e^{-1} \approx 0.367879$ is the density predicted by Boyd's probabilistic model.

which agrees with e^{-1} to the fifth decimal digit. As for the total count, we have the following observation.

OBSERVATION 2.2. Out of 348511 primes $p \in [5, 50 \times 10^5]$, 128594 are harmonic, giving a ratio $128594/348511 \approx 0.3689812$.

Returning to (2.1), let us explain how the set J_p is computed. From (2.1), we see that $H_{pn} - H_n/p$ is a p-adic integer with valuation at least one. More is true: there exists a sequence of p-adic numbers $\{c_k\}_{k\geq 1}$ such that for all integers $n \geq 1$,

$$H_{pn} - \frac{1}{p}H_n = \sum_{k=1}^{\infty} c_k p^{2k} n^{2k}. \tag{2.4}$$

The numbers c_k are p-adic integers unless $(p-1) \mid 2k$ or $p \mid k$ and, in general, we have $v_p(c_k) - 1 + v_p(1/k)$ in the first case, $v_p(c_k) = v_p(1/k)$ otherwise. This is proved in [5, Theorem 5.2]. We use (2.4) to compute harmonic numbers as follows. First, we compute the numbers $b_n = H_{pn} - H_n/p$ for $n = 1, \ldots, N$, to a p-adic precision s, and then solve the linear system

$$\sum_{k=1}^{N} c_k p^{2k} n^{2k} = b_n + O(p^s) \tag{2.5}$$

in $c_1 p^2, \ldots, c_N p^{2N}$. The matrix of this system is the Vandermonde matrix V, whose inverse satisfies $v_p(V^{-1}) > -2N/(p-1)$, so that the unknowns $c_k p^{2k}$ are obtained to

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precision $s - 2N/(p-1)$. For fixed N, the sum on the left in (2.5) represents b_n to precision [5, Remark 3]

$$s = \min_{k>N}(\nu_p(c_k) + 2k) \geq 2N + 2 - [\log_p(N+1)].$$

An algorithm to calculate J_p starts by computing $b_n = H_{pn} - H_n/p$ for $n \leq N$ and finding the coefficients $c'_k = c_k p^{2k}$ to precision $s \geq 2N + 2 - [\log_p(N+1)]$ as explained above. In the process, we have computed H_n for $1 \leq n \leq p-1$ to precision at least s and hence will know $J_{p,1}$. Then, once we have the elements at a given level $J_{p,m}$ to a precision $r \leq s$, we compute H_{pn} from (2.4) to precision $r-1$ and then compute $H_{pn+k} = H_{pn+k-1} + 1/(pn+k)$ for $k = 1, \ldots, p-1$, thus obtaining $J_{p,m+1}$ to precision $r-1$.

Notice that the precision decreases by one at each new level and so we can calculate elements in J_p up to the sth block $J_{p,s}$. If this set is empty, then we are done and we have found all elements in J_p, which is finite. If $J_{p,s}$ is not empty, we begin the computation again with a larger value of N.

As pointed out in [5, Section 5], this method is faster than the 'naive' method of computing harmonic numbers with the recursion $H_n = H_{n-1} + 1/n$. For instance, from [5], the naive method could not complete the full determination of J_{11}, which has 638 elements and contains integers as large as 11^{30}, whereas the p-adic method described above succeeds and can go much further than that.

When running the algorithm with precision s, if $J_{p,s} \neq \emptyset$, we need to go back to the beginning and repeat the computation with a higher precision. To speed up successive computations, we observe that at each level, not all nodes have children and so it is not necessary to compute all elements in J_p, but only those that have descendants in the sth block $J_{p,s}$. This yields quite a bit of time and memory saving. For instance, when $p = 1381$, we find that $J_{1381,s}$ is nonempty for all $s \leq 3800$. If we look at intermediate levels, we notice that $|J_{1381,1663}| = 2501$, but only one element in this block has descendants all the way down to $J_{1381,3800}$. Therefore, when running the algorithm for any $s \geq 3800$, it suffices to calculate one element in each $J_{1381,r}$ for all $r \leq 1663$, for a total of 1663 harmonic numbers instead of $|J_{1381,1} \cup \cdots \cup J_{1381,1663}| = 1860315$ elements.

In Figure 3, we plot the precision required to compute J_p for all primes $p \leq 16843$. This is sometimes referred to as the 'extinction time' for the branching process associated to J_p. Similarly as with the cardinality, the random model predicts that the extinction time should always be $O(p(\log\log p)^{1+\epsilon})$, but there should be infinitely many primes with extinction time larger than $p \log\log p$.

We see from Figure 3 that the extinction time is indeed often large, say larger than 400. However, very few primes have an extinction time as large as $p \log\log p$ and it does not come as a surprise that they are essentially the same ones for which the cardinality is exceptionally large. In fact, in the top part of the figure, the primes $p = 397$ and 1381 are omitted, since their extinction time is much higher than all other primes (respectively 1814 and more than 3801). We also omit the primes 2699, 4813

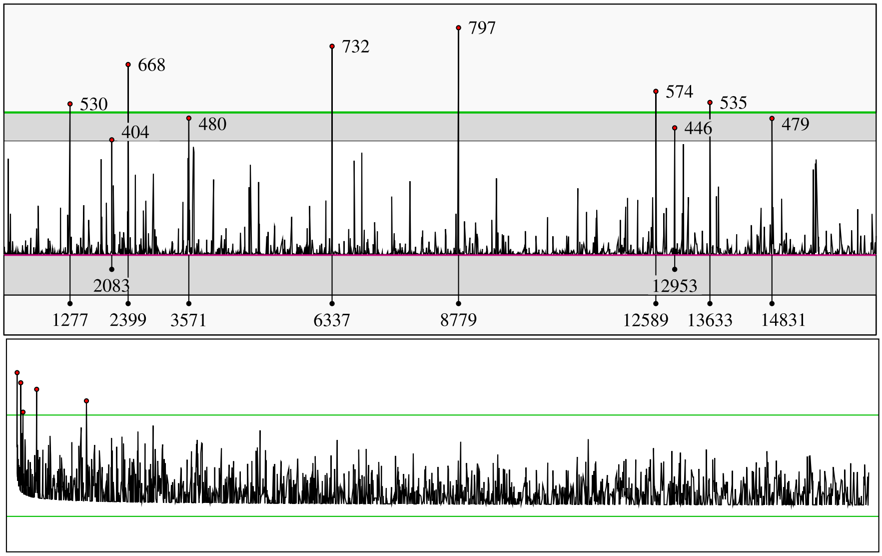

FIGURE 3. For primes $5 \le p \le 16843$, we plot the extinction time M_p (top figure: $p = 397, 1381, 2699, 4813, 11299$ are omitted) and in logarithmic scale we plot $\log M_p/\log p$ (bottom figure, including all primes).

and 11299, whose extinction times are respectively 1186, 1336 and 1214. The peaks in the bottom part of Figure 3 correspond to $p = 11, 83, 127, 397$ and 1381. Summarising, we have the following observation.

OBSERVATION 2.3. The extinction time M_p satisfies $M_p \le p$ for all primes $p \le 16843$, with the following exceptions:

$$M_{11} = 30, \quad M_{83} = 339, \quad M_{127} = 146, \quad M_{397} = 1815, \quad M_{1381} \ge 3801.$$

We conclude by discussing harmonic numbers with large p-adic valuation. For an integer n to be in J_p, we must have $\nu_p(H_n) \ge 1$ and computations reveal many integers for which $\nu_p(H_n) = 2$. Based on his model, Boyd conjectured that there are primes p for which the number of n such that $\nu_p(H_n) = 3$ is arbitrarily large, but of order between $(\log\log p)^2$ and $(\log\log p)^{2+\epsilon}$. However, he conjectured that $\nu_p(H_n) \ge 4$ never occurs. In his work, he found no element with valuation 4 or higher and only five instances of valuation 3: four when $p = 11$ and one when $p = 83$. With the new data at hand, we have the following observation.

OBSERVATION 2.4.

(i) For a given prime $5 \le p \le 16843$, the number of integers n such that $\nu_p(H_n) = 2$ can be as large as 5314. More precisely,

$$\max_{5 \le p \le 16843} |\{n \in J_p : \nu_p(H_n) = 2\}| = |\{n \in J_{1381} : \nu_{1381}(H_n) = 2\}| \ge 5314.$$

The second largest value is 1760, which is attained when $p = 397$.

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TABLE 2. For each prime $p = 11, 83, 397, 1381$, elements with valuation three are found in the intervals $J_{p,m}$ with m as listed on the right.

p	m
11	$3, 4, 4, 18$
83	$63, 108, 108, 131, 161, 207, 213, 243, 246, 291, 294$
397	567
1381	$1519, 2572, 2951, 3211, 3726$

(ii) There are 21 pairs (p, n) with $5 \leq p \leq 16843$ for which $\nu_p(H_n) = 3$ and the integers n appear in the p-adic intervals $J_{p,m}$ described in Table 2. When $p = 1381$, possible additional occurrences must have $n \geq 1381^{3801}$.

(iii) There are no pairs (p, n) with $5 \leq p \leq 16843$, $p \neq 1381$, for which $\nu_p(H_n) \geq 4$. If any such pair exists when $p = 1381$, we must have $n \geq 1381^{3801}$.

Notice that when $p = 83$, the new integers we found with valuation three are all larger than p^{107}. Since Boyd computed J_p up to p^{100}, this explains why they do not appear in his work.

References

[1] C. Altuntaş, 'On the p-adic valuation of generalized harmonic numbers', *Bull. Korean Math. Soc.* **60**(4) (2023), 933–955.

[2] M. Bayat, 'A generalization of Wolstenholme's theorem', *Amer. Math. Monthly* **104**(6) (1997), 557–560.

[3] F. Beukers, 'Irrationality of some p-adic L-values', *Acta Math. Sin. (Engl. Ser.)* **24**(4) (2008), 663–686.

[4] A. Booker, S. Hathi, M. J. Mossinghoff and T. S. Trudgian, 'Wolstenholme and Vandiver primes', *Ramanujan J.* **58**(3) (2022), 913–941.

[5] D. W. Boyd, 'A p-adic study of the partial sums of the harmonic series', *Exp. Math.* **3**(4) (1994), 287–302.

[6] F. Calegari, 'Irrationality of certain p-adic periods for small p', *Int. Math. Res. Not. IMRN* **2005**(20) (2005), 1235–1249.

[7] L. Carlitz, 'A note on Wolstenholme's theorem', *Amer. Math. Monthly* **61** (1954), 174–176.

[8] L. Carofiglio, L. De Filpo and A. Gambini, 'p-adic valuation of harmonic sums and their connections with Wolstenholme primes', *Indian J. Pure Appl. Math.* **55**(2) (2024), 555–566.

[9] G. Cherubini, https://sites.google.com/site/ggcherubini/publications.

[10] A. Eswarathasan and E. Levine, 'p-integral harmonic sums', *Discrete Math.* **91**(3) (1991), 249–257.

[11] J. W. L. Glaisher, 'On the residues of the sums of products of the first $p - 1$ numbers, and their powers, to modulus p^2 or p^3', *Quart. J. Pure Appl. Math.* **31** (1900), 321–353.

[12] J. W. L. Glaisher, 'On the residues of the sums of the inverse powers of numbers in arithmetic progression', *Quart. J. Pure Appl. Math.* **32** (1901), 271–305.

[13] C. Helou and G. Terjanian, 'On Wolstenholme's theorem and its converse', *J. Number Theory* **128**(3) (2008), 475–499.

[14] S. Hong, 'Notes on Glaisher's congruences', *Chinese Ann. Math. Ser. B* **21**(1) (2000), 33–38.

[15] C. Sanna, 'On the p-adic valuation of harmonic numbers', *J. Number Theory* **166** (2016), 41–46.

[16] Q. Sun and S. Hong, 'The p-adic approach to Wolstenholme's theorem', *Northeast. Math. J.* **17**(2) (2001), 226–230.

https://doi.org/10.1017/S0004972725100154 Published online by Cambridge University Press

[17] The PARI Group, *PARI/GP version 2.13.3* (Univ. Bordeaux, Bordeaux, 2021). http://pari.math.u-bordeaux.fr/.

[18] L. C. Washington, 'p-adic L-functions and sums of powers', *J. Number Theory* **69**(1) (1998), 50–61.

[19] J. Wolstenholme, 'On certain properties of prime numbers', *Quart. J. Pure Appl. Math.* **5** (1862), 35–39.

[20] B. L. Wu and Y. G. Chen, 'On certain properties of harmonic numbers', *J. Number Theory* **175** (2017), 66–86.

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