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RELATIONS BETWEEN THE GENERA AND BETWEEN THE HASSE-WITT INVARIANTS OF GALOIS COVERINGS OF CURVES

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To the memory of R. A. Smith

ABSTRACT. Let $G \subset \text{Aut}(C)$ be a (finite) group of automorphisms of a curve C defined over a field K and, for each subgroup $H \leq G$, let g_H denote the genus of the quotient curve $C_H = C/H$ (briefly: quotient genus of H).

In this paper we show that certain idempotent relations in the rational group ring $\mathbb{Q}[G]$ imply relations between the quotient genera $\{g_H\}_{H\leq G}$; this generalizes two theorems of Accola. Moreover, we show that in the case of char $(K) = p \neq 0$, a similar statement holds for the Hasse-Witt invariants σ_H of the curves C_H.

1. **Introduction**. Let C be a curve defined over an arbitrary field K, and let $G \subset \text{Aut}(C)$ be a finite group of automorphisms acting on C. For any subgroup $H \subseteq G$, let g_H denote the *quotient genus* of H, i.e, the genus of the quotient curve $C_H = C/H$. In his article, R. D. Accola [1] established (for $K = \mathbf{C}$) two theorems which, under certain conditions on the group G, give relations between the quotient genera $\{g_H\}_{H\leqq G}$ of the various subgroups of G.

The purpose of this note is two-fold. First, we observe that both of Accola's theorems are, in fact, special cases of a much more general theorem which shows that (certain) idempotent relations in the rational group ring $\mathbf{Q}[G]$ imply relations between the quotient genera. To be exact, if for a subgroup $H \leqq G$ we let

$$(1) \qquad \epsilon_H = \frac{1}{|H|} \sum_{h\in H} h \in \mathbf{Q}[G]$$

denote the "norm idempotent" associated to H, then we have:

THEOREM 1. *Any relation*

$$(2) \qquad \sum_H r_H \epsilon_H = 0 \qquad (r_H \in \mathbf{Q})$$

between the norm idempotents yields a relation

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(3) $$\sum_H r_H g_H = 0$$

between the quotient genera.

From Theorem 1 it is quite easy to deduce the aforementioned theorems of Accola:

COROLLARY 1. *(Accola). Suppose $H_1, \ldots, H_t \leqq G$ are subgroups of G such that $G = H_1 \cup \cdots \cup H_t$. Then:*

(4) $$|G|g_G = \sum_{r=1}^{t} (-1)^{r+1} \sum_{1 \leqq i_1 < \cdots < i_r \leqq t} |H_{i_1} \cap \cdots \cap H_{i_r}| g_{H_{i_1} \cap \cdots \cap H_{i_r}}.$$

COROLLARY 2. *(Accola). Suppose $H_1, \ldots, H_t \leqq G$ are subgroups of G satisfying the following conditions:*

(1) $H_i \cdot H_j = H_j \cdot H_i, \quad \forall i, j$
(2) *For any (complex) irreducible character χ of G there exists a subgroup $H_i \subset$* Ker χ.
Then:

(5) $$g_1 = \sum_{r=1}^{t} (-1)^{r+1} \sum_{1 \leqq i_1 < \cdots < i_r \leqq t} g_{H_{i_1} \cdot \cdots \cdot H_{i_r}}.$$

REMARK. Corollary 2 is slightly better than Accola's theorem since we need not assume that the H_i's are *normal* subgroups of G.

As we shall see below, not only are these corollaries easily deduced from Theorem 1, but this theorem itself is easily established from known properties of the (global) *Artin representation*. As a result, this gives a simpler and more transparent proof of Accola's results, and also shows that these theorems are valid in arbitrary characteristic.

The second aim of this paper concerns the case that the ground field has a *non-zero* characteristic $p \neq 0$. In that case each quotient curve C_H has besides its genus another invariant attached to it, namely its *Hasse-Witt invariant* σ_H, which may be defined by

(6) $$(J_H)_p = p^{\sigma_H},$$

where J_H denotes the Jacobian variety of C_H and $(J_H)_p$ the group of p-torsion points on J_H. We then have the following result analogous to Theorem 1:

THEOREM 2. *Any relation (2) between the norm idempotents yields a relation*

(3′) $$\sum_H r_H \sigma_H = 0$$

between the Hasse-Witt invariants of the quotient curves.

One then has the following corollaries of "Accola type":

COROLLARY 1. *In the situation of Corollary 1 to Theorem 1, we have*

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$$(4') \qquad |G|\sigma_G = \sum_{r=1}^{t} (-1)^{r+1} \sum_{1 \leqq i_1 < \cdots < i_r \leqq t} |H_{i_1} \cap \cdots \cap H_{i_r}| \sigma_{H_{i_1} \cap \cdots \cap H_{i_r}}.$$

COROLLARY 2. *In the situation of Corollary 2 to Theorem 1, we have*

$$(5') \qquad \sigma_1 = \sum_{r=1}^{t} (-1)^{r+1} \sum_{1 \leqq i_1 < \cdots < i_r \leqq t} \sigma_{H_{i_1} \cdot \cdots \cdot H_{i_r}}$$

Not only are the statements of Theorems 1 and 2 analogous; it is, in fact, possible to give a unified proof or both theorems using l-adic representations (cf. section 4). This proof also has the advantage that it generalizes to yield a theorem (Theorem 3 below) which establishes (under a hypothesis analogous to (2)) a relation between the genera (and the Hasse-Witt invariants, if applicable) of arbitrary (i.e. not necessarily galois) subcovers of C.

2. **Proof of Theorem 1 (via the Artin representation)**. As before, let C be a (smooth, irreducible, complete) curve defined over a field K of arbitrary characteristic p. (Since there is no loss of generality in assuming that K is algebraically closed, we shall do so henceforth). Recall (cf. Serre [3], p. 105) that to any finite subgroup $G \subset \text{Aut}(C)$ we can attach a complex character a_G, called the *(global) Artin character* such that the following property holds:

(*) *If* $H \leqq G$ *is any subgroup, and* $s_{G/H} = \text{Ind}_H^G 1_H$ *denotes the character of the representation of G on the coset space* G/H, *then*

$$(7) \qquad (s_{G/H}, a_G)_G = \text{deg disc}(C_H/C_G).$$

Here, as usual,

$$(8) \qquad (\phi, \psi)_G = \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g^{-1})$$

denotes the inner product of two class functions ϕ and ψ on G, and $\text{disc}(C_H/C_G) \in \text{Div}(C_G)$ denotes the discriminant divisor of the finite covering

$$\pi : C_H \to C_G$$

induced by the inclusion $H \leqq G$. Note that by using the Riemann-Hurwitz formula,

$$(9) \qquad 2g_H - 2 = \frac{|G|}{|H|}(2g_G - 2) + \text{deg disc}(C_H/C_G)$$

we can re-write (7) as

$$(7') \qquad (s_{G/H}, a_G) = 2(g_H - 1) - 2\frac{|G|}{|H|}(g_G - 1)$$

From (*) the proof of Theorem 1 follows almost immediately. To see this, we simply observe that for any class function χ on G, we have (by definition and Frobenius reciprocity):

$$\chi(\epsilon_H) = (1_H, \chi_{|H})_H = (s_{G/H}, \chi)_G$$

and so

$$a_G(\epsilon_H) = 2(g_H - 1) - 2\frac{|G|}{|H|}(g_G - 1) \tag{10}$$

Thus, if a relation (2) holds, then on the one hand we obviously have

$$\sum_H r_H a_G(\epsilon_H) = 0 \tag{11}$$

and on the other hand we have

$$\Sigma\, r_H = 1_G(\Sigma r_H \epsilon_H) = 0, \tag{12}$$

and

$$\Sigma\, r_H/|H| = \mathrm{reg}_G(\Sigma\, r_H \epsilon_H) = 0, \tag{13}$$

so the genus relation (3) follows immediately from equations (10)–(13).

3. **Proof of Corollaries 1 and 2**. Suppose first that $G = H_1 \cup \cdots \cup H_t$. Then by the usual counting procedure we have

$$\sum_{g \in G} g = \sum_{i=1}^{t} (-1)^{r+1} \sum_{1 \leqq i_1 < \cdots < i_r \leqq t} \sum_{g \in H_{i_1} \cap \cdots \cap H_{i_r}} g$$

or

$$|G|\epsilon_G = \sum_{i=1}^{t} (-1)^{r+1} \sum_{1 \leqq i_1 < \cdots < i_r \leqq t} |H_{i_1} \cap \cdots \cap H_{i_r}| \epsilon_{H_{i_1} \cap \cdots \cap H_{i_r}}$$

which gives Corollary 1.

REMARK. Corollary 1 is particularly useful when one deals with groups G which have a *partition*, i.e. for which there exist subgroups $H_1, \ldots, H_t \leqq G$ with $H_i \cap H_j = 1$ for $i \neq j$ such that $H_1 \cup \cdots \cup H_t = G$. In that case (4) simplifies to:

$$|G| g_G = \sum_{i=1}^{t} |H_i| g_{H_i} - (t-1) g_1 \tag{4'}$$

For example, elementary abelian p-groups, Frobenius groups etc. are groups with partition.

Suppose next that G satisfies conditions (1) and (2) of Corollary 2. We first observe that condition (2) implies

$$\epsilon \overset{\text{def}}{=} (1 - \epsilon_{H_1}) \cdots (1 - \epsilon_{H_t}) = 0 \tag{14}$$

To see this, note that if ρ is a representation with Ker $\rho \supset H_i$ then $\rho(\epsilon_{H_i}) = \rho(1)$ and hence $\rho(\epsilon) = 0$. Thus the hypothesis (2) implies $\rho(\epsilon) = 0$ for all irreducible representations of G, and so we have $\epsilon = 0$ as claimed.

Next, from condition (1) we infer that any two ϵ_{H_i} and ϵ_{H_j} commute and also that

$$\epsilon_{H_{i_1}\cdots H_{i_r}} = \epsilon_{H_{i_1}} \cdots \epsilon_{H_{i_r}}.$$

(Observe that by condition (1), $H_{i_1} \cdots H_{i_r}$ is a group!) We can therefore re-write (14) in the form

$$(14') \qquad 1 - \sum_{i=1}^{r} (-1)^{r+1} \sum_{1 \leqq i_1 < \cdots < i_r \leqq t} \epsilon_{H_{i_1}\cdots H_{i_r}} = 0,$$

from which (5) is immediate by Theorem 1.

4. **Proof of Theorems 1 and 2 (via *l*-adic representations)**. Let $J = J_C$ denote the Jacobian variety of C; in the sequel we shall identify the K-rational points of J with the group $\mathrm{Pic}^0(C) = \mathrm{Div}^0(C)/\mathrm{Div}_l(C)$ of divisor classes of degree 0:

$$J_C(K) = \mathrm{Pic}^0(C).$$

Any automorphism $\alpha \in \mathrm{Aut}(C)$ of C induces an automorphism α_* of $\mathrm{Div}(C)$ via

$$\alpha_*(\Sigma\, n_i P_i) = \Sigma\, n_i \alpha(P_i)$$

and hence an automorphism on J; we thus have a representation

$$j_* : \mathrm{Aut}(C) \to \mathrm{End}(J_C)$$

which extends to a representation

$$j_* : \boldsymbol{Q}[\mathrm{Aut}(C)] \to \mathrm{End}^0(J_C) \stackrel{\mathrm{def}}{=} \boldsymbol{Q} \otimes_{\mathbb{Z}} \mathrm{End}\,(J).$$

For any prime number l, we have the l-adic *Tate-module*

$$T_l(J) = \lim_{n} J_l$$

which is known to be a free $\boldsymbol{Z}_l$-module of rank $\dim_{F_l} J_l$ (cf. e.g. Mumford [2], p. 171); thus

$$(15) \qquad \mathrm{rank}_{Z_l} T_l(J) = \begin{cases} 2g_1 & \text{if } l \neq p \\ \sigma_1 & \text{if } l = p \end{cases}.$$

Each endomorphism $\alpha \in \mathrm{End}(J)$ acts on $T_l(J)$ in a natural way, so we have a representation

$$T_l : \mathrm{End}\,(J) \to \mathrm{End}_{Z_l}(T_l(J))$$

which extends to a $\boldsymbol{Q}_l$-representation

$$T_l^0 : \mathrm{End}^0(J) \to \mathrm{End}_{Q_l}(V_l(J)),$$

where $V_l(J) = \boldsymbol{Q}_l \otimes_{Z_l} T_l(J)$. Combining this with j_*, we obtain a $\boldsymbol{Q}_l$-rational representation

$$\rho_l = T_l^0 \circ j_* : \boldsymbol{Q}_l[G] \to \mathrm{End}_{Q_l}(V_l(J)),$$

whose character we denote by v_l = trace $T_l^0 \circ j_*$. Then Theorems 1 and 2 are both consequences of the following fact.

PROPOSITION 1. *For any subgroup $H \leqq G$ we have*

$$v_l(\epsilon_H) = \begin{cases} 2g_H & \text{if } l \neq p \\ \sigma_H & \text{if } l = p \end{cases}.$$

To prove this, we first observe that by (15) this is clear for $H = 1$; and hence true in general by the following more general fact.

PROPOSITION 2. *For any finite covering $\pi : C \to C'$ of curves, there exists a $\boldsymbol{Q}$-algebra homomorphism*

$$\pi^* : \mathrm{End}^\circ(J') \to \mathrm{End}^\circ(J),$$

(where J and J' denote the Jacobian varieties of C and C', respectively) such that

(1) If π is a galois covering with group G, then

$$j_*(\epsilon_G) = \pi^*(\mathrm{id}_{J'})$$

(2) For any $\alpha' \in \mathrm{End}(J')$ we have

$$\mathrm{trace}(\pi^*(\alpha')|V_l(J)) = \mathrm{trace}(\alpha'|V_l(J'))$$

PROOF. Consider the homomorphisms

$$\pi_* : J \to J'$$
$$\pi^* : J' \to J$$

which are induced by π, and put, for $\alpha' \in \mathrm{End}^\circ(J')$,

$$\pi^*\alpha' = \frac{1}{n}(\pi^* \circ \alpha \circ \pi_*), \tag{16}$$

where $n = \deg \pi$. If A denotes the connected component of Ker π_*, then we have an exact sequence of $\boldsymbol{Q}_l$-vector spaces

$$0 \to V_l(A) \to V_l(J) \overset{T_l^0(\pi_*)}{\to} V_l(J') \to 0$$

which is split by $1/n\, T_l^0(\pi^*)$ and hence yields the decomposition

$$V_l(J) = V_l(A) \oplus \pi^*V_l(J')$$

where we have written $\pi^*V_l(J')$ in place of $T_l^0(\pi^*)(V_l(J')) = 1/n\, T_l^0(\pi^*)(V_l(J'))$. Then by construction we have

$$\pi^*\alpha'|_{V_l(A)} = 0$$
$$\pi^*\alpha'|_{\pi^*V_l(J')} = \alpha'$$

(upon identifying $V_l(J') \overset{\sim}{\to} \pi^*V_l(J')$), so trace $\pi^*\alpha'$ = trace α, as claimed.

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5. **Generalization to the non-galois case**. Let C be a curve as before, and let

$$\pi_1 : C \to C_i, \qquad 1 \leqq i \leqq N$$

be a finite system of subcovers of C. To any such subcover π_i we can associate an idempotent $\epsilon_i \in \mathrm{End}^\circ(J_C)$ by

$$\epsilon_i = \pi_i^*(\mathrm{id}_{J_{C_i}})$$

where $\pi_i^* : \mathrm{End}^\circ(J_{C_i}) \to \mathrm{End}^\circ(J_C)$ is the homomorphism constructed in Proposition 2. The proof of Theorems 1 and 2 given in the previous section immediately also proves:

THEOREM 3. *Any relation*

$$\sum_{i=1}^{N} r_i \epsilon_i = 0 \tag{17}$$

between the idempotents ϵ_i *yields a relation*

$$\sum_{i=1}^{N} r_i g_i = 0 \tag{18}$$

between the genera g_i *of the subcovers* $C_i (1 \leqq i \leqq N)$ *and also, if* char $K \neq 0$, *a relation*

$$\sum_{i=1}^{N} r_i \sigma_i = 0 \tag{19}$$

between the Hasse-Witt invariants σ_i *of* C_i, $1 \leqq i \leqq N$.

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