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AMICABLE ORTHOGONAL DESIGNS OF ORDER EIGHT

DEBORAH J. STREET

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Abstract

New amicable orthogonal designs of order eight are given and they are used to construct new orthogonal designs of order 32.

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An orthogonal design of order n and type $(s_1,\dots,s_t)$, $s_i \geqslant 0$, on the commuting variables $x_1, x_2,\dots,x_t$ is an $n \times n$ matrix X with entries from $\{0, \pm x_1,\dots, \pm x_t\}$ such that

$$XX^T = \left(\sum_{i=1}^{t} s_i x_i^2\right) I_n.$$

Let X and Y be orthogonal designs of the same order n, where X is of type $(s_1,\dots,s_t)$ and Y is of type $(u_1,\dots,u_v)$. Then X and Y are amicable orthogonal designs of type $((s_1,\dots,s_t); (u_1,\dots,u_v))$ if $XY^T = YX^T$.

Amicable orthogonal designs are a useful tool in the construction of orthogonal designs. In this paper we give a summary of the known amicable orthogonal designs of order 8, including a new 'doubling' construction for such pairs. We use a design constructed in this way to give 27 new 10 variable orthogonal designs of order 32, and to give a large number of new 6 variable designs.

We use $\bar{x}$ for $-x$ and $-$ for -1 throughout.

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LEMMA 1. *Suppose A, B, C, D are orthogonal designs of order n such that (A, B) and (C, D) are both amicable pairs of type $((a_1,\dots,a_s);\ (b_1,\dots,b_t))$. Suppose further that there exists a weighing matrix $W(n, k) = W$ such that*

$$AW^T = WC^T, \qquad BW^T = -WD^T.$$

Then there exists an amicable pair of order $2n$ of type $((k, a_1,\dots,a_s);\ (k, b_1,\dots,b_t))$.

PROOF. The required designs are $\left[\begin{smallmatrix} A & xW \\ \bar{x}W & C \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} B & yW \\ yW & D \end{smallmatrix}\right]$.

COROLLARY 2. *The existence of an amicable pair of type $((a_1,\dots,a_s);\ (b_1,\dots,b_t))$ in order n implies the existence of an amicable pair of type $((1, a_1,\dots,a_s);\ (1, b_1,\dots,b_t))$ of order $2n$.*

PROOF. Let $W = I$ in the lemma.

One obvious weighing matrix to use for W is a member of the Hurwitz-Radon family for B. However this only gives examples of pairs which can be constructed using a result of Wolfe [4]. A weaker version of that result appears in Geramita and Seberry [1, Theorem 5.48].

EXAMPLE 3. The matrices

$$A = \begin{bmatrix} a & 0 & d & c \\ 0 & a & \bar{c} & d \\ \bar{d} & c & a & 0 \\ \bar{c} & \bar{d} & 0 & a \end{bmatrix}, \quad B = \begin{bmatrix} w & 0 & y & z \\ 0 & w & z & \bar{y} \\ y & z & \bar{w} & 0 \\ z & \bar{y} & 0 & \bar{w} \end{bmatrix}, \quad C = \begin{bmatrix} a & c & d & 0 \\ \bar{c} & a & 0 & d \\ \bar{d} & 0 & a & \bar{c} \\ 0 & \bar{d} & c & a \end{bmatrix},$$

$$D = \begin{bmatrix} \bar{z} & \bar{y} & \bar{w} & 0 \\ \bar{y} & z & 0 & \bar{w} \\ \bar{w} & 0 & z & y \\ 0 & \bar{w} & y & \bar{z} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}$$

satisfy the conditions of Lemma 1 and give an amicable pair of type $((1, 1, 1, 4);\ (1, 1, 1, 4))$ of order 8. This pair is new.

Using designs constructed from Wolfe's result, the $((1, 1, 1, 4);\ (1, 1, 1, 4))$ from above and [1, Theorem 5.97] gives the following 27 10-tuples which are the types of new 10 variable orthogonal designs in order 32; see Seberry [3] for the current status of the existence of 10 variable orthogonal designs of order 32. The

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10-tuples are

(1, 1, 1, 1, 1, 1, 1, 3, 3, 3),	(1, 1, 1, 2, 2, 2, 4, 4, 4, 4),
(1, 1, 1, 1, 1, 1, 2, 2, 2, 4),	(1, 1, 1, 2, 2, 2, 4, 5, 5, 5),
(1, 1, 1, 1, 1, 1, 2, 2, 4, 8),	(1, 1, 1, 3, 3, 3, 3, 3, 3, 3),
(1, 1, 1, 1, 1, 1, 2, 3, 3, 6),	(1, 1, 1, 3, 3, 3, 4, 4, 4, 4),
(1, 1, 1, 1, 1, 1, 2, 4, 4, 4),	(1, 1, 2, 2, 2, 2, 2, 2, 2, 2),
(1, 1, 1, 1, 1, 1, 3, 3, 4, 12),	(1, 1, 2, 2, 2, 2, 2, 3, 3, 3),
(1, 1, 1, 1, 1, 1, 4, 4, 4, 4),	(1, 1, 2, 2, 2, 2, 2, 3, 3, 6),
(1, 1, 1, 1, 1, 1, 4, 5, 5, 5),	(1, 1, 2, 2, 2, 2, 2, 4, 4, 4),
(1, 1, 1, 1, 1, 1, 4, 6, 6, 6),	(1, 1, 2, 2, 3, 3, 3, 3, 3, 3),
(1, 1, 1, 1, 1, 2, 2, 2, 4, 4),	(1, 1, 2, 3, 3, 3, 3, 3, 3, 3),
(1, 1, 1, 1, 1, 2, 2, 4, 4, 4),	(1, 1, 2, 3, 3, 3, 3, 4, 4, 4),
(1, 1, 1, 1, 1, 2, 2, 5, 5, 5),	(2, 2, 2, 2, 2, 2, 2, 2, 2, 2),
(1, 1, 1, 1, 1, 2, 3, 6, 6, 6),	(2, 2, 2, 2, 2, 2, 2, 4, 4, 4),
(1, 1, 1, 2, 2, 2, 2, 3, 3, 3).	

Using these designs, those in [3], and results listed in Appendix G of [1] we have the following result.

LEMMA 4. *In order* 32 *all* 6-*tuples of weight less than or equal to* 28 *are the types of orthogonal designs of order* 32 *except possibly*

(2, 2, 2, 3, 7, 9),	(1, 1, 1, 5, 8, 11),
(1, 1, 1, 1, 1, 21),	(1, 2, 2, 2, 3, 17),
(1, 1, 1, 2, 5, 17),	(1, 2, 2, 3, 8, 11),
(1, 1, 1, 5, 5, 14).	

Finally we give tables which summarize the known results about the existence and non-existence of amicable pairs of order 8. In these tables

A	means that such a pair can not exist by virtue of	[1, Theorem 5.39],
B	means that such a pair can not exist by virtue of	[1, Theorem 5.41],
C	means that such a pair can not exist by virtue of	[1, Theorem 5.45],
D	means that such a pair can not exist by virtue of	[1, Theorem 5.47],
F	means that such a pair can not exist by virtue of	[1, p. 240],
G	means that such a pair can not exist by virtue of	[1, Theorem 5.64],
48	means that such a pair can be constructed using Wolfe's construction,	
52	means that such a pair can be constructed using	[1, Theorem 5.52],
58	means that such a pair can be constructed using	[1, Theorem 5.58],
64	means that such a pair can be constructed using	[1, Theorem 5.64],
95	means that such a pair can be constructed using	[1, Theorem 5.95],

6 means that such a pair can be constructed using [1, Table 5.6],
[2] means that such a pair is given in [2], and
* means that such a pair can be constructed using Example 3.

The number of variables in each member of a pair is shown in the caption of the table.

Table 1
Both designs with 4 variables

	1 1 1 1	1 1 1 4	1 1 2 2	2 2 2 2	1 1 1 2	1 1 2 4	1 2 2 2	1 1 1 3	1 2 2 3	1 1 1 5	1 1 2 3	1 1 3 3
1 1 1 1	27			*B*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
1 1 1 4		*		*B*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
1 1 2 2			48	*F*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
2 2 2 2					*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
1 1 1 2					48	*B*	*B*	*A*	*A*	*A*	*A*	*A*
1 1 2 4						*C*	*C*	*A*	*A*	*A*	*A*	*A*
1 2 2 2								*A*	*A*	*A*	*A*	*A*
1 1 1 3									*B*	*A*	*A*	*A*
1 2 2 3									*C*	*A*	*A*	*A*
1 1 1 5										*B*	*A*	*A*
1 1 2 3												*A*
1 1 3 3												*C*

Table 2
Designs with 4 and 3 variables

	1 1 1 1	1 1 1 4	1 1 2 2	2 2 2 2	1 1 1 2	1 1 2 4	1 2 2 2	1 1 1 3	1 2 2 3	1 1 2 3	1 1 1 5	1 1 3 3
1 1 1	48	*		*B*	95	*B*			*B*	*A*	*B*	*A*
1 1 4		*		*F*		*C*			*C*	*A*	*C*	*A*
1 2 2			95	*F*	95	*C*			*C*	*A*	*C*	*A*
2 2 4	*B*	*B*		48	*B*	48		*B*	*F*	*A*	*B*	*A*
1 1 2	95		48	*F*	48	*C*		*A*	*A*		*A*	*A*
1 2 4		*		*F*		*C*		*A*	*A*		*A*	*A*
2 2 2			48					*A*	*A*		*A*	*A*
1 1 3	*A*	*A*	*A*	*A*	6	*C*			*C*		*A*	*C*
1 3 4	*A*	*A*	*A*	*A*	*B*	*C*	*C*	*B*	*C*	*C*	*A*	*C*
2 2 3	*A*	*A*	*A*	*A*	*B*	*F*		*B*	*F*		*A*	*F*
1 2 3			48	*F*	*A*	*A*	*A*		*C*		*A*	*A*
1 1 5		*		*D*	*A*	*A*	*A*	*A*			*C*	*A*
1 3 3	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*	6	*A*	*C*
1 1 6	*A*	*A*	*A*	*A*	*B*	*C*	*C*	*A*	*A*	*A*	*B*	*C*
1 2 5	*A*	*A*	*A*	*A*	*B*	*C*	*C*	*B*	*C*	*A*	*B*	*A*
2 3 3	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*B*	*F*	*A*	*A*	*D*

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Table 3
Designs with 4 and 2 variables

	1111	1114	1122	2222	1112	1124	1222	1113	1223	1123	1115	1133
11	48	*	48	F	48	C			C		C	C
14		*	95	F	6	C			C		C	C
22	6		48	48	48	48						
44	B	B		48	B	48		B	F	D	B	D
12	48	*	48	F	48	C			C		C	A
24		*	48	48		48						A
13	95		48	F	48	C			C	6	A	C
34		*								6	A	
15		*	48	F		C			C		C	A
16		*		F		C		A	A	6	C	C
17	A	A	A	A	B	C	F	B	C	D	B	C
23			95		6	F			F		F	F
25		*				F			F		F	F
26	B	B		48	B	48		B	64	A	B	D
33			48		A	A	A	48		48	A	
35	B	B			B	F		B	F	D	B	D

Table 4
Designs with 4 and 1 variables

	1111	1114	1122	2222	1112	1124	1222	1113	1223	1115	1123	1133
1	48	*	48	F	48	C			C	C	6	C
2	48	*	48	48	48	48			64			
3	48	*	48		48			48			48	
4	6	*	48	48	48	48						
5		*	95		6	F			F	F		F
6		*	48	48		48		48	64		48	
7		*									6	
8	B	B		48	B	48		B	64	B	D	D

Table 5
Both designs with 3 variables

	111	114	122	224	112	124	222	113	223	134	123	115	133	116	125	233
111	48	*	48	B	48	*		6		B		*		B	B	B
114		*		48	48	*	48			C	48	*		C	C	
122			95		48		48	6		C	95			C	C	
224				48	48	48	48	D		48		D		48	48	48
112					48		48	48		C	48		48	C	C	
124						*				C		*	6	C	C	
222							48				48					
113								6		C			6	C	C	D
223										F			6	F	F	
134										C	C	C	C	C	C	F
123											48		6	C	C	
115												*	6	C	C	D
133													6	C	C	F
116														C	C	G
125															C	F
233																

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Table 6
Designs with 3 and 2 variables

	111	114	122	224	112	124	222	113	223	134	123	115	133	116	125	233
11	48	*	48	48	48	*	48	48		*C*	48	*	48	*C*	*C*	
14	58	*	95	48	48	*	48	6		*C*	95	*	6	*C*	*C*	
22	48	48	48	48	48	48	48	48		48	48		48	48	48	48
44	*B*	48		48	48	48	48	*D*		48		*D*		48	48	48
12	48	*	48	48	48	*	48	48		*C*	48	*	48	*C*	*C*	
24	*	48	48	48	48	48	48			48	48	*	6	48	48	48
13	48	48	48	48	48	6	48	48	6	*C*	48	6	48	*C*	*C*	
34	*	*		48	6	6		6	6		6	6	58			
15	*	*	95	48	48	*	48			*C*	48	*	6	*C*	*C*	
16	*	*		48	6	6		6	6	*C*	6	6	6	*C*	*C*	
17	*B*	*F*	*F*	48		*F*		*D*	*F*	*C*	*F*	*D*		*C*	*C*	*F*
23	58	48	58		48		48	58		*F*	95		58	*F*	*F*	
25	*	*		48		*				*F*		*	6	*F*	*F*	
26	*B*	48	64	48	48	48	48	*D*	64	48	64	*D*		48	48	48
33	48		95	48	48	6	48	48	48		48	6	48			
35	*B*			48				*D*		*F*		*D*		*F*	*F*	

Table 7
Designs with 3 and 1 variables

	111	114	122	224	112	124	222	113	223	134	123	115	133	116	125	233
1	48	*	48	48	48	6	48	48	6	*C*	48	6	48	*C*	*C*	
2	48	48	48	48	48	48	48	48	64	48	48	*	48	48	48	48
3	48	*	48	48	48	6	48	48	48		48	6	48			
4	48	48	48	48	48	48	48	48	95	48	48	*	48	48	48	48
5	58	*	58	48	48	*	48	6		*F*	95	*	58	*F*	*F*	
6	48	48	48	48	48	48	48	48	48	48	48	6	48	48	48	48
7	*	*		48	6	6		6	6		6	6	58			
8	*B*	48	64	48	48	48	48	*D*	64	48	64	*D*		48	48	48

Table 8
Both designs with 2 variables

	11	12	13	14	15	16	17	22	23	24	25	26	33	34	35	44
11	48	48	48	48	48	48		48	48	48	*	48	48	48		48
12		48	48	48	48	48		48	48	48	*	48	48	48		48
13			48	48	48	48		48	48	48	6	48	48	48		48
14				6	95	6	*F*	48	58	48	*	48	48	6		48
15					*	6	*F*	48	95	48	*	48	48	6		48
16						6		48	58	48	6	48	48	58		48
17							52	48	*F*	48	*F*	48			*F*	48
22								48	48	48	48	48	48	48	48	48
23									58	48		48	48	58		48
24										48	48	48	48	48	48	48
25											*	48	48	6		48
26												48	48	48	48	48
33													48	48		48
34														58		48
35																48
44																48

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Table 9
Designs with 2 and 1 variables

	1 1	1 2	1 3	1 4	1 5	1 6	1 7	2 2	2 3	2 4	2 5	2 6	3 3	3 4	3 5	4 4
1	48	48	48	48	48	48	52	48	48	48	6	48	48	48		48
2	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48
3	48	48	48	48	48	48	[2]	48	48	48	6	48	48	48		48
4	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48
5	48	48	48	58	95	58	*F*	48	58	48	*	48	48	58		48
6	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48
7	48	48	48	6	6	58	52	48	58	48	6	48	48	58		48
8	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48

NOTE ADDED IN PROOF. Jennifer Seberry has recently informed me that she has constructed all the designs mentioned in Lemma 4 except the $(1, 1, 1, 5, 5, 14)$ design.

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Department of Mathematics
University of Queensland
St. Lucia, Queensland 4067
Australia