# GENERALIZED INTEGRALS WITH RESPECT TO FUNCTIONS OF BOUNDED VARIATION

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**Introduction.** A considerable literature has grown up around the analysis of the structure of a function in terms of its derivative, and the structure of functions $F(x)$ which are integrals of various kinds. Some of this relates to derivatives and integrals of $F(x)$ with respect to functions of bounded variation $\omega(x)$ **(1–6)** or, in the case of a paper by Ward **(4)**, with respect to a function of generalized bounded variation in the restricted sense. While functions of bounded variation have at most a denumerable set of discontinuities, yet this set can be everywhere dense, and in the studies to which we refer a consideration of these discontinuities enters, sometimes in a complicated way. In the present paper results are obtained without reference to the values of $F(x)$ or $\omega(x)$ at the points of discontinuity of $\omega$. The results lead to a descriptive definition of a Lebesgue-Stieltjes integral of a function with respect to $\omega$ and a descriptive definition of a generalized integral with respect to $\omega$. The latter involves functions $F(x)$ which are generalized absolutely continuous relatively to $\omega$.

Because $\omega(x)$ can be written as the difference of two non-decreasing functions there is no loss of generality in taking $\omega(x)$ to be non-decreasing. We shall consider a function $\omega(x)$ on a closed interval $[a, b]$ with the understanding that $w(x) = \omega(a)$, $x < a$, $\omega(x) = \omega(b)$, $x > b$. With $\omega(x)$ given we shall denote by $\mathfrak{U}$ a class of functions $F(x)$ defined at the points of continuity of $\omega(x)$ on $[a, b]$. Furthermore, if $\mathfrak{C}$ is the set over which $\omega$ is continuous, then $F(x)$ is continuous over $\mathfrak{C}$ at points of $\mathfrak{C}$, and if $x_0$ is a point of discontinuity of $\omega$ then $F(x)$ tends to a limit as $x$ tends to $x_0+$ and to $x_0-$, $x \in \mathfrak{C}$. These limits will be denoted by $F(x_0+)$, $F(x_0-)$. Also $F(x) = F(a+)$ for $x < a$ and $F(x) = F(b-)$ for $b > a$. $F(x)$ may, or may not, be defined at points of discontinuity of $\omega$.

**1. The $\omega$-measure of a set $E$ on $[a, b]$.** Let $(a', b')$ be an open interval on $[a, b]$. The $\omega$-measure of $(a', b')$ is $\omega(b'-) - \omega(a'+)$, and is denoted by $|(a', b')|_\omega$. Let $E$ be any set on $[a, b]$. Let $\alpha_1, \alpha_2, \ldots$ be a set of non-overlapping open intervals containing $E$. The outer $\omega$-measure of the set $E$ is the infimum of $\Sigma|\alpha_i|_\omega$ for all such sets of open intervals. This outer measure is denoted by $|E|_\omega^0$. Let $\tilde{E}$ be the complement of the set $E$. If for $\epsilon > 0$ there exists a set of non-overlapping open intervals $\alpha = \alpha_1 + \alpha_2 + \ldots$, $\alpha \supset E$ and a similar set $\beta \supset \tilde{E}$ for which $|\alpha\beta|_\omega^0 < \epsilon$ then the set $E$ is said to be $\omega$-measurable. The $\omega$-measure of $E$, denoted by $|E|_\omega$, is equal to $|E|_\omega^0$.

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LEMMA 1. *If $\alpha = \alpha_1 + \alpha_2 + \ldots$ is a set of non-overlapping open intervals $\alpha_1, \alpha_2, \ldots$ then $\alpha$ is $\omega$-measurable.*

Because $\omega(x)$ is $BV$ it follows that $\Sigma|\alpha_i|_\omega$ converges. Hence for $\epsilon > 0$ there exists an integer $n_0$ such that for $n > n_0$,

$$\sum_{n+1}^{\infty} |\alpha_i|_\omega < \epsilon.$$

Fix $n > n_0$. Let $\alpha_i = (a_i, b_i)$, $i \leqslant n$. Take $a_i'$, $b_i'$ with $a_i < a_i' < b_i' < b_i$ and such that

$$\sum_{i=1}^{n} \{|(a_i, a_i')|_\omega + |(b_i', b_i)|_\omega\} < \epsilon.$$

Let $\beta$ be the open set complementary to the finite set of closed intervals. $[a_1', b_i'], \ldots, [a_n', b_n']$. Then $\beta \supset \tilde{E}$ and

$$|\alpha\beta|_\omega^0 = \sum_{n+1}^{\infty} |\alpha_i|_\omega + \sum_{i=1}^{n} \{|(a_i, a_i')|_\omega + |(b_i', b_i)|_\omega\} < 2\epsilon.$$

Because $\epsilon$ is arbitrary it follows that $\alpha$ satisfies the definition of measurability.

LEMMA 2. *Let $E$ be any set on $[a, b]$. Let each point $x \in E$ be the left hand end point of a set of intervals $(x, x + h_i)$ for which $h_i \to 0$ as $i \to \infty$. Let $\mathfrak{F}$ denote the set of intervals thus associated with the set $E$. Let $\epsilon > 0$ be given.*

There exists a finite non-overlapping set $\Delta_i$ of the intervals of $\mathfrak{F}$ for which

$$\sum_{i=1}^{n} |\Delta_i|_\omega < |E|_\omega^0 + \epsilon, \quad \sum_{i=1}^{n} |\Delta_i E|_\omega^0 > |E|_\omega^0 - \epsilon.$$

Let $\eta > 0$ be given. Put $E$ in a set of open intervals $\alpha_1, \alpha_2, \ldots$ in such a way that $\Sigma|\alpha_i|_\omega < |E|_\omega{}^0 + \eta$. Now let $\alpha$ be a finite set of the intervals $\alpha_1, \alpha_2, \ldots, \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ where $n$ is sufficiently great to insure that $|\alpha E|_\omega{}^0 > |E|_\omega{}^0 - \eta$. Let $\alpha_i = (a_i, b_i)$. Let $\eta'$ be so fixed that if $b_i'$ is on $(a_i, b_i)$, $a_i < b_i' < b_i$ and $b_i - b_i' < \eta'$ then

$$\sum_{i=1}^{n} |E(b_i' b_i)|_\omega^0 < \eta. \tag{1}$$

Let $E_\delta$ be the points of $E\alpha$ which are such that if $x$ is in $E_\delta$ then there is an interval $(x, x + h)$ of $\mathfrak{F}$ with $h > \delta$ and $x, x + h$ on the same interval of the set $\alpha$. If $\delta_2 < \delta_1$ then

$$E_{\delta_2} \supset E_{\delta_1}.$$

Hence $|E_\delta|_\omega{}^0 \to |E\alpha|_\omega{}^0$ as $\delta \to 0$. Fix $\delta < \eta'$ and sufficiently near zero to insure that

$$|E_\delta|_\omega^0 > |E\alpha|_\omega^0 - \eta > |E|_\omega^0 - 2\eta. \tag{2}$$

Now consider the interval $\alpha_1 = (a_1, b_1)$. There is a point $x_1' \geqslant a_1$ which is (a) the first point of $E_\delta$ to the right of $a_1$ or (b) the infimum of points of $E_\delta$ on

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$(a_1, b_1)$. For case (a) let $x_1' = x_1$ and let $(x_1, x_1 + h_1)$ be an interval of $\mathfrak{F}$ on $(a_1, b_1)$ with $h_1 > \delta$. For case (b) take $x_1$ a point of $E_\delta$ and such that

$$(3) \qquad |E(x_1', x_1)|_\omega^0 < \epsilon_1$$

where $\epsilon_1$ is the first member of a sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ for which $\Sigma\epsilon_i < \eta/n$, and take $(x_1, x_1 + h_1)$ an interval of $\mathfrak{F}$ with $h_1 > \delta$.

In the foregoing replace $a_1$ by $x_1 + h_1$ and arrive at $x_2' \geqslant x_1 + h_1$ satisfying (a) or (b) and an interval $(x_2, x_2 + h_2)$ of $\mathfrak{F}$ on $(a_1, b_1)$ corresponding to $(x_1, x_1 + h_1)$ with $h_2 > \delta$. If case (b) holds choose $x_2$ so that

$$(4) \qquad |E(x_2', x_2)|_\omega^0 < \epsilon_2.$$

Since each $h_1 > \delta$, and $\delta < \eta'$ this procedure can be continued to get a finite non-overlapping set of intervals $[x_1, x_1 + h_1]$, $[x_2, x_2 + h_2], \ldots, (x_m, x_m + h_m)$ of the set $\mathfrak{F}$ for which no points of $\alpha_1 E_\delta$ are to the right of $x_m + h_m$ or

$$(5) \qquad b_1 - (x_m + h_m) < \eta'.$$

Also the intervals $(x_i, x_i + h_i)$, $i = 1, 2, \ldots, m$, have been so chosen that

$$(6) \qquad \sum_{i=1}^{m} |E(x_i', x_i)|_\omega < \sum \epsilon_i < \frac{\eta}{n}.$$

This procedure can be repeated for each of the remaining intervals $\alpha_2, \alpha_3, \ldots, \alpha_n$ of the set $\alpha$.

From (1), (5), (6) and the relations similar to (5) and (6) for the intervals $\alpha_2, \ldots, \alpha_n$ it follows that the total set $\Delta_i = (x_i, x_i + h_i)$ obtained by this process are on $\alpha$, are non-overlapping and

$$\sum |E_\omega \Delta_i| > |\alpha E_\delta| - 2\eta.$$

This, with (2) and the fact that $E_\delta$ is on $\alpha$ and $\Delta_i E_\delta \subset \Delta_i E$, gives $\Sigma \Delta_i E > |E|_\omega{}^0 - 4\eta$. Because $\Delta_i$ is on $\alpha$ and because $\eta$ is arbitrary the lemma follows.

*Definition* 1. A function $F(x)$ defined on $[a, b]$ and in class $\mathfrak{U}$ is *absolutely continuous relative to* $\omega$, $AC - \omega$ if for $\epsilon > 0$ there exists $\delta > 0$ such that for any set of non-overlapping intervals $(x_i, x_i')$ on $[a, b]$ with $\Sigma\{\alpha(x_i' +) - \alpha(x_i -)\} < \delta$ the relation $\Sigma|F(x_i' +) - F(x_i -)| < \epsilon$ is satisfied.

**2. The derivatives of $F(x)$ with respect to $\omega$.** Let $F(x)$ be a function in class $\mathfrak{U}$. Define the function $\psi(x, h)$ by the relation

$$\psi(x, h) = \begin{cases} \dfrac{F(x+h) - F(x-)}{\omega(x+h) - \omega(x-)}, & h > 0, \omega(x+h) - \omega(x-) \neq 0, \\[2ex] \dfrac{F(x+h) - F(x+)}{\omega(x+h) - \omega(x+)}, & h < 0, \omega(x+h) - \omega(x+) \neq 0, \\[2ex] 0, & \omega(x+h) - \omega(x\pm) = 0, \end{cases}$$

for points of continuity $x + h$ of $\omega$. If $\psi(x, h)$ tends to a limit as $h \to 0$ this limit is the derivative of $F(x)$ with respect to $\omega(x)$, $D_\omega F$. The upper and lower limits of this ratio are the upper and lower derived numbers of $F(x)$

with respect to $\omega$. It is to be noted that $D_\omega F$ exists at points of discontinuity of $\omega$ even when $F(x)$ is not defined at such a point.

*Definition* 2. Let $f(x)$ be defined on an $\omega$-measurable set $E$ on $[a, b]$. Let $e_a$ be the part of $E$ for which $f(x) < a$. If *$e_a$ is $\omega$-measurable for every real number $a$ then $f$ is $\omega$-measurable on $E$.*

*Definition* 3. Let $f(x)$ be $\omega$-measurable on an $\omega$-measurable set $E$. Let $(l_{i-1}, l_i)$ be a subdivision of the range of $f$ on $E$. Let $e_i$ be the points of $E$ for which $l_{i-1} \leqslant f < l_i$. If $\Sigma l_{i-1}|e_i|_\omega$ *tends to a limit as the supremum of* $l_i - l_{i-1} \to 0$ *then this limit is the Lebesgue-Stieltjes integral of* $f(x)$ *over the set E.*

If $x$ is a point on $[a, b]$ and $f(x)$ is $\omega$-measurable on $[a, b]$ then

$$F(x) = \int_a^x f(x)d\omega$$

is absolutely continuous relatively to $\omega$. Also $F(x)$ is in class $\mathfrak{U}$ and $D_\omega F = f$ except for a set of $\omega$-measure zero. This definition of a Lebesgue-Stieltjes integral is the usual one and the stated properties may be proved in the usual way.

## 3. Properties of functions in class $\mathfrak{U}$.

*Definition* 4. Let $F(x)$ be a function in class $\mathfrak{U}$. The set $[x, F(x)]$ is the union of the graphs $[x, F(x+)]$ and $[x, F(x-)]$. For any interval of continuity $[x, F(x)]$ is thus the graph of $F(x)$ in the usual sense.

THEOREM 1. *Let $F(x)$ and $G(x)$ be two functions in class $\mathfrak{U}$. If $F$ and $G$ are $AC - \omega$ and if $D_\omega F = D_\omega G$, except at a set of $\omega$-measure zero, then the sets $[x, F(x)]$, $[x, G(x)]$ are identical or one is a translation parallel to the y-axis of the other.*

Set $H(x) = F(x) - G(x)$. Then $H$ is $AC - \omega$ and $D_\omega H = 0$ except for a set of $\omega$-measure zero. At a point of discontinuity $x_0$ of $\omega(x)$, $H(x_0-) = H(x_0+)$. For otherwise

$$\lim_{h\to 0} \frac{H(x_0 +) - H(x_0 \pm)}{\omega(x_0 + h) - \omega(x_0 \pm)} \neq 0$$

and it follows that $D_\omega H \neq 0$ on a set of $\omega$-measure greater than zero, which is a contradiction. If at points of discontinuity of $\omega$ we set $H(x) = H(x+) = H(x-)$ then $H(x)$ is continuous on $[a, b]$. We now prove that $H(x)$ is constant on $[a, b]$.

Let $E$ be the set on $[a, b]$ at which $D_\omega H = 0$. Then $|E|_\omega = |[a, b]|_\omega$. If $x$, is a point of $E$ and $\epsilon > 0$ is given there is a sequence of intervals $[x, x + h_i]$, $h_i > 0$, $h_i \to 0$ as $i \to \infty$ such that

$$\left|\frac{H(x + h_i) - H(x -)}{\omega(x + h_i) - \omega(x -)}\right| < \epsilon. \tag{1}$$

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By Lemma 2 there is a finite set $\Delta_i$ of these intervals associated with the set $E$ such that

(2) $$|\Sigma|\Delta_i|_\omega - |[a, b]|\omega|| < \epsilon, \quad \Sigma|\bar{\Delta}_j|_\omega < \epsilon,$$

where $\bar{\Delta}_j$ is the finite set of intervals complementary to the set $\Delta_i$. It follows from the second member of (2) and the fact that $H(x)$ is $AC - \omega$ that if $\eta$ is given then $\epsilon$ can be so fixed that if $[x_j, x_j']$ are the intervals of $\bar{\Delta}_j$ then

$$\Sigma|H(x_j') - H(x_j)| < \eta.$$

It follows from (1) that for the intervals $\Delta_i$

$$\Sigma|H(x_i') - H(x_i)| \leqslant \epsilon M$$

where $M$ depends on the function of bounded variation $\omega(x)$. It then follows that $|H(b) - H(a)| < \epsilon M + \eta$. Because $\epsilon$ and $\eta$ are arbitrary, $\epsilon$ fixed after $\eta$, it follows that $F(b) = F(a)$. If $a < x < b$ it can be shown in the same way that $H(x) = H(a)$. Hence $H(x)$ is constant on $[a, b]$.

It now follows that $F(x) - G(x) = C$, a constant, at points where both functions are continuous, that is, at the points of continuity of $\omega$. Furthermore, at points $x_0$ of discontinuity of $\omega$,

$$\lim_{x \to x_0 \pm 0} [F(x) - G(x)] = C,$$

$x$ a point of continuity of $\omega$, from which it follows that at points of discontinuity of $\omega$ the jumps, if any, of $F$ and $G$ are the same and in the same direction. It then follows that the sets $[x, F(x)]$, $[x, G(x)]$ are identical or one is a translation parallel to the $y$-axis of the other.

THEOREM 2. *If $F(x)$ is in class $\mathfrak{U}$ and is $AC - \omega$ and if $D_\omega F = f(x)$, except for a set of $\omega$-measure zero, then if $a$ and $x$ are points of continuity of $\omega$*

$$F(x) - F(a) = \int_a^x f(x)\, d\omega.$$

Let

$$G(x) = \int_a^x f(x)\, d\omega.$$

Then $G(x)$ is in class $\mathfrak{U}$, is $AC - \omega$, and $D_\omega G = f(x)$ except for a set of $x$ of $\omega$-measure zero. Hence $G$ and $F$ satisfy the conditions of Theorem 1, and, except at points of discontinuity of $\omega$

$$F(x) - G(x) = C.$$

Since $G(a) = 0$ it follows that $C = F(a)$ and

$$F(x) - F(a) = G(x) = \int_a^x f(x) d\omega.$$

This theorem gives point to the following definition:

*Definition* 5. Let $f(x)$ be defined on $[a, b]$ and be measurable relative to $\omega$. Let $F(x)$ be a function in class $\mathfrak{U}$ which is $AC - \omega$ and such that $D_\omega F = f(x)$ except possibly for a set of $\omega$-measure zero. *If the series* $\Sigma|f(x_i)\{\omega(x_i+) - \omega(x_i-)\}|$, *where* $x_i$, $i = 1, 2, \ldots$, *are the points of discontinuity of* $\omega$, *converges and if* $F(x_i+) - F(x_i-) = f(x_i)\{\omega(x_i+) - \omega(x_i-)\}$ *then* $F(x)$ *is the Lebesgue-Stieltjes integral of* $f(x)$:

$$F(x) - F(a) = \int_a^x f(t)\, d\omega(t).$$

*Definition* 6. *A function* $F(x)$ *is generalized absolutely continuous with respect to* $\omega$, $ACG - \omega$, *over* $[a, b]$ *if this interval is the sum of a denumerable sequence of closed sets over each of which* $F(x)$ *is* $AC - \omega$.

THEOREM 3. *Let* $F(x)$ *and* $G(x)$ *be two functions in class* $\mathfrak{U}$ *each of which is* $ACG - \omega$ *on* $[a, b]$, *and such that* $D_\omega F = D_\omega F$ *except for a set of* $\omega$-*measure zero. Then the sets* $[x, F(x)]$, $[x, G(x)]$ *are identical or one is a translation parallel to the y-axis of the other.*

As in Theorem 1, let $H(x) = F(x) - G(x)$ at points of continuity of $\omega(x)$. Then $D_\omega H = 0$ except for set of $\omega$-measure zero, and it follows as before that $H(x_0-) = H(x_0+)$ at points of discontinuity of $\omega$. Hence if $H(x_0)$ is equal to this common value then $H(x)$ is continuous on $[a, b]$. We now have $H(x)$ continuous and $ACG - \omega$ on $[a, b]$ and $D_\omega H = 0$ except for a set of $\omega$-measure zero. We show that $H(x)$ is constant on $[a, b]$.

Because $F$ and $G$ are both $ACG - \omega$ on $[a, b]$ it follows that

$$[a, b] = \sum E_i^1, \quad [a, b] = \sum E_j^2, \quad \text{where } E_i^1, E_j^2 \text{ are closed,}$$

where $F$ is $AC - \omega$ over each $E_i^1$ and $G$ is $AC - \omega$ over each $E_j^2$. Hence $H(x)$ is $AC - \omega$ over each of the sets $E_i^1 E_j^2$, $i, j = 1, 2, \ldots$ This is a denumerable sequence $\mathfrak{E}_1, \mathfrak{E}_2, \ldots$ of closed sets which cover $[a, b]$. Let $E$ be the set on $[a, b]$ which is such that if $x \in E$ then in every interval $\omega$ with $x$ as an interior point $H(x)$ fails to be constant. The set $E$ is closed. It then follows from Baire's theorem that there is an integer $n$ and an interval $[l, m]$ such that $E[l, m]$ is not empty, $E[l, m] = \mathfrak{E}_n[l, m] = e$. The set $e$ is closed and $H$ is $AC - \omega$ over $e$. Hence, if $\epsilon$ is given, there exists $\delta$ such that if $(x_i\, x_i')$ is a set of non-overlapping intervals on $[l, m]$ with $x_i$, $x_i'$ points of $e$ and with $\Sigma|(x_i, x_i')|_\omega < \delta$ then $\Sigma|H(x_i') - H(x_i)| < \epsilon$. Now let $(x_j, x_j')$ be *any* set of non-overlapping intervals on $[l, m]$ with $\Sigma|(x_j, x_j')|_\omega < \delta$. If there are points of $e$ on $[x_j, x_j']$ let $\bar{x}_j$ be the first point of $e$ to the right of $x_j$, $\bar{x}_j = x_j$ if $x_j \in e$. Let $\bar{x}_j'$ be the first point of $e$ to the left of $x_j'$, $\bar{x}_j' = x_j'$ if $x_j' \in e$. Because $H(x)$ is continuous, and constant on intervals of $[l, m]$ which are complementary to the set $e = \mathfrak{E}_n[l, m]$, it follows that $H(x)$ is constant on $[x_j, \bar{x}_j]$ and on $[\bar{x}_j', x_j']$. We now have $H(\bar{x}_j) - H(x_j) = H(x_j') - H(\bar{x}_j') = 0$ and

$$\sum |H(x_j') - H(x_j)| \leqslant \sum |H(\bar{x}_j) - H(x_j)| + \sum |H(\bar{x}_j') - H(\bar{x}_j)| + \sum |H(x_j') - H(\bar{x}_j')| \leqslant \sum |H(\bar{x}_j') - H(\bar{x}_j)| < \epsilon.$$

The last relation follows because $\bar{x}_j, \bar{x}_j'$ are points of $\mathfrak{E}_n$ and $\Sigma|(\bar{x}_j, \bar{x}_j')|_\omega \leqslant \Sigma|(x_j, x_j')|_\omega < \delta$. Then, because $(x_j, x_j')$ is any set of intervals on $[l, m]$ it follows that $H(x)$ is $AC - \omega$ on $[l, m]$. Because $D_\omega H = 0$ except for a set of $\omega$-measure zero it now follows from Theorem 1 that $H(x)$ is constant on $[l, m]$. Consequently, there are no points of $E$ on $[l, m]$. But this contradicts the fact that $E$ is not empty. Hence the set $E$ is empty. It now follows that every point of $[a, b]$ is interior to an interval on which $H(x)$ is constant. By the Heine-Borel theorem there is a finite set of intervals covering $[a, b]$ on each of which $H(x)$ is constant. Then, because $H(x)$ is continuous it easily follows that $H(x)$ is constant on $[a,b]$. The proof of Theorem 3 may now be completed as in Theorem 1.

THEOREM 3. *Let $F(x)$ be a function in class $\mathfrak{U}$ which is $ACG - \omega$ on $[a, b]$. Let $f(x)$ be a $\omega$-measurable function on $(a, b)$ and let $D_\omega F = f$ except for at most a set of $\omega$-measure zero. Then $F(b+) - F(a-)$ can be determined in at most a denumerable set of operations.*

LEMMA 2. *Let $E$ be a closed set on $[a, b]$. If $F(x)$ satisfies the conditions of Theorem 3 there is an interval $[l, m]$ on $[a, b]$ such that $E[l, m]$ is not empty, such that $D_\omega F$ is Lebesgue-Stieltjes integrable with respect to $\omega$ over $E[l, m]$, and such that if $[\alpha_i, \beta_i]$ are the intervals on $[l, m]$ contiguous to $E[l, m]$, then $\Sigma|F(\beta_i+) - F(\alpha_i-)|$ converges.*

Let $[a, b] = \mathfrak{E}_1 + \mathfrak{E}_2 + \ldots$ where each $\mathfrak{E}_n$ is closed and $F$ is $AC - \omega$ over $\supset_n$. There is then an integer $n$ and an interval $[l, m]$ such that $E[l, m] = \mathfrak{E}_n[l, m] = e$. We now turn our attention to the function $F(x)$ on $[l, m]$. $F$ is $AC - \omega$ over $e$. It then easily follows that there exists a positive number $M$ such that for any set of non-overlapping intervals $(x_i, x_i')$ on $[a, b]$ with $x_i, x_i'$ points of $e$ the relation $\Sigma|F(x_i'+) - F(x_i-)| < M$ holds. We use this fact in showing that $|D_\omega F| = |f|$ is Lebesgue-Stieltjes integrable with respect to $\omega$ over $e$ .Let $(l_{i-1}, l_i)$, $i = 1, 2, \ldots,$ be a subdivision of the range $[0, \infty]$. Let $e_i = E(l_{i-1} < |f| \leqslant l_i, x \in e)$, $i > 1$, $e_0 = E(l_0 \leqslant f \leqslant l_1)$. Suppose that $\Sigma l_{i-1}|e_i|_\omega$ diverges. Fix $n$ so that

$$\sum_{i=1}^{n} l_{i-1}|e_i|_\omega > 2M.$$

If $x \in e_i$ there is a sequence of intervals $(x, x + h_i)$, $h_i > 0$, $h_i \to 0$ such that

$$\frac{|F(x + h_i) - F(x-)|}{|\alpha(x + h_i) - \alpha(x-)|} > l_{i-1}. \tag{3}$$

Let $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$ be a sequence of positive numbers with $\epsilon_n \to 0$. By Lemma 2 there is a finite set $\Delta_i^0$ of non-overlapping intervals of the set associated with the set $e_0$ by means of (3) for which

$$(4) \qquad \sum |\Delta_i^0|_\omega > |e_0|_\omega - \epsilon_0, \quad \sum |\tilde{e}_0 \Delta_i^0|_\omega < \epsilon_0.$$

Because of the second relation of (4) there exists a finite set $\Delta_i^1$ of the intervals associated with $e_1$ by means of (3) such that

$$\sum |\Delta_i^1|_\omega > |e_1|_\omega - \epsilon_0 - \epsilon_1, \quad \sum |\tilde{e}_1(\Delta_i^0 + \Delta_1^1)|_\omega < \epsilon_0 + \epsilon_1,$$

and $\Sigma\Delta_i^1$ does not overlap $\Sigma\Delta_i^0$. If this process is continued there is obtained a set of intervals $\Delta_i^k$, which do not overlap the set $\Sigma\Delta_i^0 + \Sigma\Delta_i^1 + \ldots + \Sigma\Delta_i^{k-1}$, such that

$$\sum |\Delta_i^k|_\omega > |e_k|_\alpha - \epsilon_0 - \epsilon_1 - \ldots - \epsilon_k,$$

and

$$\sum |\tilde{e}_k(\Delta_i^0 + \Delta_i^1 + \ldots + \Delta_i^k)|_\omega < \epsilon_0 + \epsilon_1 + \ldots + \epsilon_k.$$

Also, because of (3), it is true that if $(x_i, x_i')$ are the intervals of the set $\Delta_i^k$ then

$$\frac{F(x_i'+) - F(x_i-)}{\alpha(x_i'+) - \alpha(x_i-)} > l_{k-1},$$

and

$$\sum \{F(x_i'+) - F(x_i-)\} > l_{k-1}(|e_k|_\omega - \epsilon_0 - \epsilon_1 - \ldots - \epsilon_k).$$

Combining all the sets $\Delta_i^k$ into a single set $\Delta_i = (x_i, x_i')$ and summing over this set we get

$$\sum |F(x_i'+) - F(x_i-)| > \sum_{k=1}^{n} l_{k-1}|e_k|_\omega - nl_0\epsilon_0 - (n-1)l_1\epsilon_1 - \ldots - l_{n-1}\epsilon_n.$$

The first sum on the right is greater than $2M$. The numbers $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ are independent of $n$, and independent of the numbers $l_0, l_1, \ldots, l_{n-1}$. Hence the left side is not less than $2M$. But $x_i, x_i'$ are points of $e$ which compels the left side to be less than $M$. Thus there is a contradiction and we may conclude that $\Sigma l_{i-1}|e_i|_\omega$ converges. It then follows that $|D_\omega F|$ is Lebesgue-Stieltjes integrable with respect to $\omega$ over $e$.

Let $(\alpha_i, \beta_i)$ be the intervals on $[l, m]$ complementary to $e$. Let $e > 0$ be given. Fix $n_0$ with

$$\sum_{n_0+1}^{\infty} |F(\beta_i+) - F(\alpha_i-)| < \epsilon.$$

Take $(l_{i-1}, l_i)$ a subdivision of $(-\infty, \infty)$. Let $e_i, e_i'$ be respectively the parts of $e$ for which $l_{i-1} < D_\omega F \leqslant l_i$, $l_{i-1} \leqslant D_\omega F < l_i$. Let the subdivision $(l_{i-1}, l_i)$ and the integer $n_1$ be such that

$$\sum_{-n_1}^{n_1} l_{i-1}|e_i|_\omega, \quad \sum_{-n_1}^{n_1} l_i|e_i'|_\omega$$

differ from the Lebesgue-Stieltjes interval of $D_\omega F$ over $e$ by not more than $\epsilon$. By working as before with the sets $e_i$ we can get a set of intervals $\Delta_i$ with end points $x_i$, $x_i'$ belonging to $e$ and such that the intervals $\Delta_i$ do not overlap the intervals $(\alpha_i, \beta_i)$, $i = 1, 2, \ldots, n_0$, such that

$$\sum_{i=1}^{k} F(x_i'+) - F(x_i-) > \sum_{i=1}^{k} l_{i-1}|e_i|_\omega - \epsilon > \int_e D_\omega F d_\omega - 2\epsilon,$$

and such that the finite set of intervals $\bar{\Delta}_j$ complementary to the set $\Delta_i$ and the set $(\alpha_i, \beta_i)$, $i = 1, 2, \ldots, n_0$, satisfy $\Sigma|\bar{\Delta}_j|_\omega < \delta$, from which it follows that $\Sigma|F(x_j') - F(x_j)| < \epsilon$. Hence

$$\begin{aligned} F(m+) - F(l-) &= \sum_{1}^{n_0} \{F(\beta_i) - F(\alpha_i)\} + \sum \{F(x_i'+) - F(x_i-)\} \\ &\qquad + \sum \{F(x_j') - F(x_j)\} \\ &> \sum_{1}^{\infty} \{F(\beta_i) - F(\alpha_i)\} + \int_e D_\omega F d_\omega - 3\epsilon. \end{aligned}$$

If a similar procedure is used with the sets $e_i'$ it may be shown that

$$F(m+) - F(l-) < \sum_{1}^{\infty} \{F(\beta_i+) - F(\alpha_i-)\} + \int_e D_\omega F d_\omega + 3\epsilon.$$

Because $\epsilon$ is arbitrary we conclude that

$$F(m+) - F(l-) = \int_e D_\omega F d\omega + \sum \{F(\beta_i-) - F(\alpha_i+)\}.$$

We can now state that if $E$ is any closed set on $[a, b]$ there is an interval $[l, m]$ containing points of $E$ such that $D_\omega F$ is summable over $E[l, m]$ and such that if $(\alpha_i, \beta_i)$ are the intervals on $[l, m]$ complementary to the set $E[l, m]$ then

$$F(m+) - F(l-) = \int_{E[l,m]} D_\omega F d\omega + \sum \{F(\beta_i+) - F(\alpha_i-)\}.$$

Let $E_1$ be the points of non-summability of $D_\omega F$ over $[a, b]$, $(\alpha_i, \beta_i)$ the intervals complementary to $E_1$. If $(\alpha', \beta')$ is an interval such that $\alpha_i < \alpha' < \beta' < \beta_i$ then $D_\omega F$ is Lebesgue-Stieltjes integrable over $[\alpha', \beta']$ and

$$F(\beta'+) - F(\alpha'-) = \int_{\alpha'}^{\beta'} D_\omega F d\omega.$$

Because of the continuity properties of $F$ it follows that as $\alpha' \to \alpha_i$, $\beta' \to \beta_i$

$$F(\beta'+) - F(\alpha'-) \to F(\beta_i-) - F(\alpha_i+),$$

and

$$\begin{aligned} F(\beta_i+) - F(\alpha_i-) &= F(\beta_i-) - F(\alpha_i+) \\ &\qquad + \int_{\beta_i} D_\omega F d\omega - \int_{\alpha_i} D_\omega F d\omega. \end{aligned}$$

Thus $F(\beta_i+) - F(\alpha_i-)$ is determined for all intervals $(\alpha_i, \beta_i)$ contiguous to the set $E_1$. Now let $E_2$ be the points of $E_1$ which are such that if $x \in E_2$ there is no interval $[l, m]$ containing $x$ with $D_\omega F$ summable over $E_1[l, m]$ and $\Sigma\{F(\beta_i+) - F(\alpha_i-)\}$ converging where $(\alpha_i, \beta_i)$ are the intervals on $[l, m]$ contiguous to the set $E_1[l, m]$. The set $E_2$ is closed and, by Lemma 2, non-dense on $E_1$. If $(\alpha_i, \beta_i)$ are the intervals complementary to $E_2$ the procedure used for the intervals complementary to $E_1$ can now be used to obtain $F(\beta_i+) - F(\alpha_i-)$ for these intervals $(\alpha_i, \beta_i)$ complementary to $E_2$. This process can be continued by transfinite inductions to arrive at $F(b+) - F(a-)$ in a denumerable number of steps.

A consideration of Theorem 3 leads to the following definition.

*Definition* 7. Let $\omega$ be a non-decreasing function on $[a, b]$ and let $f(x)$ be defined on $[a, b]$ and be measurable relatively to $\omega$. If there exists a function $F(x)$ in class $\mathfrak{U}$ which is $ACG - \omega$ on $[a, b]$ and is such that $D_\omega F = f$ except for a set of $\omega$-measure zero, and for which the relations of Definition 5 are satisfied at the points of discontinuity of $\omega$, then $F(x)$ is an *indefinite Lebesgue-Stieltjes integral of f with respect to* $\omega$.

The descriptive definition of an integral with respect to a non-decreasing function $\omega$ given here appears to be equivalent to the constructive definition given in (**3**, p. 666). This requires proof. Another problem for investigation is that of extending the methods of the present paper to the case in which the base function $\omega$ is *VBG*.

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