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A NOTE ON THE RATE OF CONVERGENCE OF HERMITE-FEJÉR INTERPOLATION POLYNOMIALS*

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The Hermite-Fejér interpolation polynomial $H_n[f]$ of degree $\leq 2n-1$ is defined by

(1) $$H_n[f; x] = \sum_{k=1}^{n} f(x_{kn})(1-xx_{kn})\left[\frac{T_n(x)}{n(x-x_{kn})}\right]^2$$

where

(2) $$x_{kn} = \cos(k-\tfrac{1}{2})\frac{\pi}{n}, \qquad k = 1, 2, \ldots, n$$

are the zeroes of Chebyshev polynomial of first kind $T_n(x)=\cos n(\arccos x)$. According to L. Fejér [2] the polynomials $H_n[f]$, $n=1, 2, \ldots$ converge uniformly to a continuous function $f(x)$ defined on $[-1, 1]$. As to the rapidity of convergence E. Moldvan [4] (also O. Shisha and B. Mond [5]) has given the estimate

(3) $$\|H_n[f]-f\| \leq C\omega_f\left(\frac{\log n}{n}\right), \qquad (n \geq 4).$$

Here $\|f\|=\sup_{-1\leq x\leq 1} |f(x)|$ and ω_f is the modulus of continuity of $f(x)$.

Recently R. Bojanic [1] has given the estimate of the rate of convergence of the sequence $H_n[f]$, $n=1, 2, \ldots$ in terms of the arithmetic means of the sequence $\{\omega_f(1/n)\}$. Let Ω be an increasing subadditive and continuous function on $x(x\geq 0)$ with $\Omega(0)=0$ and let $C_M(\Omega)$ be the class of continuous functions on $[-1, 1]$ defined by

$$f \in C_M(\Omega) \Leftrightarrow \omega_f(h) \leq M\Omega(h).$$

THEOREM. (R. Bojanic). *There exist constants c and C* $(0<c<C<\infty)$ such that for $n\geq 2$ we have

(4) $$\frac{cM}{n}\sum_{k=2}^{n}\Omega\left(\frac{1}{k}\right) \leq \sup_{f\in C_M(\Omega)} \|H_n[f]-f\| \leq \frac{CM}{n}\sum_{k=1}^{n}\Omega\left(\frac{1}{k}\right).$$

In this note we show that a better local approximation can be obtained at the end points of the interval, namely we prove the following:

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THEOREM. *There exists a constant C^* such that for $n \geq 2$ and for $-1 \leq x \leq 1$ we have*

$$|H_n[f; x]-f(x)| \leq \frac{C^*M}{n}\sum_{k=1}^{n}\Omega\left(\frac{(1-x^2)^{1/2}}{k}+\frac{1}{k^2}\right). \tag{5}$$

2. For the proof of this result we shall need the following Lemma which is a modified form of a Lemma of R. Bojanic [1]:

LEMMA. *For $m \geq 2$ we have*

$$\frac{\pi}{m}\int_{\pi/m}^{\pi}\frac{\Omega(t\sin\theta)}{t^2}\,dt \leq \sum_{\gamma=1}^{m-1}\frac{1}{\gamma^2}\,\Omega\left(\frac{\gamma+1}{m}\,\pi\sin\theta\right) \leq \frac{8\pi}{m}\int_{\pi/m}^{\pi}\frac{\Omega(t\sin\theta)}{t^2}\,dt, \tag{6}$$

and

$$\frac{\pi}{m}\int_{\pi/m}^{\pi}\frac{\Omega(t^2)}{t^2}\,dt \leq \sum_{\gamma=1}^{m-1}\frac{1}{\gamma^2}\,\Omega\left(\frac{(\gamma+1)\pi^2}{m^2}\right) \leq \frac{8\pi}{m}\int_{\pi/m}^{\pi}\frac{\Omega(t^2)}{t^2}\,dt. \tag{7}$$

The proof depends on the inequalities

$$\frac{\pi}{m}\int_{\gamma\pi/m}^{(\gamma+1)\pi/m}\frac{\Omega(t\sin\theta)}{t^2}\,dt \leq \frac{1}{\gamma^2}\,\Omega\left(\frac{\gamma+1}{m}\,\pi\sin\theta\right) \leq \frac{8\pi}{m}\int_{\gamma\pi/m}^{(\gamma+1)/m}\frac{\Omega(t\sin\theta)}{t^2}\,dt \tag{8}$$

on following the same pattern as in [1].

3. **Proof of the theorem.**

We shall require the following estimate due to Vértesi [6]:

$$|f(x)-H_n[f; x]| \leq C_1\sum_{\gamma=1}^{n}\frac{1}{\gamma^2}\left[\Omega\left(\frac{\gamma+1}{n+1}\,\pi\sin\theta\right)+\Omega\left(\left(\frac{\gamma+1}{n+1}\,\pi\right)^2\right)\right] \tag{9}$$

where $x=\cos\theta$.

On using the Lemma for $m=n+1$, we get from (9)

$$|f(x)-H_n[f; x]| \leq \frac{8\pi C_1}{n+1}\left[\int_{\pi/(n+1)}^{\pi}\frac{\Omega(t\sin\theta)}{t^2}\,dt+\int_{\pi/(n+1)}^{\pi}\frac{\Omega(t^2)}{t^2}\,dt\right]. \tag{10}$$

Now

$$\begin{aligned}\int_{\pi/(n+1)}^{\pi}\frac{\Omega(t\sin\theta)}{t^2}\,dt &= \int_{1}^{n+1}\Omega\left(\frac{\pi\sin\theta}{t}\right)dt\\ &\leq C_2\int_{1}^{n}\Omega\left(\frac{\sin\theta}{t}\right)dt\\ &\leq C_3\sum_{k=1}^{n}\Omega\left(\frac{\sin\theta}{k}\right).\end{aligned} \tag{11}$$

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Similarly we can show that

$$(12)\qquad \int_{\pi/(n+1)}^{\pi} \frac{\Omega(t^2)}{t^2}\,dt \le C_4 \sum_{k=1}^{n} \Omega\left(\frac{1}{k^2}\right).$$

Thus (10), (11) and (12) complete the proof of our theorem.

Since the modulus of continuity ω_f of any continuous function f on $[-1, 1]$ has the same properties as Ω, it follows from the theorem that for any continuous function f on $[-1, 1]$ we have for $-1 \le x \le 1$

$$(13)\qquad |f(x)-H_n[f; x]| \le \frac{C}{n} \sum_{k=1}^{n} \omega_f\left[\frac{(1-x^2)^{1/2}}{k}+\frac{1}{k^2}\right].$$

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