Proceedings of the Edinburgh Mathematical Society (2014) **57**, 589–617
DOI:10.1017/S0013091513000618

COVERS OF ACTS OVER MONOIDS AND PURE EPIMORPHISMS

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(Received 15 June 2012)

Abstract In 2001, Enochs's celebrated flat cover conjecture was finally proven, and the proofs (two different proofs were presented in the same paper) have since generated a great deal of interest among researchers. The results have been recast in a number of other categories and, in particular, for additive categories. In 2008, Mahmoudi and Renshaw considered a similar problem for acts over monoids but used a slightly different definition of cover. They proved that, in general, their definition was not equivalent to that of Enochs, except in the projective case, and left open a number of questions regarding the 'other' definition. This 'other' definition is the subject of the present paper and we attempt to emulate some of Enochs's work for the category of acts over monoids, and concentrate, in the main, on strongly flat acts. We hope to extend this work to other classes of acts, such as injective, torsion free, divisible and free, in a future report.

Keywords: monoids; acts; strongly flat; covers; pure epimorphisms; colimits

2010 *Mathematics subject classification:* Primary 20M50

1. Introduction and preliminaries

Let S be a monoid. By a *right S-act* we mean a non-empty set X together with an action $X \times S \to X$ given by $(x, s) \mapsto xs$ such that, for all $x \in X$, $s, t \in S$, $x1 = x$ and $x(st) = (xs)t$. If ρ is an equivalence on a right S-act X, then we refer to it as a (right) S-congruence if, for all $s \in S$, $(x, y) \in \rho$, it follows that $(xs, ys) \in \rho$. Left S-acts and left S-congruences are defined dually. Throughout this paper, unless otherwise stated, all acts will be right S-acts and all congruences right S-congruences. We refer the reader to [**8**] for basic results and terminology in semigroups and monoids and to [**1**, **11**] for those concerning acts over monoids.

Enochs's conjecture, that all modules over a unitary ring have a flat cover, was finally proven in 2001. In 2008, Mahmoudi and Renshaw [**13**] initiated a study of flat covers of acts over monoids. Their definition of cover concerned coessential epimorphisms and, except for the case of projective covers, proved to be different to that given by Enochs. In the present paper we attempt to initiate the study of Enochs's notion of cover for the category of acts over monoids, and focus primarily on $\mathcal{SF}$-covers, where $\mathcal{SF}$ is the class of strongly flat S-acts.

After giving preliminary results and definitions, we provide some key results on directed colimits for acts over monoids. Some of these may be generally known, but there are so few references in the literature for results on direct limits of S-acts that we felt it necessary to include the more important ones here. Pure epimorphisms were studied by Stenström in [**22**] and we extend these in § 3. In § 4 we introduce the concept of an $\mathcal{X}$-cover and an $\mathcal{X}$-precover for a class of S-acts $\mathcal{X}$. This is analogous to Enochs's definition for covers of modules over rings and we prove that, for those classes that are closed under isomorphisms and directed colimits, the existence of a precover implies the existence of a cover. The inspiration for these results and their proofs is taken from the additive case (see, for example, [**4**, **6**, **23**]). We also provide a necessary and sufficient condition and a number of sufficient conditions for the existence of a precover. Finally, in § 5 we apply some of these results to the case when $\mathcal{X}$ is the class of strongly flat S-acts.

An S-act P is called *projective* if, given any S-epimorphism $f\colon A \to B$, whenever there exists an S-map $g\colon P \to B$, there exists an S-map $h\colon P \to A$ such that $hf = g$. A right S-act A is said to be *flat* if, given any monomorphism of left S-acts $f\colon X \to Y$, the induced map $1 \otimes f\colon A \otimes_S X \to A \otimes_S Y$, $a \otimes x \mapsto a \otimes f(x)$ is also a monomorphism. On the other hand, an S-monomorphism $g\colon A \to B$ is said to be *pure* (see [**17**]) if for all left S-acts X, the induced map $A \otimes_S X \to B \otimes_S X$ is also a monomorphism. Note that there are in fact two distinct notions of pure monomorphism in the literature. See [**1**, § 7.4] for more details. In 1969, Lazard proved that flat modules are directed colimits of finitely generated free modules [**12**]. In 1971, Stenström showed that the acts that satisfy the same property are different from flat acts [**22**]. In fact they are the acts A, where $A \otimes_S -$ preserves pullbacks and equalizers, or, equivalently, those that satisfy the two interpolation conditions (P) and (E). These acts have come to be known as *strongly flat* acts. A right S-act A is said to satisfy *condition (P)* if whenever $au = a'u'$ with $u, u' \in S$, $a, a' \in A$, there exist $a'' \in A$, $s, s' \in S$ with $a = a''s$, $a' = a''s'$ and $su = s'u'$, while A is said to satisfy *condition (E)* if whenever $au = au'$ with $a \in A$, $u, u' \in S$, there exist $a'' \in A$, $s \in S$ with $a = a''s$ and $su = su'$.

Throughout this paper we denote the class of all projective S-acts by $\mathcal{P}_S$, the class of all strongly flat S-acts by $\mathcal{SF}_S$, the class of all S-acts that satisfy condition (P) by $\mathcal{CP}_S$, the class of all S-acts that satisfy condition (E) by $\mathcal{E}_S$ and the class of all flat acts by $\mathcal{F}_S$. We normally simply omit the subscript.

It is well known that, in general,

$$\mathcal{P} \subsetneq \mathcal{SF} \subsetneq \mathcal{CP} \subsetneq \mathcal{F}.$$

Basic results on indecomposable acts, coproducts, pushouts and pullbacks of acts over monoids can be found in [**1**, **11**]. From [**1**, Propositions 4.1.5, 5.2.17 and 5.2.5 and Corollary 5.3.23] (see also [**11**, Lemmas III.9.3 and III.9.5]) we have the following.

Lemma 1.1. *Let S be a monoid and let $X = \dot{\bigcup} X_i$ be a coproduct of S-acts. For each of the cases $\mathcal{X} = \mathcal{P}$, $\mathcal{X} = \mathcal{SF}$, $\mathcal{X} = \mathcal{CP}$ and $\mathcal{X} = \mathcal{F}$ we have $X \in \mathcal{X}$ if and only if each $X_i \in \mathcal{X}$.*

https://doi.org/10.1017/S0013091513000618 Published online by Cambridge University Press

The following lemma will be useful in one of our main results.

Lemma 1.2. *Let S be a monoid, let X be an indecomposable S-act and let $f\colon X \to Y$ be an S-epimorphism. Then Y is indecomposable.*

Proof. Suppose that Y is not indecomposable, so there exist non-empty S-subacts $Y_1 \neq Y_2 \subseteq Y$ with $Y = Y_1 \dot\cup Y_2$. Then let $X_i = f^{-1}(Y_i)$, $i = 1, 2$, and note that X_i are non-empty S-subacts of X and that $X = X_1 \dot\cup X_2$ with $X_1 \neq X_2$, giving a contradiction. □

Let A be an S-act. We say that a projective S-act C together with an S-epimorphism $f\colon C \to A$ is a *projective cover* of A if there is no proper subact B of C such that $f|_B$ is onto. If we replace 'projective' by 'strongly flat' in this definition, then we have a *strongly flat cover*. A monoid S is called *perfect* if all S-acts have a projective cover.

We define A to be *finitely presented* if $A \cong F/\rho$ (see [**14, 22**]), where F is finitely generated free and ρ is finitely generated.

The following remark will be useful when we come to consider *precovers* in § 3.

Remark 1.3. Let S be a monoid, let A be an S-act and let θ be a congruence on A. Let ρ be a congruence on A/θ and let $\theta/\rho = \ker(\rho^\natural\theta^\natural)$. Then, clearly, θ/ρ is a congruence on A containing θ and $A/(\theta/\rho) = (A/\theta)/\rho$. Moreover, $\theta/\rho = \theta$ if and only if $\rho = 1_{F/\theta}$.

Let λ be an infinite cardinal and let $\mathcal{X}$ be a class of S-acts. By a *λ-skeleton of S-acts* $\mathcal{X}_\lambda$ we mean a set of pairwise non-isomorphic S-acts such that, for each act $A \in \mathcal{X}$ with $|A| < \lambda$, there exists a (necessarily unique) act $A_\lambda \in \mathcal{X}_\lambda$ such that $A \cong A_\lambda$.

Remark 1.4. Let S be a monoid, let $\mathcal{X}$ be a class of S-acts and suppose that there exists a cardinal λ such that every indecomposable S-act $X \in \mathcal{X}$ is such that $|X| < \lambda$. It is then reasonably clear that the class of indecomposable S-acts forms a set and so must contain a λ-skeleton.

2. Colimits and directed colimits

In the literature there is surprisingly little on direct limits and colimits of acts and, in addition, some inconsistencies of notation (see [**11, 17**]). We include here a collection of results on direct limits, some of which will be needed in later sections.

Let I be a set with a preorder (that is, a reflexive and transitive relation). A *direct system* is a collection of S-acts $(X_i)_{i\in I}$ together with S-maps $\phi_{i,j}\colon X_i \to X_j$ for all $i \leqslant j \in I$ such that

- $\phi_{i,i} = 1_{X_i}$ for all $i \in I$ and
- $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$ whenever $i \leqslant j \leqslant k$.

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The *colimit* of the system $(X_i, \phi_{i,j})$ is an S-act X together with S-maps $\alpha_i\colon X_i \to X$ such that

- $\alpha_j \circ \phi_{i,j} = \alpha_i$ whenever $i \leqslant j$ and
- if Y is an S-act and $\beta_i\colon X_i \to Y$ are S-maps such that $\beta_j \circ \phi_{i,j} = \beta_i$ whenever $i \leqslant j$, then there exists a unique S-map $\psi\colon X \to Y$ such that the diagram

$$\begin{tikzcd} X_i \arrow[r, "\alpha_i"] \arrow[d, "\beta_i"'] & X \arrow[dl, "\psi"] \\ Y & \end{tikzcd}$$

  commutes for all $i \in I$.

If the indexing set I satisfies the property that for all $i, j \in I$ there exists $k \in I$ such that $k \geqslant i, j$, then we say that I is *directed*. In this case we call the colimit a *directed colimit*.

As with all universal constructions, the colimit, if it exists, is unique up to isomorphism. That colimits of S-acts do indeed exist is easy to demonstrate. In fact, let $\lambda_i\colon X_i \to \dot{\bigcup}_i X_i$ be the natural inclusion and let ρ be the right congruence on $\dot{\bigcup}_i X_i$ generated by

$$R = \{(\lambda_i(x_i), \lambda_j(\phi_{i,j}(x_i))) \mid x_i \in X_i,\ i \leqslant j \in I\}.$$

Then, $X = (\dot{\bigcup}_i X_i)/\rho$ and $\alpha_i\colon X_i \to X$ given by $\alpha_i(x_i) = \lambda_i(x_i)\rho$ are such that (X, α_i) is a colimit of $(X_i, \phi_{i,j})$. In addition, if the index set I is directed, then

$$\rho = \{(\lambda_i(x_i), \lambda_j(x_j)) \mid \text{there exists } k \geqslant i, j \text{ with } \phi_{i,k}(x_i) = \phi_{j,k}(x_j)\}.$$

See [**16**, Theorems I.3.1 and I.3.17] for more details. We subsequently talk of *the* (directed) colimit of a direct system.

Lemma 2.1 (Renshaw [17, Lemma 3.5 and Corollary 3.6]). *Let $(X_i, \phi_{i,j})$ be a direct system of S-acts with directed index set, and let (X, α_i) be the directed colimit. Then, $\alpha_i(x_i) = \alpha_j(x_j)$ if and only if $\phi_{i,k}(x_i) = \phi_{j,k}(x_j)$ for some $k \geqslant i, j$. Consequently, α_i is a monomorphism if and only if $\phi_{i,k}$ is a monomorphism for all $k \geqslant i$.*

In fact the following is now easy to establish.

Theorem 2.2. *Let S be a monoid, let $(X_i, \phi_{i,j})$ be a direct system of S-acts with directed index set I, and let X be an S-act and $\alpha_i\colon X_i \to X$ be such that*

$$\begin{tikzcd} X_i \arrow[rr, "\phi_{i,j}"] \arrow[dr, "\alpha_i"'] & & X_j \arrow[dl, "\alpha_j"] \\ & X & \end{tikzcd}$$

commutes for all $i \leqslant j$ in I. Then, (X, α_i) is the directed colimit of $(X_i, \phi_{i,j})$ if and only if

(1) *for all $x \in X$ there exist $i \in I$ and $x_i \in X_i$ such that $x = \alpha_i(x_i)$,*

(2) *for all $i, j \in I$, $\alpha_i(x_i) = \alpha_j(x_j)$ if and only if $\phi_{i,k}(x_i) = \phi_{j,k}(x_j)$ for some $k \geqslant i, j$.*

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We use these two basic properties of directed colimits without further reference.

Lemma 2.3. *Let S be a monoid and let $(X_i, \phi_{i,j})$ be a direct system of S-acts with directed index set I and directed colimit (X, α_i). For every family $y_1, \dots, y_n \in X$ and the relations*

$$y_{j_i} s_i = y_{k_i} t_i, \quad 1 \leqslant i \leqslant m,$$

there exist some $l \in I$ and $x_1, \dots, x_n \in X_l$ such that $\alpha_l(x_r) = y_r$ for $1 \leqslant r \leqslant n$, and

$$x_{j_i} s_i = x_{k_i} t_i \quad \text{for all } 1 \leqslant i \leqslant m.$$

Proof. Given $y_1, \dots, y_n \in X$ there exist $m(1), \dots, m(n) \in I$ and $y'_r \in X_{m(r)}$ such that $\alpha_{m(r)}(y'_r) = y_r$ for all $1 \leqslant r \leqslant n$. So for all $1 \leqslant i \leqslant m$ we have that

$$\alpha_{m(j_i)}(y'_{j_i} s_i) = \alpha_{m(j_i)}(y'_{j_i}) s_i = \alpha_{m(k_i)}(y'_{k_i}) t_i = \alpha_{m(k_i)}(y'_{k_i} t_i),$$

so there exist $l_i \geqslant m(j_i), m(k_i)$ such that, for all $1 \leqslant i \leqslant m$,

$$\phi_{m(j_i),l_i}(y'_{j_i}) s_i = \phi_{m(j_i),l_i}(y'_{j_i} s_i) = \phi_{m(k_i),l_i}(y'_{k_i} t_i) = \phi_{m(k_i),l_i}(y'_{k_i}) t_i.$$

Let $l \geqslant l_1, \dots, l_m$. There then exist $\phi_{m(1),l}(y'_1), \dots, \phi_{m(n),l}(y'_n) \in X_l$ such that

$$\alpha_l(\phi_{m(r),l}(y'_r)) = \alpha_{m(r)}(y'_r) = y_r$$

for all $1 \leqslant r \leqslant n$, and

$$\phi_{m(j_i),l}(y'_{j_i}) s_i = \phi_{l_i,l}(\phi_{m(j_i),l_i}(y'_{j_i})) s_i = \phi_{l_i,l}(\phi_{m(k_i),l_i}(y'_{k_i})) t_i = \phi_{m(k_i),l}(y'_{k_i}) t_i$$

for all $1 \leqslant i \leqslant m$, and the result follows. □

The following result shows that, in a certain sense, directed colimits preserve monomorphisms.

Lemma 2.4. *Let S be a monoid, let $(X_i, \phi_{i,j})$ be a direct system of S-acts with directed index set and let (X, α_i) be the directed colimit. Suppose that Y is an S-act and that $\beta_i \colon X_i \to Y$ are monomorphisms such that $\beta_i = \beta_j \phi_{i,j}$ for all $i \leqslant j$. There then exists a unique monomorphism $h \colon X \to Y$ such that $h\alpha_i = \beta_i$ for all i.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
X_i \arrow[rr, "\phi_{i,j}"] \arrow[dr, "\alpha_i"] \arrow[ddr, "\beta_i"'] & & X_j \arrow[dl, "\alpha_j"'] \arrow[ddl, "\beta_j"] \\
& X \arrow[d, "h"'] & \\
& Y &
\end{tikzcd}$$

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where h is the unique map guaranteed by the directed colimit property. Suppose that $h(x) = h(x')$. There then exist i, j and $x_i \in X_i$, $x_j \in X_j$ such that $x = \alpha_i(x_i)$ and $x' = \alpha_j(x_j)$. Hence, there exists $k \geqslant i, j$, so

$$\beta_k \phi_{i,k}(x_i) = h\alpha_k \phi_{i,k}(x_i) = h\alpha_i(x_i) = h\alpha_j(x_j) = h\alpha_k \phi_{j,k}(x_j) = \beta_k \phi_{j,k}(x_j).$$

Since β_k is a monomorphism, $\phi_{i,k}(x_i) = \phi_{j,k}(x_j)$, so $x = x'$, as required. □

Lemma 2.5. *Let S be a monoid, let X be an S-act and let $\{\rho_i \colon i \in I\}$ be a set of congruences on X, partially ordered by inclusion, with the property that the index set is directed and has a minimum element 0. Let $\phi_{i,j} \colon X/\rho_i \to X/\rho_j$ be the S-map defined by $a\rho_i \mapsto a\rho_j$ whenever $\rho_i \subseteq \rho_j$, so $(X/\rho_i, \phi_{i,j})$ is a direct system. Let $\rho = \bigcup_{i \in I} \rho_i$. Then, X/ρ is the directed colimit of $(X/\rho_i, \phi_{i,j})$.*

Proof. First note that ρ is transitive, since I is directed. Clearly, we can define S-maps $\alpha_i \colon X/\rho_i \to X/\rho$, $a\rho_i \mapsto a\rho$, such that $\alpha_i = \alpha_j \phi_{i,j}$ for all $i \leqslant j$. Now suppose that there exist an S-act Q and S-maps $\beta_i \colon X/\rho_i \to Q$ such that $\beta_i = \beta_j \phi_{i,j}$ for all $i \leqslant j$. Define $\psi \colon X/\rho \to Q$ by $\psi(a\rho) = \beta_0(a\rho_0)$. To see that this is well defined, let $a\rho = a'\rho$ in X/ρ, that is, $(a, a') \in \rho$, so there must exist some $k \in I$ such that $(a, a') \in \rho_k$, and we get that

$$\beta_0(a\rho_0) = \beta_k \phi_{0,k}(a\rho_0) = \beta_k(a\rho_k) = \beta_k(a'\rho_k) = \beta_k \phi_{0,k}(a'\rho_0) = \beta_0(a'\rho_0),$$

so $\psi(a\rho) = \psi(a'\rho)$ and ψ is well defined. It is easy to see that ψ is also an S-map. Because 0 is the minimum element, we have that $\beta_0(a\rho_0) = \beta_0 \phi_{i,0}(a\rho_i) = \beta_i(a\rho_i)$, so $\psi\alpha_i = \beta_i$ for all $i \in I$. Finally, let $\psi' \colon X/\rho \to Q$ be an S-map such that $\psi'\alpha_i = \beta_i$ for all $i \in I$; then $\psi'(a\rho) = \psi'(\alpha_0(a\rho_0)) = \beta_0(a\rho_0) = \psi(a\rho)$, and we are done. □

Remark 2.6. In particular, this holds when we have a chain of congruences $\rho_1 \subset \rho_2 \subset \cdots$ and $\rho = \bigcup_{i \geqslant 1} \rho_i$.

Example 2.7. If S is an inverse monoid, which we consider as a right S-act, then for any $e \leqslant f \in E(S)$ it follows that $\ker \lambda_f \subseteq \ker \lambda_e$, where $\lambda_e(s) = es$. Hence, there exists a set of right congruences on S, partially ordered by inclusion, where the identity relation $\ker \lambda_1$ is a least element in the ordering. We can now construct a direct system of S-acts $S/\ker \lambda_f \to S/\ker \lambda_e$, $s \ker \lambda_f \mapsto s \ker \lambda_e$, whose directed colimit, by the previous lemma, is S/σ, where $\sigma = \bigcup_{e \in E(S)} \ker \lambda_e$, which is easily seen to be the minimum group congruence on S (see [**8**, p. 159]).

Proposition 2.8 (Stenström [22, Proposition 5.2]). *Let S be a monoid. Every directed colimit of a direct system of strongly flat acts is strongly flat.*

The following easily proved result is probably well known.

Proposition 2.9. *Let S be a monoid. Every directed colimit of a direct system of acts that satisfy condition (P) satisfies condition (P).*

The situation for projective acts is slightly different.

Proposition 2.10 (Fountain [7]). *Let S be a monoid. Every directed colimit of a direct system of projective acts is projective if and only if S is perfect.*

3. Purity and epimorphisms

Let $\psi\colon X \to Y$ be an S-epimorphism. We say that ψ is *a pure epimorphism* if for every finitely presented S-act M and every S-map $f\colon M \to Y$ there exists $g\colon M \to X$ such that

$$\begin{tikzcd} X \arrow[r, "\psi"] & Y \\ & M \arrow[u, "f"'] \arrow[ul, "g"] \end{tikzcd}$$

commutes.

Theorem 3.1 (Stenström [22, Proposition 4.3]). *Let S be a monoid and let $\psi\colon X \to Y$ be an S-epimorphism. The following are then equivalent:*

(1) *ψ is pure;*

(2) *for every family $y_1, \dots, y_n \in Y$ and the relations*

$$y_{j_i} s_i = y_{k_i} t_i \quad (1 \leqslant i \leqslant m)$$

there exist $x_1, \dots, x_n \in X$ such that $\psi(x_r) = y_r$ for $1 \leqslant r \leqslant n$, and

$$x_{j_i} s_i = x_{k_i} t_i \quad \textit{for all } 1 \leqslant i \leqslant m.$$

Example 3.2. Let S be an inverse monoid and let σ be the minimum group congruence on S as in Example 2.7. The right S-map $S \to S/\sigma$ is then a pure S-epimorphism. To see this let $y_1 = x_1\sigma, \dots, y_n = x_n\sigma \in S/\sigma$ and suppose that we have the relations

$$y_{j_i} s_i = y_{k_i} t_i \quad (1 \leqslant i \leqslant m).$$

For $1 \leqslant i \leqslant m$ we then have $(x_{j_i} s_i, x_{k_i} t_i) \in \sigma$, so there exist $e_i \in E(S)$ $(1 \leqslant i \leqslant m)$ such that $e_i x_{j_i} s_i = e_i x_{k_i} t_i$. Now let $e = e_1 \cdots e_m$ and note that, for $1 \leqslant i \leqslant m$, $e x_{j_i} s_i = e x_{k_i} t_i$ and, for $1 \leqslant l \leqslant n$, $\sigma^\natural(e x_l) = (e x_l)\sigma = x_l\sigma = y_l$, as required.

It is clear that if the epimorphism ψ splits with splitting monomorphism $\phi\colon Y \to X$, then $\phi f\colon M \to X$ is such that $\psi\phi f = f$, so ψ is pure. The converse is not in general true. For example, let $S = \mathbb{N}$ with multiplication given by

$$n.m = \max\{m, n\} \quad \text{for all } m, n \in S.$$

Let $\Theta_S = \{\theta\}$ be the 1-element right S-act and note that $S \to \Theta_S$ is a pure epimorphism by Theorem 3.1. However, as S does not contain a fixed point, it does not split.

From Lemma 2.3 we can immediately deduce the following.

Corollary 3.3. *Let S be a monoid and let $(X_i, \phi_{i,j})$ be a direct system of S-acts with directed index set I and directed colimit (X, α_i). The natural map $\dot\bigcup X_i \to X$ is then a pure epimorphism.*

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Suppose that $(X_i, \phi_{i,j})$ and $(Y_i, \theta_{i,j})$ are direct systems of S-acts and S-maps, suppose that for each $i \in I$ there exists an S-map $\psi_i \colon X_i \to Y_i$ and suppose that (X, β_i) and (Y, α_i), the directed colimits of these systems, are such that

$$\begin{tikzcd} X_i \arrow[r, "\psi_i"] \arrow[d, "\beta_i"'] & Y_i \arrow[d, "\alpha_i"] \\ X \arrow[r, "\psi"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[r, "\phi_{i,j}"] \arrow[d, "\psi_i"'] & X_j \arrow[d, "\psi_j"] \\ Y_i \arrow[r, "\theta_{i,j}"'] & Y_j \end{tikzcd} \tag{$*$}$$

commute for all $i \leqslant j \in I$. We then refer to ψ as the *directed colimit of the* ψ_i. It was shown in [**16**] that directed colimits of (monomorphisms) epimorphisms are (monomorphisms) epimorphisms.

Corollary 3.4. *Let S be a monoid. Directed colimits of pure S-epimorphisms are pure.*

Proof. Suppose that $(X_i, \phi_{i,j})$ and $(Y_i, \theta_{i,j})$ are direct systems and that for each $i \in I$ there exists a pure epimorphism $\psi_i \colon X_i \to Y_i$, and suppose that (X, β_i) and (Y, α_i), the directed colimits of these systems, are such that the diagrams in $(*)$ commute for all $i \leqslant j \in I$.

Suppose that there exist $y_1, \dots, y_n \in Y$, $s_1, \dots, s_m, t_1, \dots t_m \in S$ and the relations

$$y_{j_i} s_i = y_{k_i} t_i \quad (1 \leqslant i \leqslant m).$$

By Lemma 2.3 there exist $l \in I$ and $z_1, \dots, z_n \in Y_l$ such that $\alpha_l(z_r) = y_r$ for $1 \leqslant r \leqslant n$, and

$$z_{j_i} s_i = z_{k_i} t_i \quad \text{for all } 1 \leqslant i \leqslant m.$$

Since ψ_l is pure, there exist $x_1, \dots, x_n \in X_l$ such that $\psi_l(x_r) = z_r$ for $1 \leqslant r \leqslant n$, and

$$x_{j_i} s_i = x_{k_i} t_i \quad \text{for all } 1 \leqslant i \leqslant m.$$

Hence,

$$\beta_l(x_{j_i}) s_i = \beta_l(x_{k_i}) t_i \quad \text{for all } 1 \leqslant i \leqslant m,$$

and $\psi \beta_l(x_r) = \alpha_l \psi_l(x_r) = \alpha_l(z_r) = y_r$ for $1 \leqslant r \leqslant n$, so ψ is pure. $\square$

Lemma 3.5. *Let S be a monoid, let*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\psi"'] & D \end{tikzcd}$$

be a pullback diagram of S-acts, and suppose that ψ is a pure epimorphism. Then, ϕ is also a pure epimorphism.

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Proof. That ϕ is onto is clear. Suppose that M is finitely presented and that $f\colon M \to B$ is a morphism. There then exists $g\colon M \to C$ such that $\psi g = \beta f$. Since A is a pullback, there exists a unique $h\colon M \to A$ such that $\phi h = f$ and $\alpha h = g$. $\square$

Although not every pure epimorphism splits, we can deduce the following.

Theorem 3.6. *Let S be a monoid and let $\psi\colon X \to Y$ be an epimorphism. Then, ψ is pure if and only if it is a directed colimit of split epimorphisms.*

Proof. Suppose that ψ is pure. We know (see [**22**, Proposition 4.1]) that Y is a directed colimit of finitely presented acts $(Y_i, \phi_{i,j})$, so let $\alpha_i\colon Y_i \to Y$ be the canonical maps. For each Y_i let

$$\begin{tikzcd} X_i \arrow[r, "\psi_i"] \arrow[d, "\beta_i"'] & Y_i \arrow[d, "\alpha_i"] \\ X \arrow[r, "\psi"'] & Y \end{tikzcd}$$

be a pullback diagram, so by Lemma 3.5 ψ_i is pure. Hence, since Y_i is finitely presented, it easily follows that ψ_i splits. Note that $X_i = \{(y_i, x) \in Y_i \times X \mid \alpha_i(y_i) = \psi(x)\}$, $\psi_i(y_i, x) = y_i$ and $\beta_i(y_i, x) = x$, and that, since ψ is onto, $X_i \neq \emptyset$.

For $i \leqslant j$ define $\theta_{i,j}\colon X_i \to X_j$ by $\theta_{i,j}(y_i, x) = (\phi_{i,j}(y_i), x)$ and note that $\beta_j\theta_{i,j} = \beta_i$ and that $\psi_j\theta_{i,j} = \phi_{i,j}\psi_i$. Suppose now that there exist Z and $\gamma_i\colon X_i \to Z$ with $\gamma_j\theta_{i,j} = \gamma_i$ for all $i \leqslant j$. Define $\gamma\colon X \to Z$ by $\gamma(x) = \gamma_i(y_i, x)$, where i and y_i are chosen such that $\alpha_i(y_i) = \psi(x)$. Then, γ is well defined since if $\psi(x) = \alpha_j(y_j)$, then there exists $k \geqslant i, j$ with $\phi_{i,k}(y_i) = \phi_{j,k}(y_j)$ and

$$\begin{aligned} \gamma_i(y_i, x) &= \gamma_k\theta_{i,k}(y_i, x) = \gamma_k(\phi_{i,k}(y_i), x) \\ &= \gamma_k(\phi_{j,k}(y_j), x) = \gamma_k\theta_{j,k}(y_j, x) \\ &= \gamma_j(y_j, x). \end{aligned}$$

Then, γ is an S-map and, clearly, $\gamma\beta_i = \gamma_i$. Finally, if $\gamma'\colon X \to Z$ is such that $\gamma'\beta_i = \gamma_i$ for all i, then $\gamma'(x) = \gamma'\beta_i(y_i, x) = \gamma_i(y_i, x) = \gamma(x)$, so γ is unique. We therefore have that (X, β_i) is the directed colimit of $(X_i, \theta_{i,j})$, as required.

Conversely, since split epimorphisms are pure, then ψ is pure by Corollary 3.4. $\square$

Example 3.7. Let S be as in Example 2.7. Note that, for all $e \in E(S)$, $S \to S/\ker\lambda_e$ splits with splitting map $s\ker\lambda_e \mapsto es$. Moreover,

$$\begin{tikzcd} S \arrow[r] \arrow[d, "1_S"'] & S/\ker\lambda_e \arrow[d] \\ S \arrow[r, "\sigma^\natural"'] & S/\sigma \end{tikzcd}$$

commutes for all $e \in E(S)$ and $\sigma^\natural$ is a directed colimit of split epimorphisms.

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Theorem 3.8 (Stenström [22, Theorem 5.3]). *Let S be a monoid. An S-act Y is then strongly flat if and only if every epimorphism $X \to Y$ is pure.*

In [**15**], Normak defines an epimorphism $\phi\colon X \to Y$ to be 1-*pure* if for every element $y \in Y$ and the relations $ys_i = yt_i$, $i = 1, \dots, n$, there exists an element $x \in X$ such that $\phi(x) = y$ and $xs_i = xt_i$ for all i. He proves the following.

Proposition 3.9 (Normak [15, Proposition 1.17]). *Let S be a monoid. An epimorphism $\phi\colon X \to Y$ is* 1*-pure if and only if for all cyclic finitely presented S-acts C and every morphism $f\colon C \to Y$ there exists $g\colon C \to X$ with $f = \phi g$.*

Proposition 3.10 (Normak [15, Proposition 2.2]). *Let S be a monoid. Y satisfies condition (E) if and only if every epimorphism $X \to Y$ is* 1*-pure.*

As a generalization, we say that an epimorphism $g\colon B \to A$ of S-acts is n-*pure* if for every family of n elements $a_1, \dots, a_n \in A$ and every finite family of relations $a_{\alpha_i} s_i = a_{\beta_i} t_i$, $i = 1, \dots, m$, there exist $b_1, \dots, b_n \in B$ such that $g(b_i) = a_i$ and $b_{\alpha_i} s_i = b_{\beta_i} t_i$ for all i.

We are interested in the cases $n = 1$ and $n = 2$. Clearly, pure $\Rightarrow$ 2-pure $\Rightarrow$ 1-pure.

Proposition 3.11. *Let S be a monoid and let $\psi\colon X \to Y$ be an S-epimorphism in which X satisfies condition (E). Then, Y satisfies condition (E) if and only if ψ is* 1*-pure.*

Proof. Suppose that ψ is 1-pure and that $y \in Y$, $s, t \in S$ are such that $ys = yt$ in Y. Hence, there exists $x \in X$ such that $\psi(x) = y$ and $xs = xt$. Since X satisfies condition (E) there exist $x' \in X$, $u \in S$ such that $x = x'u$, $us = ut$ and so $y = \psi(x')u$, $us = ut$ and Y satisfies condition (E).

The converse holds by Proposition 3.10. □

Proposition 3.12. *Let S be a monoid and let $\psi\colon X \to Y$ be an S-epimorphism in which X satisfies condition (P). If ψ is* 2*-pure, then Y satisfies condition (P).*

Proof. Suppose that ψ is 2-pure and suppose that $y_1, y_2 \in Y$, $s_1, s_2 \in S$ are such that $y_1 s_1 = y_2 s_2$ in Y. Hence, there exist $x_1, x_2 \in X$ with $\psi(x_i) = y_i$ and $x_1 s_1 = x_2 s_2$ in X. Since X satisfies condition (P), there exist $x_3 \in X$, $u_1, u_2 \in S$ such that $x_1 = x_3 u_1$, $x_2 = x_3 u_2$ and $u_1 s_1 = u_2 s_2$. Consequently, $y_1 = \psi(x_3) u_1$, $y_2 = \psi(x_3) u_2$ and $u_1 s_1 = u_2 s_2$, so Y satisfies condition (P). □

The converse of this last result is false. For example, let $S = (\mathbb{N} \cup \{0\}, +)$ and let $\Theta_S = \{\theta\}$ be the 1-element S-act. Let $x = y = \theta \in \Theta_S$; then $x0 = y0$ and $x0 = y1$ but there cannot exist $x', y' \in S$ such that $x' + 0 = y' + 0$ and $x' + 0 = y' + 1$, so $S \to \Theta_S$ is not 2-pure, but it is easy to check that Θ_S does satisfy condition (P).

From Theorem 3.8 and Propositions 3.11 and 3.12 we deduce the following.

Corollary 3.13. *Let S be a monoid and let $\psi\colon X \to Y$ be an S-epimorphism with X strongly flat. The following are equivalent:*

(1) *Y is strongly flat;*

(2) *ψ is pure;*

(3) *ψ is* 2*-pure.*

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Let X be an S-act and let θ be a congruence on X. Say that θ is *pure* if $X \to X/\theta$ is pure. As a corollary to Theorem 3.1 we have the following.

Corollary 3.14. *Let S be a monoid, let X be an S-act and let θ be a congruence on X. Then, θ is pure if and only if for every family $x_1, \dots, x_n \in X$ and the relations*

$$x_{j_i} s_i \theta x_{k_i} t_i \quad (1 \leqslant i \leqslant m)$$

on X there exists $y_1, \dots, y_n \in X$ such that $y_i \theta x_i$ and

$$y_{j_i} s_i = y_{k_i} t_i \quad \text{for all } 1 \leqslant i \leqslant m.$$

Corollary 3.15. *Let ρ be a right S-congruence on a monoid S. Then, ρ is pure if and only if S/ρ is strongly flat.*

Example 3.16. It now follows easily from Example 3.2 that if S is an inverse monoid with minimum group congruence σ, then S/σ is a strongly flat right S-act.

Let $f\colon X \to Y$ be an S-monomorphism. Renshaw [**20**] defined f to be P-*unitary* if

$$(\forall y, y' \in Y \setminus \operatorname{im}(f),\ \forall s, t \in S) \quad ys, y't \in \operatorname{im}(f) \Rightarrow ys = y't.$$

This is obviously equivalent to saying that whenever $y, y' \in Y$, $s, t \in S$ are such that $ys \neq y't$ but $ys, y't \in \operatorname{im}(f)$, either $y \in \operatorname{im}(f)$ or $y' \in \operatorname{im}(f)$.

In the same way, Renshaw defined f to be E-*unitary* if

$$(\forall y \in Y \setminus \operatorname{im}(f),\ \forall s, t \in S) \quad ys, yt \in \operatorname{im}(f) \Rightarrow ys = yt,$$

which is obviously equivalent to saying that whenever $y \in Y$, $s, t \in S$ are such that $ys \neq yt$ but $ys, yt \in \operatorname{im}(f)$, then $y \in \operatorname{im}(f)$.

Theorem 3.17. *Let S be a monoid, let $f\colon X \to Y$ be a monomorphism and suppose that $Y \to Y/X$ is a 2-pure epimorphism. Then, f is P-unitary. Moreover, for all $s, t \in S$ there exists $x, x' \in X$ with $xs = x't$.*

Proof. Let $\rho = \operatorname{im}(f) \times \operatorname{im}(f) \cup 1_Y$, so $Y/X = Y/\rho$. Let $y, y' \in Y \setminus \operatorname{im}(f)$ and suppose that $ys, y't \in \operatorname{im}(f)$. Then $ys \;\rho\; y't$, and so, by assumption, it easily follows that $ys = y't$, as required.

Let $x \in X$ so that $f(x)s \;\rho\; f(x)t$. There then exists $x_1, x_2 \in X$ with

$$f(x_1) \;\rho\; f(x) \;\rho\; f(x_2) \quad \text{and} \quad f(x_1)s = f(x_2)t.$$

Hence, $x_1 s = x_2 t$, as required. □

It then follows from [**20**, Theorems 4.1 and 4.3] that if $Y \to Y/X$ is a pure epimorphism, then $f\colon X \to Y$ is a pure monomorphism. In fact, following the remark after the proof of [**20**, Theorem 4.1], we see that f splits. In addition, we see from [**20**, Theorem 4.22] that if every epimorphism is pure, then S is a group. Actually, from Theorem 3.8 we see that all S-acts are strongly flat, so S is the trivial group.

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Theorem 3.18. *Let S be a monoid, let $f\colon X \to Y$ be a monomorphism and suppose that $Y \to Y/X$ is a 1-pure epimorphism. Then, f is E-unitary. Moreover, for all $s, t \in S$ there exists $x \in X$ with $xs = xt$.*

Proof. Let $\rho = \operatorname{im}(f) \times \operatorname{im}(f) \cup 1_Y$, so $Y/X = Y/\rho$. Let $y \in Y \setminus \operatorname{im}(f)$, $s, t \in S$ and suppose that $ys, yt \in \operatorname{im}(f)$. Then $(y\rho)s = (y\rho)t$, so there exist $z \in Y$, $u \in S$ with $y\rho = (z\rho)u$ and $us = ut$. Hence, $y = zu$, so $ys = yt$, as required.

Let $x \in X$ so that $f(x)\rho s = f(x)\rho t$. There then exists $y \in Y$ with $y\rho = f(x)\rho$ and $ys = yt$. Hence, $y = x_1$ for some $x_i \in X$, so $x_1 s = x_1 t$, as required. □

Theorem 3.19. *Let S be a monoid, let $f\colon X \to Y$ be a monomorphism and suppose that $Y \to Y/X$ is a split epimorphism. Then f is P-unitary. Moreover, for all $s, t \in S$ there exists $x \in X$ with $xs = xt$.*

Proof. Let $\rho = \operatorname{im}(f) \times \operatorname{im}(f) \cup 1_Y$, so $Y/X = Y/\rho$. Let $g\colon Y/X \to Y$ be the splitting map. Note that if $y \notin \operatorname{im}(f)$, then $g(y\rho) = y$. Let $y, y' \in Y \setminus \operatorname{im}(f)$, $s, t \in S$ and suppose that $ys, y't \in \operatorname{im}(f)$. Then $(y\rho)s = (y'\rho)t$, so $g(y\rho)s = g(y'\rho)t$. Consequently, $ys = y't$, as required.

Let $x \in X$ such that $f(x)\rho s = f(x)\rho t$. Then $g(f(x)\rho)s = g(f(x)\rho)t$, so there exists $x_1 \in X$ with $g(f(x)\rho) = f(x_1)$, so $x_1 s = x_1 t$, as required. □

4. Covers and precovers

Let S be a monoid, and let A be an S-act. Unless otherwise stated, in the rest of this section, $\mathcal{X}$ is a class of S-acts closed under isomorphisms. By an *$\mathcal{X}$-precover* of A we mean an S-map $g\colon P \to A$ for some $P \in \mathcal{X}$ such that, for every S-map $g'\colon P' \to A$, for $P' \in \mathcal{X}$, there exists an S-map $f\colon P' \to P$ with $g' = gf$.

$$\begin{tikzcd} P \arrow[r, "g"] & A \\ & P' \arrow[ul, "f"] \arrow[u, "g'"'] \end{tikzcd}$$

If, in addition, the precover satisfies the condition that each S-map $f\colon P \to P$ with $gf = g$ is an isomorphism, then we call it an *$\mathcal{X}$-cover*. We, of course, frequently identify the (pre)cover with its domain. Obviously, an S-act A is an $\mathcal{X}$-cover of itself if and only if $A \in \mathcal{X}$. Note that this definition of cover is different from that given in [**13**].

Theorem 4.1 (Mahmoudi and Renshaw [13, Theorem 5.8]). *Let S be a monoid. If $g_1\colon X_1 \to A$ and $g_2\colon X_2 \to A$ are both $\mathcal{X}$-covers of an S-act A, then there exists an isomorphism $h\colon X_1 \to X_2$ such that $g_2 h = g_1$.*

Theorem 4.2 (Mahmoudi and Renshaw [13, Theorem 5.7]). *Let S be a monoid. An S-map $g\colon P \to A$, with $P \in \mathcal{P}$, is a $\mathcal{P}$-cover of A if and only if it is a projective cover.*

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It was demonstrated in [**13**] that the previous result is not true for condition (P). We show in § 5 that it is also false for strongly flat acts.

Recall from [**11**, Theorem II.3.16] that an S-act G is called a *generator* if there exists an S-epimorphism $G \to S$.

Proposition 4.3. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts that contains a generator G. If $g\colon C \to A$ is an $\mathcal{X}$-precover of A, then g is an epimorphism.*

Proof. Let $h\colon G \to S$ be an S-epimorphism. There then exists an $x \in G$ such that $h(x) = 1$. For all $a \in A$ define the S-map $\lambda_a\colon S \to A$ by $\lambda_a(s) = as$. By the $\mathcal{X}$-precover property, there exists an S-map $f\colon G \to C$ such that $gf = \lambda_a h$. Hence, $g(f(x)) = a$, so $\operatorname{im}(g) = A$ and g is epimorphic. □

Obviously, if every S-act has an epimorphic $\mathcal{X}$-precover, then S has an epimorphic $\mathcal{X}$-precover, which by definition is then a generator in $\mathcal{X}$, so we have the following corollary.

Corollary 4.4. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that every S-act has an $\mathcal{X}$-precover. Every S-act then has an epimorphic $\mathcal{X}$-precover if and only if $\mathcal{X}$ contains a generator.*

Note that, for any class of S-acts containing S, S is a generator in $\mathcal{X}$, so $\mathcal{X}$-precovers are always epimorphic. In particular, this is true for the classes $\mathcal{P}$, $\mathcal{SF}$, $\mathcal{CP}$ and $\mathcal{F}$.

Lemma 4.5. *Let S be a monoid and let $h\colon X \to A$ be a homomorphism of S-acts where $A = \dot\bigcup_{i\in I} A_i$ is a coproduct of non-empty subacts $A_i \subseteq A$. There then exist $J \subseteq I$ and $X_j \subseteq X$ for each $j \in J$ such that $X = \dot\bigcup_{j\in J} X_j$ and $\operatorname{im}(h|_{X_j}) \subseteq A_j$ for each $j \in J$. Moreover, if h is an epimorphism, then $J = I$.*

Proof. For each $i \in I$ let $X_i = \{x \in X\colon h(x) \in A_i\}$ and define $J = \{i \in I\colon X_i \neq \emptyset\}$. For all $x_j \in X_j$ and $s \in S$, $h(x_j s) = h(x_j)s \in A_j$, so $x_j s \in X_j$ and X_j is a subact of X. Since A_j are disjoint and h is a well-defined S-map, X_j are also disjoint and $X = \dot\bigcup_{j\in J} X_j$. Clearly, $\operatorname{im}(h|_{X_j}) \subseteq A_j$ for each $j \in J$. If h is an epimorphism, then none of the X_i are empty, so $J = I$. □

Proposition 4.6. *Let S be a monoid and let $\mathcal{X}$ satisfy the property that, for each $i \in I$, $\dot\bigcup_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$. Each A_i then has an $\mathcal{X}$-precover if and only if $\dot\bigcup_{i\in I} A_i$ has an $\mathcal{X}$-precover.*

Proof. For each $i \in I$, let $g_i\colon C_i \to A_i$ be an $\mathcal{X}$-precover of A_i. Define $g\colon \dot\bigcup_{i\in I} C_i \to \dot\bigcup_{i\in I} A_i$ to be the obvious induced map where $g|_{C_i} = g_i$ for each $i \in I$. We claim that this is an $\mathcal{X}$-precover of $\dot\bigcup_{i\in I} A_i$. Let $X \in \mathcal{X}$ and let $h\colon X \to \dot\bigcup_{i\in I} A_i$. By Lemma 4.5, there exists a subset $J \subseteq I$ such that $X = \dot\bigcup_{j\in J} X_j$ and $\operatorname{im}(h|_{X_j}) \subseteq A_j$ for each $j \in J$. Now, by the hypothesis, $X_j \in \mathcal{X}$, so, since C_j is an $\mathcal{X}$-precover of A_j, there exists $f_j \in \operatorname{Hom}_S(X_j, C_j)$ such that $h|_{X_j} = g_j f_j$. So define $f\colon \dot\bigcup_{j\in J} X_j \to \dot\bigcup_{i\in I} C_i$ to be the obvious induced map with $f|_{X_j} = f_j$ for each $j \in J$, and clearly $gf = h$.

Conversely, let $g\colon C \to \dot\bigcup_{i\in I} A_i = A$ be an $\mathcal{X}$-precover of A. Let $i \in I$ and define $C_i = \{c \in C\colon g(c) \in A_i\}$, and let $g_i = g|_{C_i}$. Suppose that X is an S-act and suppose that $h \in \mathrm{Hom}_S(X, A_i)$. Then, clearly $h \in \mathrm{Hom}_S(X, A)$, so by the $\mathcal{X}$-precover property there exists an $f \in \mathrm{Hom}_S(X, C)$ such that $h = gf$. In fact $g(f(X)) = h(X) \subseteq A_i$, so $f \in \mathrm{Hom}_S(X, C_i)$ and $h_i = g_i f$. By the hypothesis, $C_i \in \mathcal{X}$, and hence $g_i \colon C_i \to A_i$ is an $\mathcal{X}$-precover of A_i. $\square$

By Lemma 1.1, the classes $\mathcal{P}$, $\mathcal{SF}$, $\mathcal{CP}$ and $\mathcal{F}$ all satisfy this property, so, for any of these classes, to show that all S-acts have $\mathcal{X}$-precovers it is enough to show that the indecomposable S-acts have $\mathcal{X}$-precovers.

Lemma 4.7. *Let S be a monoid. The one element S-act Θ_S has an $\mathcal{X}$-precover if and only if there exists an S-act $A \in \mathcal{X}$ such that $\mathrm{Hom}_S(X, A) \neq \emptyset$ for all $X \in \mathcal{X}$.*

Proof. Let $\Theta_S = \{\theta\}$, let $A \in \mathcal{X}$ and let $g \colon A \to \Theta_S$ be given by $g(a) = \theta$. Given any S-act $X \in \mathcal{X}$ with S-map $h \colon X \to \Theta_S$, clearly $gf = h$ for every $f \in \mathrm{Hom}_S(X, A)$. $\square$

We now show that the colimits of $\mathcal{X}$-precovers are $\mathcal{X}$-precovers. To be more precise, we have the following.

Lemma 4.8. *Let S be a monoid, let $\mathcal{X}$ be a class of S-acts closed under colimits and let A be an S-act. Suppose that $(X_i, \phi_{i,j})$ is a direct system of S-acts with $X_i \in \mathcal{X}$ for each $i \in I$ and with colimit (X, α_i). Suppose also that, for each $i \in I$, $f_i \colon X_i \to A$ is an $\mathcal{X}$-precover of A such that, for all $i \leqslant j$, $f_j \phi_{i,j} = f_i$. There then exists an $\mathcal{X}$-precover $f \colon X \to A$ such that $f\alpha_i = f_i$ for all $i \in I$.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
X_i \arrow[rr, "\phi_{i,j}"] \arrow[dr, "\alpha_i"] \arrow[ddr, "f_i"'] & & X_j \arrow[dl, "\alpha_j"'] \arrow[ddl, "f_j"] \\
& X & \\
& A &
\end{tikzcd}$$

so there exists a unique S-map $f \colon X \to A$ such that $f\alpha_i = f_i$ for all $i \in I$. If $F \in \mathcal{X}$ and if $g \colon F \to A$, then for each $i \in I$ there exists $h_i \colon F \to X_i$ such that $f_i h_i = g$. Choose any $i \in I$ and let $h \colon F \to X$ be given by $h = \alpha_i h_i$. Then $fh = g$, as required. $\square$

The motivation for the next few results comes mainly from [**23**].

Lemma 4.9. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts closed under directed colimits. Let A be an S-act and suppose that $k \colon C \to A$ is an $\mathcal{X}$-precover of A. There then exists an $\mathcal{X}$-precover $\bar{k} \colon \bar{C} \to A$ and an S-map $g \colon C \to \bar{C}$ with $\bar{k}g = k$ such that for any $\mathcal{X}$-precover $k^* \colon C^* \to A$ and any S-map $h \colon \bar{C} \to C^*$ with $k^* h = \bar{k}$, $h|_{\mathrm{im}(g)}$ is a monomorphism.*

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Proof. Suppose, by way of contradiction, that for all $\mathcal{X}$-precovers $\bar{k}\colon \bar{C} \to A$ and S-maps $g\colon C \to \bar{C}$ with $\bar{k}g = k$ there exist an $\mathcal{X}$-precover $k^*\colon C^* \to A$ and an S-map $h\colon \bar{C} \to C^*$ with $k^*h = \bar{k}$ and such that $h|_{\mathrm{im}(g)}$ is not a monomorphism. So, in particular, when $\bar{C} = C$, $\bar{k} = k$ and $g = 1_C$, there exists an $\mathcal{X}$-precover $k_1\colon C_1 \to A$ and an S-map $g_{1,0}\colon C \to C_1$ with $k_1 g_{1,0} = k$ and such that $g_{1,0}|_{\mathrm{im}(1_C)}$ is not a monomorphism.

Now let $\kappa \geqslant 2$ be an ordinal and suppose that for all ordinals $\alpha < \kappa$ there exist an $\mathcal{X}$-precover $k_\alpha\colon C_\alpha \to A$ and S-maps $g_{\alpha,\beta}\colon C_\beta \to C_\alpha$ for $\beta < \alpha$ such that for any triple $\gamma < \delta < \alpha$, $g_{\alpha,\gamma} = g_{\alpha,\delta} g_{\delta,\gamma}$ and

$$\ker(g_{1,0}) \subsetneq \cdots \subsetneq \ker(g_{\alpha,0}) \subsetneq \cdots \subseteq C \times C.$$

We proceed by transfinite induction. First, if κ is not a limit ordinal, then on setting $\bar{C} = C_{\kappa-1}$, $\bar{k} = k_{\kappa-1}$ and $g = g_{\kappa-1,0}$ we deduce that there exist an $\mathcal{X}$-precover $k_\kappa\colon C_\kappa \to A$ and an S-map $g_{\kappa,\kappa-1}\colon C_{\kappa-1} \to C_\kappa$ with $k_\kappa g_{\kappa,\kappa-1} = g_{\kappa-1,0}$ such that $g_{\kappa,\kappa-1}|_{\mathrm{im}(g_{\kappa-1,0})}$ is not a monomorphism. For $\beta < \kappa - 1$ let $g_{\kappa,\beta} = g_{\kappa,\kappa-1} g_{\kappa-1,\beta}$, so $\ker(g_{\kappa-1,0}) \subsetneq \ker(g_{\kappa,0})$ and, for $\gamma < \delta < \kappa$, $g_{\kappa,\gamma} = g_{\kappa,\delta} g_{\delta,\gamma}$, as required.

Now, if κ is a limit ordinal, then let $(C_\kappa, g_{\kappa,\alpha}\colon C_\alpha \to C_\kappa)$ be the directed colimit of the system $(C_\alpha, g_{\alpha,\beta})$ and consider the diagram

$$\begin{tikzcd}
C_\beta \arrow[rr, "g_{\alpha,\beta}"] \arrow[dr, "g_{\kappa,\beta}"] \arrow[ddr, "k_\beta"'] & & C_\alpha \arrow[dl, "g_{\kappa,\alpha}"'] \arrow[ddl, "k_\alpha"] \\
& C_\kappa \arrow[d, "k_\kappa"] & \\
& A &
\end{tikzcd}$$

where $k_\kappa\colon C_\kappa \to A$ is the unique S-map that makes the diagram commutative. By Lemma 4.8 we then deduce that $k_\kappa\colon C_\kappa \to A$ is an $\mathcal{X}$-precover for A. In addition, we see that, for $\gamma < \delta < \kappa$, $g_{\kappa,\gamma} = g_{\kappa,\delta} g_{\delta,\gamma}$ and that $\ker(g_{\delta,0}) \subseteq \ker(g_{\kappa,0})$. But $\ker(g_{\delta,0}) \subsetneq \ker(g_{\delta+1,0}) \subseteq \ker(g_{\kappa,0})$, so $\ker(g_{\delta,0}) \subsetneq \ker(g_{\kappa,0})$, as required.

It then follows that $|C \times C|$ is greater than the cardinality of every ordinal, which is a clear contradiction. $\square$

Lemma 4.10. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts closed under directed colimits. Let A be an S-act and suppose that $k\colon C \to A$ is an $\mathcal{X}$-precover of A. There then exists an $\mathcal{X}$-precover $\bar{k}\colon \bar{C} \to A$ such that, for any $\mathcal{X}$-precover $k^*\colon C^* \to A$ and any S-map $h\colon \bar{C} \to C^*$ with $k^*h = \bar{k}$, h is a monomorphism.*

Proof. By Lemma 4.9 there exist an $\mathcal{X}$-precover $k_1\colon C_1 \to A$ and an S-map $g_{1,0}\colon C \to C_1$ with $k_1 g_{1,0} = k$ such that, for any $\mathcal{X}$-precover $k^*\colon C^* \to A$ and any S-map $h\colon C_1 \to C^*$ with $k^*h = k_1$, $h|_{\mathrm{im}(g_{1,0})}$ is a monomorphism. Now, let $n > 1$ and suppose by way of induction that there exist an $\mathcal{X}$-precover $k_{n-1}\colon C_{n-1} \to A$ and a map $g_{n-1,n-2}\colon C_{n-2} \to C_{n-1}$ with $k_{n-1} g_{n-1,n-2} = k_{n-2}$ and such that, for any

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$\mathcal{X}$-precover $k^*\colon C^* \to A$ and any S-map $h\colon C_{n-1} \to C^*$ with $k^*h = k_{n-1}$, $h|_{\mathrm{im}(g_{n-1,n-2})}$ is a monomorphism (here, we obviously assume that $C_0 = C$ and $k_0 = k$):

$$\begin{tikzcd} C_{n-2} \arrow[r, "g_{n-1,n-2}"] & C_{n-1} \arrow[r, "k_{n-1}"] \arrow[dr, "h"'] & A \\ & & C \arrow[u, "k^*"'] \end{tikzcd}$$

By Lemma 4.9 we then deduce that there exist an $\mathcal{X}$-precover $k_n\colon C_n \to A$ and a map $g_{n,n-1}\colon C_{n-1} \to C_n$ with $k_n g_{n,n-1} = k_{n-1}$ and such that, for any $\mathcal{X}$-precover $k^*\colon C^* \to A$ and any S-map $h\colon C_n \to C^*$ with $k^*h = k_n$, $h|_{\mathrm{im}(g_{n,n-1})}$ is a monomorphism.

Now, let $(C_\omega, g_{\omega,n}\colon C_n \to C_\omega)$ be the directed colimit of the system $(C_n, g_{n,n-1})$ and consider the diagram

$$\begin{tikzcd} C_{n-1} \arrow[rr, "g_{n,n-1}"] \arrow[dr, "g_{\omega,n-1}"] \arrow[ddr, "k_{n-1}"'] & & C_n \arrow[dl, "g_{\omega,n}"'] \arrow[ddl, "k_n"] \\ & C_\omega \arrow[d, "k_\omega"] & \\ & A & \end{tikzcd}$$

where $k_\omega\colon C_\omega \to A$ is the unique S-map that makes the diagram commutative. By Lemma 4.8 we then deduce that $k_\omega\colon C_\omega \to A$ is an $\mathcal{X}$-precover for A. We claim that this $\mathcal{X}$-precover has the desired properties. So let $k^*\colon C^* \to A$ be an $\mathcal{X}$-precover of A and let $h\colon C_\omega \to C^*$ be an S-map with $k^*h = k_\omega$. Suppose also that $h(x) = h(y)$ for $x, y \in C_\omega$. There then exist $m, n > 0$ and $x_m \in C_m$, $y_n \in C_n$ such that $g_{\omega,m}(x_m) = x$ and $g_{\omega,n}(y_n) = y$. Assume, without loss of generality, that $m \leqslant n$ and let $z_n = g_{n,m}(x_m)$. Then

$$hg_{\omega,n+1}(g_{n+1,n}(z_n)) = hg_{\omega,n}(z_n) = hg_{\omega,n}(y_n) = hg_{\omega,n+1}(g_{n+1,n}(y_n)).$$

But $hg_{\omega,n+1}\colon C_{n+1} \to C^*$ and $hg_{\omega,n+1}|_{\mathrm{im}(g_{n+1,n})}$ is, therefore, a monomorphism. Hence, $g_{n+1,n}(z_n) = g_{n+1,n}(y_n)$, so

$$x = g_{\omega,m}(x_m) = g_{\omega,n+1}(g_{n+1,n}(z_n)) = g_{\omega,n+1}(g_{n+1,n}(y_n)) = g_{\omega,n}(y_n) = y,$$

as required. □

We can now deduce one of our main theorems.

Theorem 4.11. *Let S be a monoid, let A be an S-act and let $\mathcal{X}$ be a class of S-acts closed under directed colimits. If A has an $\mathcal{X}$-precover, then A has an $\mathcal{X}$-cover.*

Proof. By Lemma 4.10 there exists an $\mathcal{X}$-precover $k_0\colon C_0 \to A$ such that, for any $\mathcal{X}$-precover $k^*\colon C^* \to A$ and any S-map $h\colon C_0 \to C^*$ with $k^*h = k_0$, h is a monomorphism. We show that $k_0\colon C_0 \to A$ is in fact an $\mathcal{X}$-cover of A.

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Assume by way of contradiction that A does not have an $\mathcal{X}$-cover. Let $C_1 = C_0$ and $k_1 = k_0$. There then exists $g_{1,0} \colon C_0 \to C_1$ with $k_1 g_{1,0} = k_0$ and such that $g_{1,0}$ is a monomorphism but not an epimorphism. It follows that

$$\operatorname{im}(g_{1,0}) \subsetneq C_1 = C_0.$$

By way of transfinite induction, suppose that $\kappa \geqslant 2$ is an ordinal such that, for all ordinals $\alpha < \kappa$, there exists an $\mathcal{X}$-precover $k_\alpha \colon C_\alpha \to A$ such that the following hold:

(1) for any $\mathcal{X}$-precover $k^* \colon C^* \to A$ and any S-map $h \colon C_\alpha \to C^*$ with $k^* h = k_\alpha$, h is a monomorphism;

(2) for all ordinals $\beta < \alpha$ there exist S-maps $g_{\alpha,\beta} \colon C_\beta \to C_\alpha$, which are monomorphisms but not epimorphisms, and $\operatorname{im}(g_{\alpha,\beta}) \subsetneq C_\alpha$;

(3) for all ordinals $\gamma < \beta < \alpha$, $g_{\alpha,\gamma} = g_{\alpha,\beta} g_{\beta,\gamma}$ and

$$\operatorname{im}(g_{\alpha,\gamma}) \subsetneq \operatorname{im}(g_{\alpha,\beta}).$$

We show that κ also possesses these properties. If κ is not a limit ordinal, then let $C_\kappa = C_{\kappa-1}$ and $k_\kappa = k_{\kappa-1}$. Then, clearly, $k_\kappa \colon C_\kappa \to A$ satisfies the condition of (1) above. There also exists $g_{\kappa,\kappa-1} \colon C_{\kappa-1} \to C_\kappa$ with $k_\kappa g_{\kappa,\kappa-1} = k_{\kappa-1}$, which is a monomorphism but not an epimorphism. For each $\beta < \kappa$ let $g_{\kappa,\beta} = g_{\kappa,\kappa-1} g_{\kappa-1,\beta}$. Then, since $g_{\kappa,\kappa-1}$ is not onto, it follows that $g_{\kappa,\beta}$ is not an epimorphism but is a monomorphism, so $\operatorname{im}(g_{\kappa,\beta}) \subsetneq C_\kappa$. By the inductive hypothesis, if $\gamma < \beta < \kappa$, $g_{\kappa,\gamma} = g_{\kappa,\beta} g_{\beta,\gamma}$ and, in addition, $\operatorname{im}(g_{\kappa,\gamma}) \subsetneq \operatorname{im}(g_{\kappa,\beta})$.

Now suppose that κ is a limit ordinal, let $(C_\kappa, g_{\kappa,\beta} \colon C_\beta \to C_\kappa)$ be the directed colimit of the system $(C_\beta, g_{\beta,\gamma})$ and consider the diagram

$$\begin{tikzcd}
C_\gamma \arrow[rr, "g_{\beta,\gamma}"] \arrow[dr, "g_{\kappa,\gamma}"] \arrow[ddr, "k_\gamma"'] & & C_\beta \arrow[dl, "g_{\kappa,\beta}"'] \arrow[ddl, "k_\beta"] \\
& C_\kappa \arrow[d, "k_\kappa"] & \\
& A &
\end{tikzcd}$$

where $k_\kappa \colon C_\kappa \to A$ is the unique S-map that makes the diagram commute. By Lemma 4.8 we then deduce that $k_\kappa \colon C_\kappa \to A$ is an $\mathcal{X}$-precover for A. In addition, we see that, for $\gamma < \beta < \kappa$, $g_{\kappa,\gamma} = g_{\kappa,\beta} g_{\beta,\gamma}$, and that since each $g_{\beta,\gamma}$ is a monomorphism, so is each $g_{\kappa,\beta}$. Suppose that $g_{\kappa,\gamma}$ is onto for some $\gamma < \kappa$. Then, for each $\gamma < \beta < \kappa$, since $g_{\kappa,\beta}$ is a monomorphism, it follows that $g_{\beta,\gamma}$ is also onto, which is a contradiction, so $g_{\kappa,\gamma}$ is not an epimorphism for any $\gamma < \kappa$. It is then clear that

$$\operatorname{im}(g_{\kappa,\gamma}) \subsetneq \operatorname{im}(g_{\kappa,\beta}) \subsetneq C_\kappa.$$

Finally, let $k^*\colon C^* \to A$ be an $\mathcal{X}$-precover and let $h\colon C_\kappa \to C^*$ be such that $k^*h = k_\kappa$. For each $\beta < \kappa$ we then have a commutative diagram

$$\begin{tikzcd} C_\beta \arrow[r, "g_{\kappa,\beta}"] \arrow[drr, "hg_{\kappa,\beta}"'] & C_\kappa \arrow[r, "k_\kappa"] \arrow[dr, "h"] & A \\ & & C^* \arrow[u, "k^*"'] \end{tikzcd}$$

and by assumption $hg_{\kappa,\beta}$ is a monomorphism. Hence, by Lemma 2.4 it follows that h is a monomorphism. In particular, we can deduce that there exists a monomorphism $C_\kappa \to C_0$.

Consequently, we see that for any ordinal κ we have a chain of length κ,

$$\operatorname{im}(g_{\kappa,0}) \subsetneq \cdots \subsetneq \operatorname{im}(g_{\kappa,\beta}) \subsetneq \cdots \subsetneq C_\kappa \subseteq C_0,$$

which is a contradiction. □

It is clear that a necessary condition for an S-act A to have an $\mathcal{X}$-precover is that there exists $X \in \mathcal{X}$ with $\operatorname{Hom}_S(X, A) \neq \emptyset$. This condition is always satisfied in the category of modules over a ring (or indeed any category with a zero object), as every Hom-set is always non-empty, but this is not always the case for S-acts.

Let S be a monoid and let $\mathcal{X}$ be a class of S-acts. We say that $\mathcal{X}$ satisfies the *(weak) solution set condition* if for all S-acts A there exists a set $S_A \subseteq \mathcal{X}$ such that for all (indecomposable) $X \in \mathcal{X}$ and all S-maps $h\colon X \to A$ there exist $Y \in S_A$, $f\colon X \to Y$ and $g\colon Y \to A$ such that $h = gf$.

Theorem 4.12. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that $\dot{\bigcup}_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$. Every S-act then has an $\mathcal{X}$-precover if and only if*

(1) *for every S-act A there exists an X in $\mathcal{X}$ such that $\operatorname{Hom}_S(X, A) \neq \emptyset$,*

(2) *$\mathcal{X}$ satisfies the weak solution set condition.*

Proof. Suppose that $\mathcal{X}$ satisfies the given conditions. Let A be an S-act and let $S_A = \{C_i\colon i \in I\}$ be as given in the weak solution set condition. Note that, by property (1), $S_A \neq \emptyset$. Moreover, we can assume that, for all $Y \in S_A$, $\operatorname{Hom}_S(Y, A) \neq \emptyset$ as $S_A \setminus \{Y \in S_A \mid \operatorname{Hom}_S(Y, A) = \emptyset\}$ will also satisfy the requirements of the solution set condition.

For each $i \in I$ and for each S-map $g\colon C_i \to A$ let $C_{i,g}$ be an isomorphic copy of C_i with the isomorphism $\phi_{i,g}\colon C_{i,g} \to C_i$ (recall that we are assuming that $\mathcal{X}$ is closed under isomorphisms). Let

$$C_A = \dot{\bigcup}_{i\in I,\, g\in \operatorname{Hom}_S(C_i,A)} C_{i,g}.$$

By the hypothesis, $C_A \in \mathcal{X}$ and we can define an S-map $\bar{g}\colon C_A \to A$ by $\bar{g}|_{C_{i,g}} = g\phi_{i,g}$ for each $i \in I$, $g \in \operatorname{Hom}_S(C_i, A)$. We claim that $(C_A, \bar{g})$ is an $\mathcal{X}$-precover for A. Let $X \in \mathcal{X}$ and let $h\colon X \to A$ be an S-map. By the hypothesis, $X = \dot{\bigcup}_{j\in J} X_j$ is a coproduct of indecomposable S-acts with $X_j \in \mathcal{X}$ for each $j \in J$. Furthermore, by the

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hypothesis, there exist $C_{i_j} \in S_A$, $f_j\colon X_j \to C_{i_j}$ and $g_j\colon C_{i_j} \to A$ such that $g_j f_j = h|_{X_j}$. Now $\bar{g}|_{C_{i_j,g_j}} \phi^{-1}_{i_j,g_j} = g_j$, so both triangles and the outer square in the following diagram commute (where the unlabelled arrows are the obvious inclusion maps):

$$
\begin{tikzcd}
C_A \arrow[r, "\bar{g}"] & A \\
C_{i_j,g_j} \arrow[u] & X = \dot{\bigcup} X_j \arrow[u, "h"'] \\
C_{i_j} \arrow[u, "\phi^{-1}_{i_j,g_j}"] \arrow[uur, "g_j"] & X_j \arrow[l, "f_j"] \arrow[u]
\end{tikzcd}
$$

So define $f\colon X \to C_A$ by $f|_{X_j} = \phi^{-1}_{i_j,g_j} f_j$ and note that $\bar{g} f = h$, as required.

Conversely, if A is an S-act with an $\mathcal{X}$-precover C_A, then $\mathrm{Hom}_S(C_A, A) \neq \emptyset$, and on setting $S_A = \{C_A\}$ we see that $\mathcal{X}$ satisfies the (weak) solution set condition. $\square$

Note from the proof of Theorem 4.12 that we can also deduce the following.

Theorem 4.13. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that $X_i \in \mathcal{X}$ for each $i \in I \Rightarrow \dot{\bigcup}_{i \in I} X_i \in \mathcal{X}$. Every S-act then has an $\mathcal{X}$-precover if and only if*

(1) *for every S-act A there exists an X in $\mathcal{X}$ such that $\mathrm{Hom}_S(X, A) \neq \emptyset$,*

(2) *$\mathcal{X}$ satisfies the solution set condition.*

Corollary 4.14. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that*

(1) *$\dot{\bigcup}_{i \in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$,*

(2) *for every S-act A there exists an X in $\mathcal{X}$ such that $\mathrm{Hom}_S(X, A) \neq \emptyset$,*

(3) *there exists a cardinal λ such that for every indecomposable X in $\mathcal{X}$, $|X| < \lambda$.*

Then, every S-act has an $\mathcal{X}$-precover.

Proof. By (3), there exists a λ-skeleton $C = \{C_i\colon i \in I\}$ for the indecomposable S-acts in $\mathcal{X}$. Suppose that A is an S-act and let $S_A = C$. If $X \in \mathcal{X}$ is indecomposable and if $h\colon X \to A$ is an S-map, then there exists an isomorphism $\phi\colon X \to C_i$ for some $C_i \in C$, we have an S-map $h\phi^{-1}\colon C_i \to A$ and clearly $h = h\phi^{-1}\phi$, so $\mathcal{X}$ satisfies the weak solution set condition. $\square$

Let A be an S-act and let ρ be a congruence on A. We say that ρ is *$\mathcal{X}$-pure* if $A/\rho \in \mathcal{X}$. The inspiration for some of the following results comes from [**23**].

Theorem 4.15. *Let S be a monoid, let $\mathcal{X}$ be a class of S-acts and suppose that A is an S-act such that $\psi\colon F \to A$ is an $\mathcal{X}$-precover. Suppose also that the set of $\mathcal{X}$-pure congruences on F is closed under unions of chains. There then exists an $\mathcal{X}$-precover $\phi\colon G \to A$ of A such that there exists no non-identity $\mathcal{X}$-pure congruence $\rho \subset \ker(\phi)$ on G.*

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Proof. First, if there does not exist a non-identity $\mathcal{X}$-pure congruence $\sigma \subseteq \ker(\psi)$ on F, then we let $G = F$ and $\phi = \psi$. Otherwise, by assumption, any chain of $\mathcal{X}$-pure congruences on F contained in $\ker(\psi)$ has an upper bound, so by Zorn's lemma there is a maximum σ, say. Let $G = F/\sigma$ and let $\phi\colon G \to A$ by the natural map that makes

$$\begin{tikzcd} F \arrow[r, "\psi"] \arrow[d, "\sigma^\natural"'] & A \\ G \arrow[ur, "\phi"'] & \end{tikzcd}$$

commute. It is then easy to check that $\phi\colon G \to A$ is an $\mathcal{X}$-precover, as, if $H \in \mathcal{X}$ and $f\colon H \to A$, then there exists $g\colon H \to F$ such that $\psi g = f$. So $\sigma^\natural g\colon H \to G$ and $\phi\sigma^\natural g = \psi g = f$, and $\phi\colon G \to A$ is an $\mathcal{X}$-precover.

Finally, suppose that $\rho \neq 1_G$ is an $\mathcal{X}$-pure congruence on G such that $\rho \subset \ker(\phi)$. By Remark 1.3, σ/ρ is then an $\mathcal{X}$-pure congruence on F containing σ and $\sigma/\rho = \ker(\rho^\natural\sigma^\natural) \subseteq \ker(\psi)$. By the maximality of σ it follows that $\sigma = \sigma/\rho$, so $\rho = 1_G$, a contradiction, as required. $\square$

Following [**2**], we can extend this result as follows.

Proposition 4.16. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts. If A is an S-act such that $\psi\colon F \to A$ is an $\mathcal{X}$-cover, then there exists no non-identity $\mathcal{X}$-pure congruence $\rho \subset \ker\psi$ on F.*

Proof. Let $\rho \subset \ker\psi$ be an $\mathcal{X}$-pure congruence on F. There then exists an induced S-map $\phi\colon F/\rho \to A$ such that $\phi\rho^\natural = \psi$. Since (F, ψ) is a precover, there exists an S-map $\theta\colon F/\rho \to F$ such that $\psi\theta = \phi$:

$$\begin{tikzcd} F \arrow[r, "\psi"] & A \\ & F/\rho \arrow[ul, "\theta"] \arrow[u, "\phi"'] \end{tikzcd}$$

Hence, $\psi\theta\rho^\natural = \phi\rho^\natural = \psi$, so $\theta\rho^\natural$ is an automorphism of F. Hence, $\rho^\natural$ is a monomorphism, so $\rho = 1_A$, as required. $\square$

Let $\mathcal{X}$ be a class of S-acts. We say that $\mathcal{X}$ is *(weakly) congruence pure* if for each cardinal λ there exists a cardinal $\kappa > \lambda$ such that for every (indecomposable) $X \in \mathcal{X}$ with $|X| \geqslant \kappa$ and every congruence ρ on X with $|X/\rho| \leqslant \lambda$ there exists an $\mathcal{X}$-pure congruence $1_X \neq \theta \subseteq \rho$ of X.

Theorem 4.17. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts such that*

(1) $\dot{\bigcup}_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ *for each* $i \in I$,

(2) *for every S-act A there exists an X in $\mathcal{X}$ such that* $\mathrm{Hom}_S(X, A) \neq \emptyset$,

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(3) *for each $X \in \mathcal{X}$ the set of all $\mathcal{X}$-pure congruences on X is closed under unions of chains,*

(4) *$\mathcal{X}$ is weakly congruence pure.*

Then, $\mathcal{X}$ satisfies the weak solution set condition, so every S-act has an $\mathcal{X}$-precover.

Proof. Let A be an S-act, let $\lambda = \max\{|A|, \aleph_0\}$, let κ be as given in the weakly congruence pure condition and let S_A be any κ-skeleton of $\mathcal{X}$ consisting of S-acts of cardinalities less than κ. Suppose that $X \in \mathcal{X}$ is an indecomposable S-act and that $h\colon X \to A$ is an S-map. If $|X| < \kappa$, then let $Y \in S_A$ be an isomorphic copy of X and let $f\colon X \to Y$ be an isomorphism and define $g\colon Y \to A$ by $g = hf^{-1}$ such that $h = gf$.

Suppose now that $|X| \geqslant \kappa$. Then $|X/\ker(h)| = |\mathrm{im}(h)| \leqslant \lambda$, so there exists an $\mathcal{X}$-pure congruence $1_X \neq \theta \subseteq \ker(h)$ on X with $X/\theta \in \mathcal{X}$. In fact, using a combination of Zorn's lemma and the hypothesis that the set of $\mathcal{X}$-pure congruences on X is closed under unions of chains, we can assume that θ is maximal with respect to this property. Now let $\bar{h}\colon X/\theta \to A$ be the unique map such that

$$\begin{tikzcd} X \arrow[r, "\theta^\natural"] \arrow[dr, "h"'] & X/\theta \arrow[d, "\bar{h}"] \\ & A \end{tikzcd}$$

commutes. Note that, since $\mathrm{im}(\bar{h}) = \mathrm{im}(h)$,

$$|(X/\theta)/\ker(\bar{h})| = |X/\ker(h)| \leqslant \lambda.$$

Now suppose, by way of contradiction, that $1_{X/\theta} \neq \rho \subseteq \ker(\bar{h})$ is an $\mathcal{X}$-pure congruence on X/θ such that $(X/\theta)/\rho \in \mathcal{X}$. Then by Remark 1.3 and since $X \in \mathcal{X}$ it follows that θ/ρ is an $\mathcal{X}$-pure congruence on X containing θ, and since $\rho \subseteq \ker(\bar{h})$ it easily follows that $\theta/\rho \subseteq \ker(h)$. Hence, by the maximality of θ we deduce that $\theta/\rho = \theta$, so $\rho = 1_{X/\theta}$. Therefore, it follows that X/θ does not contain a non-identity $\mathcal{X}$-pure congruence contained in $\ker(\bar{h})$ and, since by Lemma 1.2 X/θ is indecomposable and since $\mathcal{X}$ is weakly congruence pure, we deduce that $|X/\theta| < \kappa$. Consequently, it follows that there exist $Y \in S_A$ and an isomorphism $\bar{f}\colon X/\theta \to Y$; so define $f\colon X \to Y$ by $f = \bar{f}\theta^\natural$ and $g\colon Y \to A$ by $g = \bar{h}\bar{f}^{-1}$ such that $gf = h$.

Hence, $\mathcal{X}$ satisfies the weak solution set condition and the result follows from Theorem 4.12. $\square$

A similar condition to this is considered in [**3**, **4**] and forms the basis of one of the proofs of the flat cover conjecture.

5. Strongly flat and condition (P) covers

In this section we apply some of the previous results to the specific classes $\mathcal{X} = \mathcal{SF}$ and $\mathcal{X} = \mathcal{CP}$. In particular, note from Lemma 1.1 that $\dot{\bigcup}_{i\in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$

holds for both $\mathcal{X} = \mathcal{SF}$ and $\mathcal{X} = \mathcal{CP}$. Also, since S is strongly flat (and hence satisfies condition (P)), given any S-act A there exists an X in $\mathcal{X}$ such that $\mathrm{Hom}_S(X, A) \neq \emptyset$.

Let A be an S-act and let ρ be a congruence on A. Recall that we say that ρ is $\mathcal{X}$*-pure* if $A/\rho \in \mathcal{X}$. So, by Propositions 3.11 and 3.12, Corollary 3.13 and [**1**, Corollary 4.1.3 and Theorem 4.1.4] we deduce the following.

Corollary 5.1. *Let S be a monoid, let X be an S-act and let ρ be a congruence on X.*

(1) *If $X \in \mathcal{E}$, then ρ is $\mathcal{E}$-pure if and only if it is 1-pure.*

(2) *If $X \in \mathcal{CP}$, then ρ is $\mathcal{CP}$-pure if it is 2-pure.*

(3) *If $X \in \mathcal{SF}$, then ρ is $\mathcal{SF}$-pure if and only if it is pure if and only if it is 2-pure.*

(4) *If $X \in \mathcal{P}$, then ρ is $\mathcal{P}$-pure if and only if $\rho^\natural$ splits.*

From Lemma 2.5 and Propositions 2.8 and 2.9 we can immediately deduce the following important result.

Theorem 5.2. *Let S be a monoid and let $\mathcal{X}$ be a class of S-acts closed under directed colimits. Then, $\mathcal{X}$ is closed under chains of $\mathcal{X}$-pure congruences. In particular, this is true for the classes $\mathcal{X} = \mathcal{SF}$ and $\mathcal{X} = \mathcal{CP}$.*

Recall that an act X is said to be *locally cyclic* if for all $x, y \in X$ there exists $z \in X$, $s, t \in S$ with $x = zs$, $y = zt$. By [**19**, Theorem 3.7] the indecomposable acts in $\mathcal{CP}$ and $\mathcal{SF}$ are the locally cyclic acts.

Lemma 5.3. *Let S be a monoid, suppose that X satisfies condition (P) and suppose that we have the system of equations*

$$\begin{aligned} x_1 s_1 &= x_2 t_2, \\ x_2 s_2 &= x_3 t_3, \\ &\vdots \\ x_{n-1} s_{n-1} &= x_n t_n, \end{aligned}$$

where $x_i \in X$, $s_i, t_i \in S$. There then exist $y \in X$, $u_i \in S$ such that, for $1 \leqslant i \leqslant n-1$, we have that $x_i = yu_i$ and $u_i s_i = u_{i+1} t_{i+1}$.

Proof. We prove this by induction on n. Suppose then that $n = 2$. Our system is then

$$x_1 s_1 = x_2 t_2,$$

and condition (P) means there exist $y \in X$, $u_1, u_2 \in S$ with $x_1 = yu_1$, $x_2 = yu_2$ and $u_1 s_1 = u_2 t_2$, as required.

Suppose then that the result is true for $i \leqslant n$ and suppose that we have the system of equations

$$\begin{aligned} x_1 s_1 &= x_2 t_2, \\ x_2 s_2 &= x_3 t_3, \\ &\vdots \\ x_{n-1} s_{n-1} &= x_n t_n, \\ x_n s_n &= x_{n+1} t_{n+1}. \end{aligned}$$

By induction there exist $y \in X$, $u_i \in S$ such that, for $1 \leqslant i \leqslant n$, we have that $x_i = yu_i$ and, for $1 \leqslant i \leqslant n-1$, that $u_i s_i = u_{i+1} t_{i+1}$. In addition, condition (P) means that there exist $y' \in X$, $u'_n, v'_n \in S$ with $x_n = y'u'_n$, $x_{n+1} = y'v'_n$ and $u'_n s_n = v'_n t_{n+1}$. But then $x_n = yu_n = y'u'_n$, so there exist $z \in X$, $p, q \in S$ with $y = zp$, $y' = zq$ and $pu_n = qu'_n$. Hence, for $1 \leqslant i \leqslant n$ it follows that $x_i = z(pu_i)$ and, for $1 \leqslant i \leqslant n-1$, $(pu_i)s_i = (pu_{i+1})t_{i+1}$, while $x_{n+1} = z(qv'_n)$ and $(pu_n)s_n = qu'_n s_n = (qv'_n)t_{n+1}$, as required. $\square$

The following was suggested to us by Philip Bridge (personal communication, 2010). For a version involving more general categories see [**5**].

Proposition 5.4 (Bridge [5, Theorem 5.21]). *Let S be a monoid and suppose that S satisfies the following property:*

$$\forall s \in S,\ \exists k \in \mathbb{N} \text{ such that } \forall m \in S,\ |\{p \in S \mid ps = m\}| \leqslant k.$$

Every S-act then has an $\mathcal{SF}$-cover and a $\mathcal{CP}$-cover.

Proof. We show that every indecomposable S-act that satisfies condition (P) (and hence every strongly flat indecomposable S-act) has a bound on its cardinality. Let X be an indecomposable S-act that satisfies condition (P). It is then locally cyclic, so for all $x, y \in X$ there exist $z \in X$, $s, t \in S$ such that $x = zs$, $y = zt$:

$$\begin{tikzcd} x & & y \\ & z \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

We now fix $x \in X$ and consider how many possible $y \in X$ could satisfy these equations. Firstly, we take a fixed $s \in S$ and consider how many possible $z \in X$ could satisfy $x = zs$. By the hypothesis, there exists $k \in \mathbb{N}$ such that, for any $m \in S$, $|\{p \in S \colon ps = m\}| \leqslant k$. We suppose that there exist at least $k+1$ distinct z such that $x = zs$. That is, $x = z_1 s =$

$z_2s = \cdots = z_{k+1}s$. By Lemma 5.3 there then exist $w \in X$, $p_1, \dots, p_{k+1} \in S$ such that $p_1s = \cdots = p_{k+1}s$ and $z_i = wp_i$ for each $i \in \{1, \dots, k+1\}$:

$$\begin{tikzcd}
 & & x & & \\
z_1 \arrow[urr, "s"] & z_2 \arrow[ur, "s"] & \cdots & z_k \arrow[ul, "s"'] & z_{k+1} \arrow[ull, "s"'] \\
 & & w \arrow[ull, "p_1"] \arrow[ul, "p_2"] \arrow[ur, "p_k"'] \arrow[urr, "p_{k+1}"'] & &
\end{tikzcd}$$

However, by the hypothesis, this means at least two p_i are equal and, hence, at least two z_i are equal, which is a contradiction. So, given some fixed $s \in S$, there exist at most k possible z such that $x = zs$. Hence, there exist no more than $\aleph_0|S|$ possible $z \in X$, $s \in S$ such that $x = zs$. Similarly, given a fixed $z \in X$, there exist at most $|S|$ possible $t \in S$ such that $zt = y$, and, hence, there exist no more than $\aleph_0|S|^2$ possible elements in X. So the result follows by Corollary 4.14. □

A finitely generated monoid that satisfies this property is said to have *finite geometric type* (see [**21**]). Let B be the bicyclic monoid and let $(s, t) \in B$. Suppose that $(m, n) \in B$ is fixed and suppose that $(p, q) \in B$ is such that $(p, q)(s, t) = (m, n)$. We count the number of solutions to this equation. Recall that

$$(p, q)(s, t) = (p - q + \max(q, s), t - s + \max(q, s)) = (m, n).$$

If $q \geqslant s$, then $(p, q) = (m, n - (t - s))$ and there exists at most one solution to the equation. Otherwise, $(p, q) = (m - s + q, q)$, where q ranges between 0 and $s - 1$. There exist, therefore, at most $s + 1$ possible values of (p, q) that satisfy the equation, so B has finite geometric type. Hence, we deduce the following.

Proposition 5.5. *Let S be the bicyclic monoid. All S-acts then have an $\mathcal{SF}$-cover and a $\mathcal{CP}$-cover.*

On letting $k = 1$ in Proposition 5.4, we can deduce the following corollary.

Corollary 5.6. *Let S be a right cancellative monoid. Every right S-act then has an $\mathcal{SF}$-cover and a $\mathcal{CP}$-cover.*

It also now follows that not every $\mathcal{SF}$-cover is a strongly flat cover, as it was shown in [**13**, Remark 3.6] that $(\mathbb{N}, \cdot)$ is a monoid in which the 1-element act Θ does not have a strongly flat cover. It is, however, obviously right cancellative.

Recall [**9**] that a monoid S is said to satisfy condition (A) if all right S-acts satisfy the ascending chain condition for cyclic subacts. This is equivalent to saying that every locally cyclic right S-act is cyclic.

Proposition 5.7. *Let S be a monoid that satisfies condition (A). Every right S-act then has an $\mathcal{SF}$-cover and a $\mathcal{CP}$-cover.*

Proof. By [**19**, Theorem 3.7] the indecomposable acts in $\mathcal{CP}$ and $\mathcal{SF}$ are the locally cyclic acts, but since S satisfies condition (A) all the locally cyclic acts are cyclic. If S/ρ is cyclic, then clearly $|S/\rho| \leqslant |S|$ and the result follows from Corollary 4.14. □

It is well known that not every monoid that satisfies condition (A) is perfect, so we can then deduce that $\mathcal{P}$-covers are, in general, different from $\mathcal{SF}$-covers and $\mathcal{CP}$-covers. Also, given that indecomposable projective acts are cyclic, the indecomposable S-acts are bounded in size, so by Corollary 4.14 we can deduce the following.

Proposition 5.8. *Let S be a monoid. Every S-act has a $\mathcal{P}$-precover.*

Lemma 5.9. *Let S be a monoid. If A is a right S-act and if $k\colon C \to A$ is an SF-cover with C projective, then C is a $\mathcal{P}$-cover.*

Proof. If P is projective and if $g\colon P \to A$ is an S-map, then P is strongly flat, so there exists $h\colon P \to C$ with $kh = g$, so P is a projective cover. □

Since right perfect monoids satisfy condition (A), we have the following.

Corollary 5.10. *Let S be a right perfect monoid. Every right S-act then has an $\mathcal{SF}$-cover.*

In addition, since S is right perfect if and only if all strongly flat S-acts are projective, we have the following.

Corollary 5.11. *S is right perfect if and only if every right S-act has a projective $\mathcal{SF}$-cover.*

From [**10**, Examples 2.9 and 2.10] we can deduce the following.

Theorem 5.12. *The following classes of monoids satisfy condition (A), so every right S-act over such a monoid has an $\mathcal{SF}$-cover and a $\mathcal{CP}$-cover:*

(1) *finite monoids;*

(2) *rectangular bands with a 1 adjoined;*

(3) *right groups with a 1 adjoined;*

(4) *right simple semigroups with a 1 adjoined;*

(5) $(\mathbb{N}, \max)$.

The previous results rely on us showing that the indecomposable strongly flat S-acts are bounded in size, and hence that the class of indecomposable strongly flat S-acts forms a set. We show that there exists a monoid S with a proper class of indecomposable strongly flat acts by constructing an indecomposable strongly flat act of arbitrary cardinality.

Example 5.13. Let $S = \mathcal{T}(\mathbb{N})$ be the full transformation monoid over the set of natural numbers and let $\phi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection of sets. For convenience, we write maps on the right. Given any set $X \neq \emptyset$, let $A_X = \{f\colon X \to \mathbb{N}\}$ be the set of all maps from X to $\mathbb{N}$. We can make A_X into an S-act by composition of maps: for $f \in A_X$, $s \in S$ define $fs \in A_X$ by $x(fs) = (xf)s$. Given any $f, g \in A_X$, let $h \in A_X$ be defined as $xh = (xf, xg)\phi$. Then define $u, v \in S$ to be $u = \phi^{-1}\pi_1$ and $v = \phi^{-1}\pi_2$, where $(x, y)\pi_1 = x$ and $(x, y)\pi_2 = y$. Therefore, $f = hu$, $g = hv$ and A_X is locally cyclic (hence indecomposable) and has cardinality at least $|X|$. We now show that A_X is strongly flat. Let $f, g \in A_X$, $s, t \in S$ such that $fs = gt$. Define $h \in A_X$ as before, pick some $x \in X$ and define $u_x, v_x \in S$ by

$$nu_x = \begin{cases} nu & \text{if } n \in \operatorname{im}(h),\\ xf & \text{otherwise,}\end{cases}$$
$$nv_x = \begin{cases} nv & \text{if } n \in \operatorname{im}(h),\\ xg & \text{otherwise.}\end{cases}$$

Then $f = hu_x$, $g = hv_x$ and $u_x s = v_x t$, so A_X satisfies condition (P). Let $f \in A_X$, $s, t \in S$ such that $fs = ft$. Pick some $x \in X$ and define $w \in S$,

$$nw = \begin{cases} n & \text{if } n \in \operatorname{im}(f),\\ xf & \text{otherwise.}\end{cases}$$

Then $f = fw$ and $ws = wt$, so A_X satisfies condition (E) and is strongly flat.

Let T be a monoid and let S be a submonoid of T. If X is an S-act that satisfies condition (P), then $X \otimes_S T$ is a T-act and $X \to X \otimes_S T$, given by $x \mapsto x \otimes 1$, is an S-monomorphism (since X is flat). Moreover, if X is locally cyclic, then so is $X \otimes_S T$, since if $x_1 \otimes t_1$, $x_2 \otimes t_2 \in X \otimes_S T$, then there exist $z \in X$, $u_1, u_2 \in S$ with $x_1 = zu_1$, $x_2 = zu_2$. So $x_1 \otimes t_1 = z \otimes u_1t_1 = (z \otimes 1)u_1t_1$ and, similarly, $x_2 \otimes t_2 = (z \otimes 1)u_2t_2$.

Finally, we can also deduce that $X \otimes_S T$ satisfies condition (P) as, if $(x \otimes t_1)r_1 = (x' \otimes t_2)r_2$, then there exist $x_2, \dots, x_n \in X$, $u_2, \dots, u_n, v_2 \dots, v_n \in S$, $p_2, \dots p_{n-1} \in T$ such that

$$\begin{aligned} x &= x_2u_2, & u_2t_1r_1 &= v_2p_2,\\ x_2v_2 &= x_3u_3, & u_3p_2 &= v_3p_3,\\ &\;\vdots & &\;\vdots\\ x_{n-1}v_{n-1} &= x_nu_n, & u_np_{n-1} &= v_nt_2r_2,\\ x_nv_n &= x'. \end{aligned}$$

So, by Lemma 5.3, there exist $y \in X$ and $w_i \in S$ such that $x_i = yw_i$ and $w_i v_i = w_{i+1}u_{i+1}$ ($x = yw_1$, $w_1 = w_2u_2$ and $x' = yw_{n+1}$, $w_n v_n = w_{n+1}$), so we have a scheme of the form

$$\begin{aligned} x = yw_1 &= yw_2u_2, & u_2t_1r_1 &= v_2p_2,\\ yw_2v_2 &= yw_3u_3, & u_3p_2 &= v_3p_3,\\ &\vdots & &\vdots\\ yw_{n-1}v_{n-1} &= yw_nu_n, & u_np_{n-1} &= v_nt_2r_2,\\ yw_nv_n &= yw_{n+1} = x'. \end{aligned}$$

Hence, $x \otimes t_1 = (y \otimes 1)w_1t_1$, $x' \otimes t_2 = (y \otimes 1)w_{n+1}t_2$ and

$$\begin{aligned} (w_1t_1)r_1 &= w_2u_2t_1r_1 = w_2v_2p_2 = w_3u_3p_2 = w_3v_3p_3 = \cdots\\ &= w_{n-1}v_{n-1}p_{n-1} = w_nu_np_{n-1} = w_nv_nt_2r_2 = (w_{n+1}t_2)r_2. \end{aligned}$$

In a similar way, if X is strongly flat, then whenever $(x \otimes t)r_1 = (x \otimes t)r_2$ in $X \otimes_S T$ we can proceed as above and deduce the existence of a scheme

$$\begin{aligned} x = yw_1 &= yw_2u_2, & u_2tr_1 &= v_2p_2,\\ yw_2v_2 &= yw_3u_3, & u_3p_2 &= v_3p_3,\\ &\vdots & &\vdots\\ yw_{n-1}v_{n-1} &= x_nu_n, & u_np_{n-1} &= v_ntr_2,\\ yw_nv_n &= yw_{n+1} = x. \end{aligned}$$

Now, since $yw_1 = yw_{n+1}$ and since X satisfies condition (E), there exist $z \in X$, $u \in S$ with $y = zu$ and $uw_1 = uw_{n+1}$, so $x \otimes t = (z \otimes 1)uw_1t$ and, as before, $(uw_1t)r_1 = \cdots = uw_{n+1}tr_2 = (uw_1t)r_2$. So X also satisfies condition (E).

Let T be a monoid that satisfies condition (A), let S be a left pure submonoid of T (in the sense that the inclusion $S \to T$ is a left pure S-monomorphism) and let X be a locally cyclic right S-act. Then, from above, we see that $X \otimes_S T$ is a locally cyclic right T-act, so is cyclic. Hence, there exist $x_0 \in X$, $t_0 \in T$ such that $X \otimes_S T \cong (x_0 \otimes t_0)T$. We show that X is also cyclic. First, we say that a left S-monomorphism $f\colon C \to D$ is *stable* if, for all right S-monomorphisms $\lambda\colon A \to B$,

$$\operatorname{im}(1_B \otimes f) \cap \operatorname{im}(\lambda \otimes 1_D) = \operatorname{im}(\lambda \otimes f).$$

It was shown in [**18**, Theorem 3.1] that left pure monomorphisms are stable. In particular, the above remarks hold when $\lambda\colon x_0S \to X$, $f\colon S \to T$ are the inclusions. Consequently, if $x \in X$, then $x \otimes 1 = x_0 \otimes t$ in $X \otimes_S T$ for some $t \in T$. Hence, there exists $s \in S$ such that $x \otimes 1 = x_0s \otimes 1$ in $X \otimes_S T$, and since $X \to X \otimes_S T$ is a monomorphism, by left purity of $S \to T$, $x = x_0s$, as required. Hence, we can deduce the following.

Proposition 5.14. *The class of monoids that satisfy condition (A) is closed under the taking of left pure submonoids.*

https://doi.org/10.1017/S0013091513000618 Published online by Cambridge University Press

We can also deduce the following theorem.

Theorem 5.15. *Let T be a monoid and let $\mathcal{X}_T = \mathcal{SF}_T$ or $\mathcal{X}_T = \mathcal{CP}_T$. Let $\mathcal{M}$ be the class of monoids such that, for all $T \in \mathcal{M}$, there exists a cardinal κ with $|X| < \kappa$ for all locally cyclic right T-acts $X \in \mathcal{X}_T$. Then $\mathcal{M}$ is closed under submonoids. In addition, for any monoid $S \in \mathcal{M}$, every right S-act has an $\mathcal{SF}$-cover and a $\mathcal{CP}$-cover.*

Proof. Let $T \in \mathcal{M}$ and let S be a submonoid of T. If $X \in \mathcal{X}_S$ is a locally cyclic right S-act, then $X \otimes_S T \in \mathcal{X}_T$ is a locally cyclic right T-act. By assumption, there exists a cardinal κ such that $|X \otimes_S T| < \kappa$, so, since $X \to X \otimes_S T$ is a monomorphism, $|X| < \kappa$, and hence $S \in \mathcal{M}$. □

Corollary 5.16. *Let S be any submonoid of the bicyclic monoid. Every S-act then has an $\mathcal{SF}$-cover and a $\mathcal{CP}$-cover.*

Many of the results in this paper involve monoids belonging to $\mathcal{M}$. However, Example 5.13 demonstrates that $\mathcal{M}$ is not the class of all monoids. One of the proofs of the flat cover conjecture in [**4**] involved showing that every module over a unitary ring satisfied a condition very similar to that given in Theorem 4.17. We feel that a similar situation should hold in the category of S-acts.

We hope to consider the classes of torsion-free, divisible, injective and free acts in a subsequent paper.

Acknowledgements. Parts of this work will be included in a PhD thesis to be submitted to the University of Southampton by A.B., who thanks his supervisor, J.H.R., for his help and advice.

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